Superpotentials, A_∞ Relations and WDVV Equations for Open Topological Strings

M. Herbst, C. Lazaroiu, W. Lerche, hep-th/0402110

1. Closed string TCFT

2. Basic quantities in open string TCFT

3. Consistency relations

   (a) A_∞ relations

   (b) Bulk-boundary crossing relation

   (c) Cardy relation

4. Application: LG minimal models

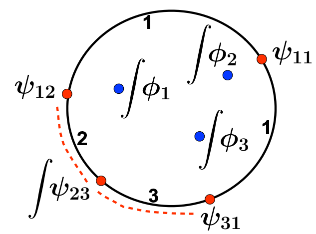
Notation/Language/WhatIsItAbout

- "bulk" sector = closed string sector: operators ϕ_i, deformation parameters t_i

- "boundary", "brane" sector = open string sector; D-branes = boundary conditions;
  operators ψ, deformation parameters s:
  "boundary preserving": $\psi_a \equiv \psi_{aa} \sim$ Hom(a, a)
  "boundary changing": $\psi_{ab} \sim$ Hom(a, b)

- objective: compute $\mathcal{W}(t_i, s_a)$, which is understood here as generating function of deformed disk correlators $\mathcal{F}$

$$\begin{aligned}\mathcal{W}(t_i, s_a) &\sim \langle e^{t_i \int_D d^2z\, \phi_i}\, P e^{s_a \int_{\partial D} dx\, \psi_a} \rangle \\ &= \sum s_{a_m} \ldots s_{a_1} \partial_{t_{i_n}} \ldots \partial_{t_{i_n}} \mathcal{F}_{a_1, \ldots, a_n}(t_k)\end{aligned}$$



- Derive conditions on $\mathcal{A}$ (and thus, $\mathcal{W}$) from TFT consistency conditions (Moore, Segal, Lazaroiu)

TCFT on the sphere (I)

cohomological TCFT from twisting a $\mathcal{N} = (2,2)$ SCFT

$$Q^2 = 0$$
$$T(z) = [Q, G(z)]$$

physical fields and descendants:

$$[Q, \phi_i] = 0$$

$$[Q, \phi_i^{(1,0)}] = \partial \phi_i$$
$$[Q, \phi_i^{(0,1)}] = \bar{\partial} \phi_i$$
$$[Q, \phi_i^{(2)}] = d\phi_i^{(1)}$$

integrated insertions:

$$[Q, \int_{S^2} \phi_i^{(2)}] = 0$$

basic deformed correlation functions (fix $SL(2,C)$):

$$C_{i_1 \ldots i_n} := \langle \phi_{i_1} \ldots \phi_{i_3} \int \phi_{i_4}^{(2)} \ldots \int \phi_{i_n}^{(2)} \rangle_{S^2}$$

depend on Q-cohomology classes and are independent of world-sheet metric and are constant.

TCFT on the sphere (II)

write in terms of deformed 3-point correlation functions:

$$C_{i_1 i_2 i_3}(t) = \langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \; e^{\sum_i t_i \int_{S^2} \phi_i^{(2)}} \rangle$$

Ward identities of the current $G(z)$ and $\bar{G}(\bar{z})$ fix contact terms:

- correlation functions are symmetric under exchange of all fields.
- the 2-point function is independent of perturbations.

integrability (DVV):

$$\partial_{i_0} C_{i_1 i_2 i_3}(t) = \partial_{i_1} C_{i_0 i_2 i_3}(t)$$

prepotential $\mathcal{F}$ (eff. lagrangian in $N = 2$ supergravity):

$$C_{i_1 i_2 i_3}(t) = \partial_{i_1} \partial_{i_2} \partial_{i_3} \mathcal{F}(t)$$

factorization - WDVV equations:

$$\partial_i \partial_j \partial_m \mathcal{F} \; \eta^{mn} \; \partial_n \partial_k \partial_l \mathcal{F} = \partial_i \partial_k \partial_m \mathcal{F} \; \eta^{mn} \; \partial_n \partial_j \partial_l \mathcal{F}$$

What is the open string analog ?

TCFT on the disk (I)

boundary conditions - $\mathcal{N}=(2,2)\to\mathcal{N}=2$:

$$T(z) = \bar{T}(\bar{z})\big|_{z=\bar{z}} \,, \qquad G(z) = \bar{G}(\bar{z})\big|_{z=\bar{z}}$$

physical fields and descendants:

$$[Q,\psi_a] = 0$$

$$[Q,\psi_a^{(1)}] = d\psi_a$$

integrated insertions:

$$[Q,\int_{D^2}\phi_i^{(2)}] = \int_{\partial D^2}\phi_i^{(1)} \,, \qquad [Q,\int_{\tau_L}^{\tau_R}\psi_a^{(1)}] = \psi_a\Big|_{\tau_L}^{\tau_R}$$

Q maps to the boundary of the (super)integration domain!

The boundary contributions are a new feature, and complication, as compared to the closed string correlators on the sphere

TCFT on the disk (II)

basic correlation functions (fix $SL(2,R)$):

$$B_{a_0\dots a_m;i_1\dots i_n} := -\Big\langle\ \phi_{i_1}\psi_{a_0}\ P\int\psi_{a_1}^{(1)}\dots\int\psi_{a_m}^{(1)}\ \int\phi_{i_2}^{(2)}\dots\int\phi_{i_n}^{(2)}\ \Big\rangle$$

$$= (-1)^{\sum_{\ell=1}^{m-1}(a_\ell+1)}\Big\langle\ \psi_{a_0}\psi_{a_1}\ P\int\psi_{a_2}^{(1)}\dots\int\psi_{a_{m-1}}^{(1)}\ \psi_{a_m}\int\phi_{i_1}^{(2)}\dots\int\phi_{i_n}^{(2)}\ \Big\rangle$$

Ward identity of the current $G(z)$:

- the metric $\omega_{ab} := \langle\psi_a\psi_b\rangle$ does not get corrections from integrated insertions
- the correlators are constant, independent of WS–metric
- Ward identity relates the two kinds of correlation functions (exceptions: $B_{a;i}$, $B_{ab;i}$ and B_{abc})
- the correlators are symmetric in bulk fields: integrability wrt t_i

$$\mathcal{F}_{a_0\dots a_m}(t) := \pm\,\langle\psi_{a_0}\psi_{a_1}P\int\psi_{a_2}\dots\int\psi_{a_{m-1}}\psi_{a_m}e^{\sum_i t_i\int_{D^2}\phi_i^{(2)}}\rangle\ ,$$

$$\partial_i\mathcal{F}_a(t) := -\,\langle\phi_i\ \psi_a\ e^{\sum_i t_i\int_{D^2}\phi_i^{(2)}}\rangle\ ,$$

$$\partial_i\mathcal{F}_{ab}(t) := -\,\langle\phi_i\ \psi_a\ P\int\psi_b^{(1)}\ \ e^{\sum_i t_i\int_{D^2}\phi_i^{(2)}}\rangle$$

- Note: tadpoles $\mathcal{F}_a(t)$ and $\mathcal{F}_{ab}(t)$ vanish, if the bulk moduli t_i are turned off (ie., switching them on requires adjusting boundary deformations).

- Grassmann grading:

  While integrated and unintegrated bulk fields have the same Z_2 grade, the grades of a physical boundary field and its integrated descendant differ

  "suspended" grading: $\tilde{a} = |\psi_a| + 1 \pmod 2$

  assoc with deformation parameters ($s_a \int \psi_a^{(1)}$): $|s_a| = \tilde{a}$

- the correlators are invariant under cyclic permutations of the boundary fields:

$$\mathcal{F}_{a_0 \ldots a_m}(t) = (-1)^{\tilde{a}_m(\tilde{a}_0 + \ldots + \tilde{a}_{m-1})} \mathcal{F}_{a_m a_0 \ldots a_{m-1}}(t)$$

  The suspended grading $\tilde{a}_l$ is the natural one!

Superpotential $\mathcal{W}$

Infinite sequence of t-dependent prepotentials:

$\mathcal{F}(t)$ prepotential (on the sphere)

$$\left.\begin{array}{c} \mathcal{F}_{a_1}(t) \\ \mathcal{F}_{a_1 a_2}(t) \\ \mathcal{F}_{a_1 a_2 a_3}(t) \\ \mathcal{F}_{a_1 a_2 a_3 a_4}(t) \\ \vdots \end{array}\right\}$$

cyclic correlators (on the disk) do in general not integrate

Encode in generating function $\mathcal{W}$:

$$\begin{aligned} \mathcal{W}(s,t) &= \sum_{m \geq 1} \frac{1}{m} s_{a_m} \ldots s_{a_1} \; \mathcal{F}_{a_1 \ldots a_m}(t) \\ &= \sum_{m \geq 1} \frac{1}{m!} s_{a_m} \ldots s_{a_1} \; \mathcal{A}_{a_1 \ldots a_m}(t) \end{aligned}$$

symmetrized string amplitude (for boundary preserving sectors):

$$\mathcal{A}_{a_1 \ldots a_m}(t) := (m-1)! \; \mathcal{F}_{a_1 (a_2 \ldots a_m)}(t)$$

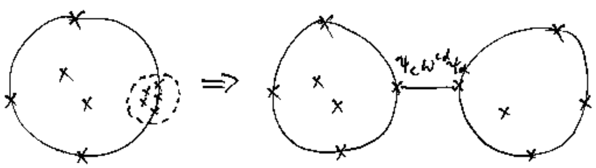
ordering matters for the boundary changing open string "moduli" s_a $\sim$ non-commutative (cyclic derivatives)

A_∞ relations (I)

$$\langle [Q, \phi_{i_0}\psi_{a_0} P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_m} \int \phi^{(2)}_{i_1} \ldots \int \phi^{(2)}_{i_n}] \rangle = 0$$

Q maps to the boundary of the (super)integration domain:

- $m \geq 2$ boundary fields and $n \geq 0$ bulk fields approach each other
- $m \geq 0$ boundary fields and $n \geq 1$ bulk fields approach each other



The factorization by insertion of a complete system of boundary fields leads to a cyclic A_∞ structure:

$$\sum_{\substack{k,j=0\\ k\leq j}}^{m} (-1)^{\tilde{a}_1+\ldots+\tilde{a}_k}\ \mathcal{F}^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m}(t)\ \mathcal{F}^{c}{}_{a_{k+1}\ldots a_j}(t) \ = \ 0$$

...derivation quite technical!

A_∞ relations (II)

Define open string scattering products $r_m : H_o^{\otimes m} \to H_o$ through the relations:

$$\begin{array}{rcl} r_m(\psi_{a_1}\ldots\psi_{a_m}) & := & \mathcal{F}^a{}_{a_1\ldots a_m}(t)\ \psi_a \quad \text{for} \quad m\geq 1 \\ r_o(1) & := & \mathcal{F}^a(t)\ \psi_a \quad . \end{array}$$

Then the A_∞ algebra becomes:

$$\sum_{\substack{k,j=0\\ k\leq j}}^{m} (-1)^{\tilde{a}_1+\ldots+\tilde{a}_k} r_{m-j+k}(\psi_{a_1}\ldots\psi_{a_k}, r_{j-k}(\psi_{a_{k+1}}\ldots\psi_{a_j}), \psi_{a_{j+1}}\ldots\psi_{a_m}) = 0$$

minimal A_∞	if	$r_0 = r_1 = 0$	undef. theory
(strong) A_∞	if	$r_0 = 0$	
DGA	if	$r_1, r_2 \neq 0$ only	
weak A_∞	if	$r_m \neq 0$ for $m = 0, 1, \ldots$	def. theory

The bulk field insertions deform a minimal A_∞ algebra into a weak A_∞ algebra ($\mathcal{F}_a|_{t=0} = \mathcal{F}_{ab}|_{t=0} = 0$). (Hochschild cohomology)

open string field theory

Gaberdiel, Zwiebach (hep-th/9705038):
OSFT has the structure of a cyclic A_∞ algebra

Witten: (hep-th/9207094):
topological open strings described by Chern-Simons theory as OSFT

Define a string field

$$\psi = \sum_a s_a \psi_a \in H_o$$

and non-degenerate bilinear form on H_o:

$$\omega(\psi_a, \psi_b) := \omega_{ab} = \langle \psi_a \psi_b \rangle$$

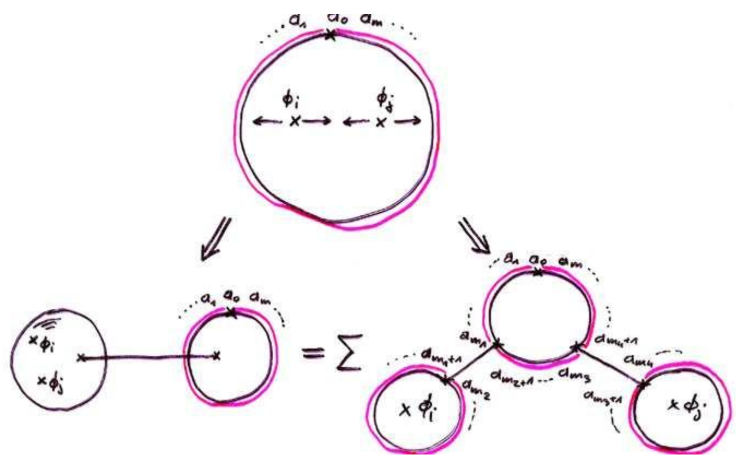
then the superpotential can (formally) be written as string field theory action:

$$\mathcal{W}(s,t) = \sum_{m \geq 0} \frac{1}{m+1}\, \omega(\psi, r_m(\psi^{\otimes m}))$$

... tree diagrams, bubbling off disks (Kajiura)

bulk-boundary crossing symmetry

bulk fact: boundary fact:



$$\partial_i \partial_j \partial_k \mathcal{F}(t)\ \eta^{kl}\ \partial_l \mathcal{F}_{a_0 a_1 \ldots a_m}(t) =$$

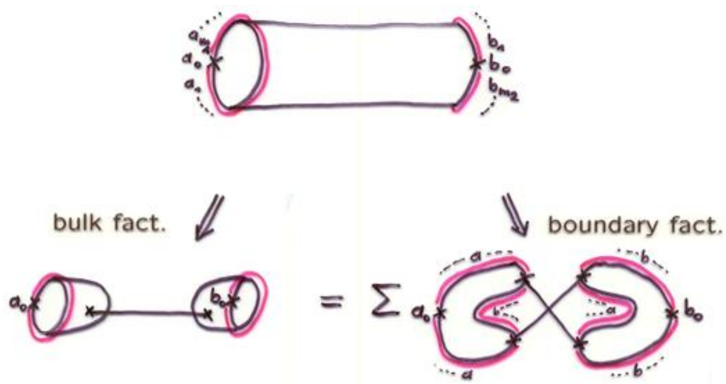
$$= \sum_{0 \leq m_1 \leq \ldots m_4 \leq m} (-1)^s\ \mathcal{F}_{a_0 \ldots a_{m_1} b a_{m_2+1} \ldots a_{m_3} c a_{m_4+1} \ldots a_m}(t)\ \partial_i \mathcal{F}^b{}_{a_{m_1+1} \ldots a_{m_2}}(t)\ \partial_j \mathcal{F}^c{}_{a_{m_3+1} \ldots a_{m_4}}(t)$$

(sign: $s = \tilde{a}_{m_1+1} + \ldots + \tilde{a}_{m_3}$)

These equations relate the bulk prepotential $\mathcal{F}(t)$ to the disk correlation functions $\mathcal{F}_{a_1 \ldots a_m}(t)$!

topological Cardy relation

factorize cylinder diagram:



$$\partial_i \mathcal{F}_{a_0\ldots a_n}\eta^{ij}\partial_j\mathcal{F}_{b_0\ldots b_m} =$$

$$= \sum_{\substack{0\le n_1\le n_2\le n\\ 0\le m_1\le m_2\le m}} (-1)^{s+\tilde c_1+\tilde c_2}\ \omega^{c_1d_1}\ \omega^{c_2d_2}\ \mathcal{F}_{a_0\ldots a_{n_1}d_1 b_{m_1+1}\ldots b_{m_2}c_2 a_{n_2+1}\ldots a_n}\ \mathcal{F}_{b_0\ldots b_{m_1}c_1 a_{n_1+1}\ldots a_{n_2}d_2 b_{m_2+1}\ldots b_m}$$

This powerful and important relation too relates bulk with boundary correlators

Summary: TFT consistency relations

(WDVV) $$\partial_i\partial_j\partial_m\mathcal{F}\ \eta^{mn}\ \partial_n\partial_k\partial_l\mathcal{F} = \partial_i\partial_k\partial_m\mathcal{F}\ \eta^{mn}\ \partial_n\partial_j\partial_l\mathcal{F}$$

(A_∞) $$\sum_{\substack{k,j=0\\ k\le j}}^{m} (-)^{\tilde a_1+\ldots+\tilde a_k}\ \mathcal{F}^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m}\ \mathcal{F}^{c}{}_{a_{k+1}\ldots a_j} = 0$$

(Crossing) $$\partial_i\partial_j\partial_k\mathcal{F}\ \eta^{kl}\ \partial_l\mathcal{F}_{a_0a_1\ldots a_m} =$$

$$= \sum_{0\le m_1\le\ldots m_4\le m} (-)^{\tilde a_{m_1+1}+\ldots+\tilde a_{m_3}}\mathcal{F}_{a_0\ldots a_{m_1} b a_{m_2+1}\ldots a_{m_3} c a_{m_4+1}\ldots a_m}\ \partial_i\mathcal{F}^{b}{}_{a_{m_1+1}\ldots a_{m_2}}\ \partial_j\mathcal{F}^{c}{}_{a_{m_3+1}\ldots a_{m_4}}$$

(Cardy) $$\partial_i\mathcal{F}_{a_0\ldots a_n}\ \eta^{ij}\ \partial_j\mathcal{F}_{b_0\ldots b_m} =$$

$$= \sum_{\substack{0\le n_1\le n_2\le n\\ 0\le m_1\le m_2\le m}} (-)^{s+\tilde c_1+\tilde c_2}\ \omega^{c_1d_1}\ \omega^{c_2d_2}\ \mathcal{F}_{a_0\ldots a_{n_1}d_1 b_{m_1+1}\ldots b_{m_2}c_2 a_{n_2+1}\ldots a_n}\ \mathcal{F}_{b_0\ldots b_{m_1}c_1 a_{n_1+1}\ldots a_{n_2}d_2 b_{m_2+1}\ldots b_m}$$

These form an in general infinite system of algebraic and differential equations ... can we ever hope to solve them explicitly ?

Example: B-branes in topological minimal models

boundary Landau-Ginzburg action:

$$S \sim \int_{D^2} d^2z d^2\theta\, W_{LG}(x) + \int_{\partial D^2} d\tau d\theta\, \Pi J(x)\ , \quad \big(D\Pi = E(x)\big)$$

(Warner, Kapustin, BHLS)

BRST operator/SUSY charge: $Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}$

The action is supersymmetric iff:

$$W_{LG} = \frac{1}{2}Q^2 = JE.$$

Consider first undeformed theory (parameter k="level"):

the bulk sector is governed by

$$W_{LG}(x) = \frac{x^{k+2}}{k+2}\ , \quad k \geq 1$$

the B-type D0-branes M_ℓ are given by the polynomial matrix factorizations of W_{LG}:

$$J(x) = x^{\ell+1}\ , \quad E(x) = \frac{x^{k-\ell+1}}{k+2}\ , \quad \ell = -1, 0, \ldots, \left[\frac{k}{2}\right]$$

Physical spectrum: Q-cohomology

boundary preserving physical fields (Hom(M_ℓ, M_ℓ)): composed of x, ω = even/odd generators of boundary cohomology with relations

$$x^{\ell+1} = 0, \qquad \omega^2 = x^{k-2\ell}$$

fields	parameters	Q-exact
$\phi_i = \{1, x, \ldots, x^k\}$	$\{t_{k+2}, t_{k+1}, \ldots t_2\}$	$\partial_x W_{LG} \sim 0$
$\phi_a = \{1, x, \ldots, x^\ell\}$	$\{\xi_{(k+2)/2}, \ldots, \xi_{(k+2)/2-\ell}\}$	gcd$(J, E) \sim 0$
$\psi_a = \omega \otimes \{1, x, \ldots, x^\ell\}$	$\{s_{\ell+1}, s_\ell, \ldots, s_1\}$	gcd$(J, E) \sim 0$

(def. parameters s grassmann even, ξ odd def params)

boundary changing fields (Hom(M_{ℓ_1}, M_{ℓ_2})) between two branes M_{ℓ_1} and M_{ℓ_2}:

fields	parameters	Q-exact
$\phi_a^{\ell_1,\ell_2} = \beta^{\ell_1,\ell_2} \otimes \{1, x, \ldots, x^{\ell_{12}}\}$	$\{\xi_a^{\ell_1,\ell_2}\}$	gcd$(J_i, E_i) \sim 0$
$\psi_a^{\ell_1,\ell_2} = \omega^{\ell_1,\ell_2} \otimes \{1, x, \ldots, x^{\ell_{12}}\}$	$\{s_a^{\ell_1,\ell_2}\}$	gcd$(J_i, E_i) \sim 0$

($\ell_{12} \equiv$ min(ℓ_1, ℓ_2))

Kontsevich's triangulated category $C_{\mathcal{W}}$

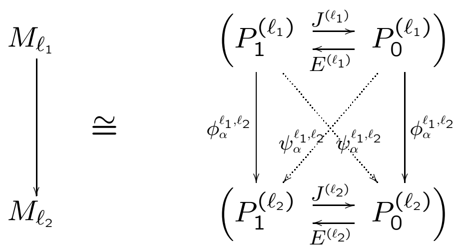
The Landau-Ginzburg model provides a concrete physical realization of Kontsevich's proposal for a certain Z_2 graded derived category (worked out by Orlov, Kapustin, BHLS)

...the objects correspond to our branes M_ℓ:

$$M_\ell \cong \left(P_1^{(\ell)} \underset{E^{(\ell)}}{\overset{J^{(\ell)}}{\rightleftarrows}} P_0^{(\ell)} \right)$$

(graded modules P_0, $P_1 \sim \mathbb{C}[x]$)

...the morphisms correspond precisely to the boundary LG fields introduced above:



All maps J, E, ϕ, ψ have an explicit realization in terms of Landau-Ginzburg quantities (boundary potential, perturbations)

Deforming the theory

infinitesimal perturbations:

$$\begin{aligned} \delta W_{LG}(x) &= -\sum_{i=0}^{k} t_{k+2-i} x^i \\ \delta J(x) &= -\sum_{a=0}^{\ell} u_{\ell+1-a} x^a \\ \delta E(x) &= -x^{k-2\ell} \left(\sum_{a=0}^{\ell} u_{\ell+1-a} x^a \right) \end{aligned}$$

Effects:

- The supersymmetry is generically broken, since $W_{LG} \neq JE$. It can be restored on submanifolds of the t, u parameters space.
- The spectrum of topological boundary fields is generically truncated, since deg(gcd(J, E)) $<$ deg(J).
- Branes M_ℓ can decay/bind to other ones.

Applying the consistency constraints

... only a finite number of polynomials equations.

All correlation functions for the minimal model are uniquely determined once the constraint equations are imposed ! (The A_∞ relations by themselves do **not** suffice.)

Concise result:

$$\mathcal{W}(s,t) \;=\; \oint W_{LG}(x,t)\,\log\det\mathbb{J}(x,s)$$

where flat bulk LG potential (DVV):

$$W_{LG}(t) \;=\; \frac{x^{k+2}}{k+2} \;-\; \sum_{i=0}^{k} g_{k+2-i}(t)\;x^i$$

and where (for a pair of branes M_{ℓ_1}, M_{ℓ_2}):

$$\mathbb{J} \;=\; \begin{pmatrix} x^{\ell_1+1} - \sum_{\alpha=0}^{\ell_1} s^{[11]}_{\ell_1+1-\alpha}x^\alpha & -\sum_{\gamma=0}^{\ell_{12}} s^{[12]}_{\frac{1}{2}(\ell_1+\ell_2)+1-\gamma}x^\gamma \\ -\sum_{\gamma=0}^{\ell_{21}} s^{[21]}_{\frac{1}{2}(\ell_1+\ell_2)+1-\gamma}x^\gamma & x^{\ell_2+1} - \sum_{\alpha=0}^{\ell_2} s^{[22]}_{\ell_2+1-\alpha}x^\alpha \end{pmatrix}$$

...makes direct contact to the categorial description !

NB: Superpotential can also be rewritten as

$$\mathcal{W}(s,t) \;=\; \mathrm{Tr}\,V(X(s),t)$$

where

$\partial_x V(x,t) = W_{LG}(x,t)$

$X(s) = \mathrm{diag}(x_1(s), ...x_{\ell_1+\ell_2+2}(s))$

$\det\mathbb{J}(x,s) = \prod(x - x_i(s))$

...Kontsevich matrix model !

Properties of deformation space

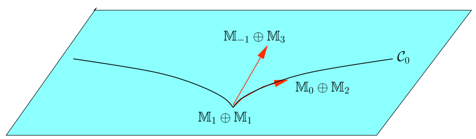
Matrix factorization $\longleftrightarrow$ crit. locus of $\mathcal{W}(s,t)$

$$\mathbb{E}(x,s,t) \;=\; W_{LG}(x,t)/\mathbb{J}(x,s)$$

If we require this to be polynomial ($\mathbb{E}_- = 0$), then (v.v.):

$$\partial_s \mathcal{W}(s,t) \;=\; \oint \mathrm{Tr}\left[\mathbb{E}(x,s,t)\partial_s \mathbb{J}(x,s)\right] \;=\; 0$$

This allows to systematically and exactly study composite brane formation ("tachyon condensation", "boundary flows")



Physical realization of cone construction:

$$\text{triangle}: \qquad M_{\ell_1} \xrightarrow{s} M_{\ell_2} \longrightarrow C(s) \longrightarrow M_{\ell_1}[1]$$

$$\text{cone}: \qquad C(s) = \left(P_1^{(\ell_1)} \oplus P_1^{(\ell_2)} \underset{\mathbb{E}(s)}{\overset{\mathbb{J}(s)}{\rightleftarrows}} P_0^{(\ell_1)} \oplus P_0^{(\ell_2)} \right)$$

anti-brane (shift functor): swap J, E

$$M_\ell[1] \;=\; \left(P_0^{(\ell)} \; \underset{-J^{(\ell)}}{\overset{-E^{(\ell)}}{\rightleftarrows}} \; P_1^{(\ell)} \right) \;\cong\; M_{k-\ell} \quad .$$

The topological LG model is a nice (the simplest?) toy lab for studying D-brane categories !