D-Branes and Matrix Factorizations I

W.Lerche, Cargese 06/2006

- Part I: Introduction, boundary LG models,
  D-branes in minimal models, deformations

  hep-th/0210296 Kapustin, Li
  hep-th/0305133 Brunner, Herbst, WL, Scheuner
  hep-th/0402110 Herbst, Lazaroiu, WL
  hep-th/0604189 Knapp, Omer

  Prior work by Kontsevich, Orlov

- Part II: homological mirror symmetry,
  D-branes on elliptic curve,
  eff potential from instantons

  hep-th/0408243 Brunner, Herbst, WL, Walcher
  hep-th/0512208 Govindarajan, Jockers, WL, Warner
  hep-th/0603085 Herbst, WL, Nemeschansky

  Prior work by Kontsevich, Polishchuk, Zaslow

  Further introductory/review/background material

  hep-th/0403166 Aspinwall
  hep-th/0011017 Douglas
  hep-th/0409204 Hori, Walcher
  hep-th/0010269, 0107162 Lazaroiu

Overview of Part I

- Motivation: study non-perturbative phenomena
  (quantum geometry of D-branes)

- Properties of open string TFT (A_∞ relations)

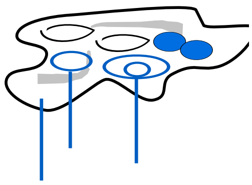
- New approach: boundary LG theory ....translates abstract
  mathematical notions into concrete physical terms

- Example computations: minimal models

- Effective superpotential from obstruction theory

Motivation: D-brane worlds

Typical brane + flux configuration on a Calabi-Yau space



closed string (bulk) moduli t

open string (brane location + bundle) moduli u

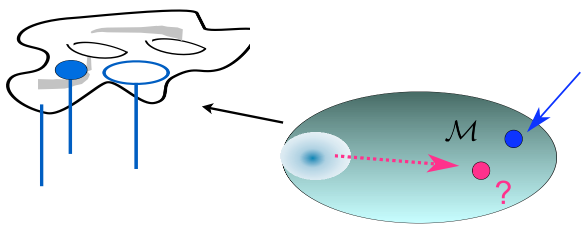
3+1 dim world volume with effective N=1 SUSY theory

What is the exact effective superpotential, the vacuum states, etc ?

$$\mathcal{W}_{\rm eff}(\Phi, t, u) \ = ?$$

Quantum geometry of D-branes

Classical geometry ("branes wrapping p-cycles", gauge bundle configurations on top of them) makes sense only at weak coupling/large radius:



Quantum corrected geometry: (instanton) corrections wipe out notions of classical geometry

....well developed techniques (mirror symmetry)

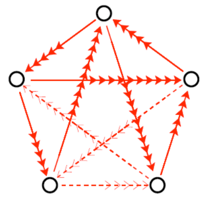
for non-intersecting branes only !

and mostly for non-compact geometries.

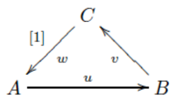
The Derived Category Db(Coh(M))

Mathematicians (Kontsevich) tell us that the proper mathematical language for describing B-branes is the (bounded) derived category (of coherent sheaves on CY)

What does it mean for physicists ?

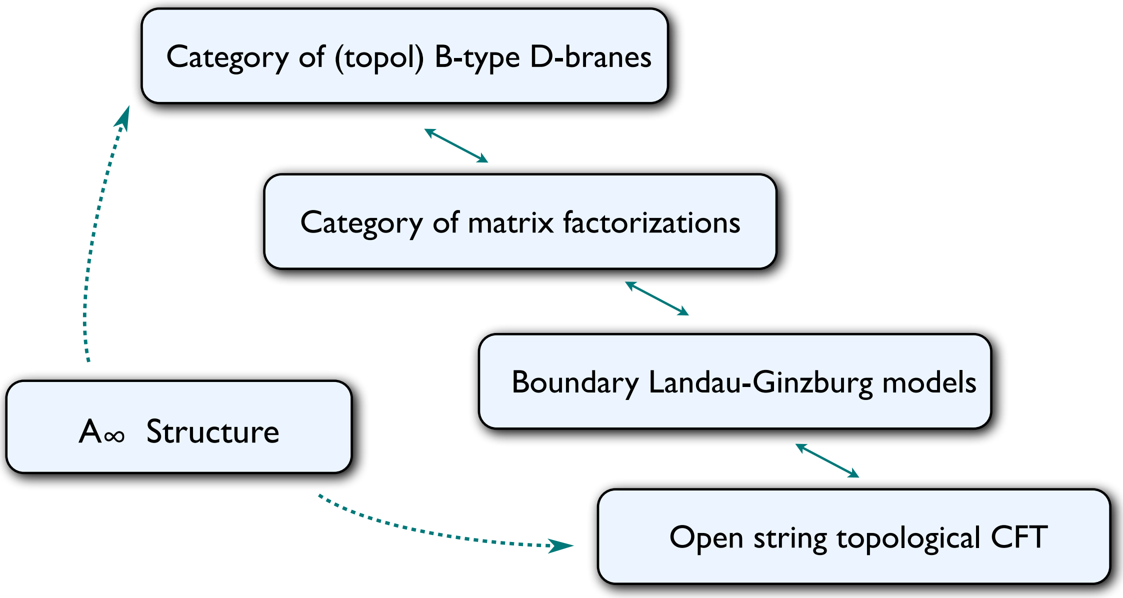


- treats branes and anti-branes on equal footing
- more general than cohomology/ K-theory (RR charges)
- keeps track of brane positions
  robust under continuous deformations (want: moduli dependence),
- describes bound state formation/tachyon condensation (triangulated category)



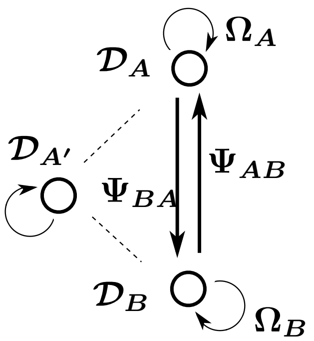
...we will to translate this language to one that is more familiar to physicists: boundary Landau-Ginzburg theory

Roadmap

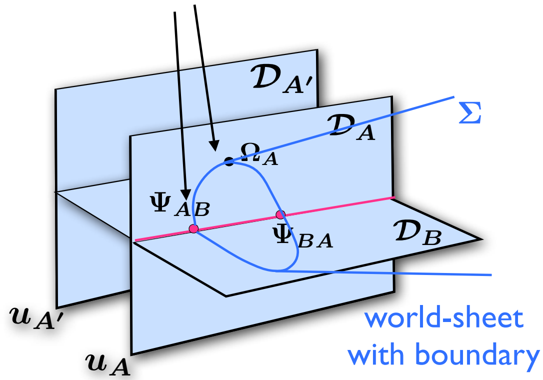


The category of topological D-branes

- objects: $\mathcal{D}$ ⟷ boundary conditions, D-branes
- morphisms (maps): Ω, Ψ ⟷ boundary preserving/changing open string vertex operators

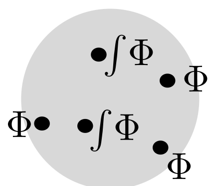


Quiver diagram



Recap closed string TFT: (twisted) N=2 SCFT

- A typical correlator on S_2 looks like:



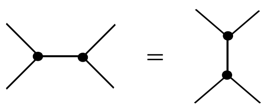
$$C_{ijk}(t) = \partial_{t_i}\partial_{t_j}\partial_{t_k}\mathcal{F}_{\text{eff}}(t)$$

Generating function (N=2 prepotential):
(t_i = deformation params, moduli)

$$\mathcal{F}_{\text{eff}}(t_i, u_a) = \left\langle e^{t_i \int_D \Phi_i^{(2)}} \right\rangle = \sum t_{i_n} ... t_{i_1} C_{i_1 ... i_n}(t)$$

WDVV equations from factorization:

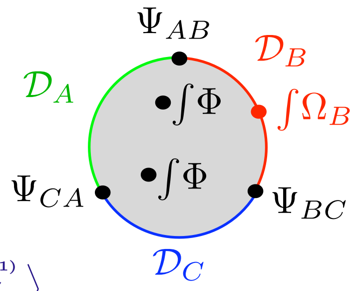
$$C_{ijm}\eta^{mn}C_{nkl} = C_{ikm}\eta^{mn}C_{njl}$$



governed by "N=2 special geometry"

Open string TFT: (twisted) N=2 boundary SCFT

- A typical disk correlator looks like:



- Generating function (N=1 superpotential):
  (t_i = bulk, u_a = boundary deformation params)

$$\mathcal{W}_{\text{eff}}(t_i,u_a) = \left\langle e^{t_i\int_D \Phi_i^{(2)}}\, P e^{u_a\int_{\partial D}\Psi_a^{(1)}}\right\rangle$$
$$= \sum u_{a_m}...u_{a_0} t_{i_n}...t_{i_1} B_{a_0....a_m;i_1...i_n}(t)$$

where:

$$B_{a_0....a_m;t_1....t_n}(t) = \left\langle \Psi_{a_0}\Psi_{a_1}\Psi_{a_2}\, P\int\Psi_{a_3}^{(1)}\dots\int\Psi_{a_m}^{(1)}\int\Phi_{i_1}^{(2)}\dots\int\Phi_{i_n}^{(2)}\right\rangle$$
$$= \partial_{t_{i_n}}...\partial_{t_{i_1}}\mathcal{F}_{a_1....a_n}(t)$$

Sequence of t-dependent cyclic "prepotentials":
..in general not integrable wrto u

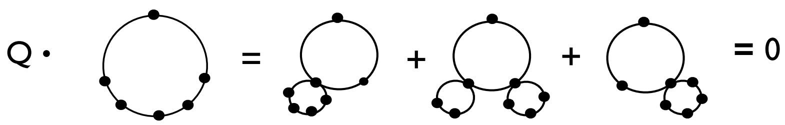
$$\left\{\begin{array}{c}\mathcal{F}_{a_1}(t)\\ \mathcal{F}_{a_1a_2}(t)\\ \mathcal{F}_{a_1a_2a_3}(t)\\ \mathcal{F}_{a_1a_2a_3a_4}(t)\\ \vdots\end{array}\right.$$

Open/closed top. string consistency conditions I

- Boundary TFT: Q-closedness and factorization

$$\left\langle [Q,\Psi\Psi\Psi\int G^-\Psi\dots\int G^-\Psi]\right\rangle \stackrel{!}{=} 0$$

Contact terms from $\{Q,G^-\} = \partial_x$



lead to "A∞ relations" for correlators

$$\sum_{\substack{k,j=0\\k\leq j}}^{m}(-1)^{\tilde a_1+\ldots+\tilde a_k}\lambda_{m-j+k}(\psi_{a_1}\dots\psi_{a_k},\lambda_{j-k}(\psi_{a_{k+1}}\dots\psi_{a_j}),\psi_{a_{j+1}}\dots\psi_{a_m})=0$$

$$\lambda_m(\Psi_{a_1}...\Psi_{a_m}) \equiv \Psi_{a_0}B^{a_0}_{a_1...a_m}$$ "higher products" $\lambda_m:\ \mathcal{H}^{\otimes m}\to\mathcal{H}$

Kontsevich: D-branes indeed form a cyclic A∞ category

....but there is more.

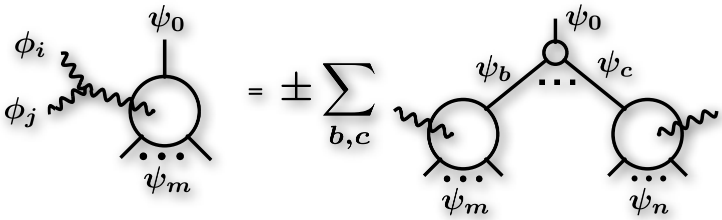
Open/closed top. string consistency conditions II

- Beyond A∞ we have extra constraints, involving bulk operator insertions

....they deform $B^{a_0}_{a_1\dots a_m} \;\longrightarrow\; B^{a_0}_{a_1\dots a_m}(t)$

(deformation theory: "Hochschild complex")

- Bulk-boundary crossing symmetry:

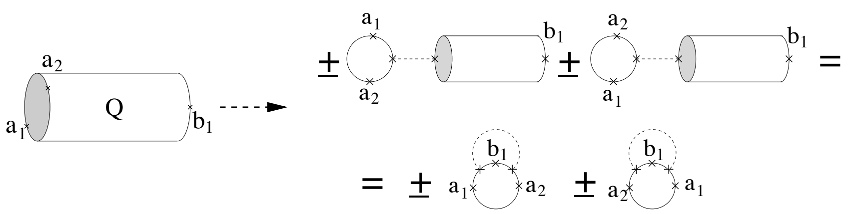


$$\partial_i\partial_j\partial_k\mathcal{F}(t)\ \eta^{kl}\ \partial_l\mathcal{F}_{a_0a_1\dots a_m}(t) =$$

$$=\sum_{0\le m_1\le\dots m_4\le m} (-1)^s\ \mathcal{F}_{a_0\dots a_{m_1} b a_{m_2+1}\dots a_{m_3} c a_{m_4+1}\dots a_m}(t)\ \partial_i \mathcal{F}^b{}_{a_{m_1+1}\dots a_{m_2}}(t)\ \partial_j\mathcal{F}^c{}_{a_{m_3+1}\dots a_{m_4}}(t)$$

Open/closed top. string consistency conditions III

- Annulus factorization



$$\sum_{c,d}\Big((-)^{\bar a_1+\bar d\bar a_2}\ \mathcal{F}^{0,1}_{a_1ca_2}\ \eta^{cd}\ \mathcal{F}^{0,2}_{d|\ b_1} + (-)^{\bar a_1+\bar a_2}\ \mathcal{F}^{0,1}_{a_1a_2c}\ \eta^{cd}\ \mathcal{F}^{0,2}_{d|\ b_1}\Big)$$

$$= \sum_{c,d}\Big((-)^{\bar a_1+\bar b_1(\bar d+\bar a_2)}\eta^{cd}\mathcal{F}^{0,1}_{a_1cb_1da_2} + (-)^{\bar a_1+\bar a_2+\bar b_1\bar d}\eta^{cd}\mathcal{F}^{0,1}_{a_1a_2cb_1d}\Big)$$

Summary: open/closed factorization axioms

WDVV: $\mathcal{F}_{ijm}\eta^{mn}\mathcal{F}_{nkl} \;=\; \mathcal{F}_{ikm}\eta^{mn}\mathcal{F}_{njl}$

A∞: $\sum_{\substack{k,j=0\\ k\leq j}}^{m}(-1)^{\tilde{a}_1+\ldots+\tilde{a}_k} r_{m-j+k}(\psi_{a_1}\ldots\psi_{a_k}, r_{j-k}(\psi_{a_{k+1}}\ldots\psi_{a_j}), \psi_{a_{j+1}}\ldots\psi_{a_m}) = 0$

Crossing: $\partial_i\partial_j\partial_k\mathcal{F}(t)\;\eta^{kl}\;\partial_l\mathcal{F}_{a_0a_1\ldots a_m}(t) =$

$$= \sum_{0\leq m_1\leq\ldots m_4\leq m} (-1)^s\; \mathcal{F}_{a_0\ldots a_{m_1}ba_{m_2+1}\ldots a_{m_3}ca_{m_4+1}\ldots a_m}(t)\;\partial_i\mathcal{F}^b{}_{a_{m_1+1}\ldots a_{m_2}}(t)\;\partial_j\mathcal{F}^c{}_{a_{m_3+1}\ldots a_{m_4}}(t)$$

Annulus: $\sum_{c,d}((-)^{\bar{a}_1+\bar{d}\bar{a}_2}\;\mathcal{F}^{0,1}_{a_1ca_2}\;\eta^{cd}\;\mathcal{F}^{0,2}_{d|\;b_1} + (-)^{\bar{a}_1+\bar{a}_2}\;\mathcal{F}^{0,1}_{a_1a_2c}\;\eta^{cd}\;\mathcal{F}^{0,2}_{d|\;b_1})$

$$= \sum_{c,d}((-)^{\bar{a}_1+\bar{b}_1(\bar{d}+\bar{a}_2)}\eta^{cd}\mathcal{F}^{0,1}_{a_1cb_1da_2} + (-)^{\bar{a}_1+\bar{a}_2+\bar{b}_1\bar{d}}\eta^{cd}\mathcal{F}^{0,1}_{a_1a_2cb_1d})$$

- This is an (in general) infinite system of differential and algebraic equations... can we ever hope to (recursively) solve them explicitly for a given model ?

Apart from spectrum, we need extra input,
in particular the three-point functions.... ⇒ Landau-Ginzburg theory

Recap: topological Landau-Ginzburg models

- Consider bulk d=2 LG model with N=(2,2) supersymmetries:

$$S_{LG} = \int d^2z d\theta^4 K(x,\bar{x}) + \int d^2 d\theta^2\; W_{LG}(x) + \mathrm{cc.}$$

- If W_{LG} = quasi-homogeneous holomorphic superpotential

$$W_{LG}(s^{q_i}x_i) = \; sW_{LG}(x_i)$$

then in the IR, theory flows to a superconformal fixed point (SCFT) entirely determined by the singularity type of W_{LG} !

$$c_{N=2} = 3\sum(1-2q_i)$$

- Upon topologically twisting, the theory turns into a TFT with a finite dimensional Hilbert space

The spectrum of physical operators, the chiral ring, is represented as polynomial ring modulo the eqs of motion:

$$\mathcal{R} \;\cong\; C[x_i]/\partial_i W_{LG}$$

Recap: topological minimal models

- The simplest theories are the (twisted) N=2 minimal models

  They can be realized by LG models with

$$
\begin{aligned}
A_{k+1}: &\quad W_{LG} = x^{k+2} \\
D_k: &\quad W_{LG} = {x_1}^{k-1} + x_1 {x_2}^2 \\
E_6: &\quad W_{LG} = {x_1}^3 + {x_2}^4 \\
E_7: &\quad W_{LG} = {x_1}^3 + x_1 {x_2}^3 \\
E_8: &\quad W_{LG} = {x_1}^3 + {x_2}^5
\end{aligned}
$$

("simple singularities" of ADE type)

- We will focus on A_{k+1} models for which

$$\mathcal{R} \cong C[x]/x^{k+1} = \{1, x, x^2, \ldots, x^k\}$$

$$c_{N=2} = \frac{3k}{k+2} \quad \text{(central charge)}$$

Landau-Ginzburg description of B-type D-branes

- Consider bulk LG model with superpotential:

$$\int_\Sigma d^2z d\theta^+ d\theta^- \, W_{LG}(x) + \text{cc.}$$

B-type SUSY variations induce boundary ("Warner")-term:

$$\int_\Sigma d^2z d\theta^+ d\theta^- (\bar{Q}_+ + \bar{Q}_-) \, W_{LG} = \int_\Sigma d^2z d\theta^+ d\theta^- (\theta^+ \partial_+ + \theta^- \partial_-) W_{LG}$$

$$= \int_{\partial\Sigma} d\sigma d\theta \, W_{LG}$$

- Restore SUSY by adding boundary fermions $\Pi = (\pi + \theta^+ \ell)$

  (... not quite chiral: $\bar{D}\,\Pi = E(x)|_{\partial\Sigma}$)

via a boundary potential: $\delta S = \int_{\partial\Sigma} d\sigma d\theta \, \Pi \, J(x)$

Condition for SUSY: $J(x)E(x) = W_{LG}(x)$

Matrix factorizations

- BRST operator: $Q(x) = \pi\, J(x) + \bar{\pi}\, E(x) = \begin{pmatrix} & J(x) \\ E(x) & \end{pmatrix}$

  thus SUSY condition implies a matrix factorization of W:

$$Q(x) \cdot Q(x) \;=\; W_{LG}(x)\, 1$$

  Total BRST operator $\mathcal{Q} \;=\; Q + Q_{bulk}$

  then squares to zero: $\mathcal{Q}^2 = 0$

- Generalization for n LG fields: need $N=2^n$ boundary fermions, and

$$J_{N\times N} \cdot E_{N\times N} \;=\; E_{N\times N} \cdot J_{N\times N} = \; W_{LG}\, 1_{N\times N}$$

Anti- and trivial branes

- anti-brane $D[1] \equiv \bar{D}$ is described by swapping E, J

$$Q_D = \begin{pmatrix} & J \\ E & \end{pmatrix} \;,\; Q_{\bar{D}} = \begin{pmatrix} & -E \\ -J & \end{pmatrix}$$

- trival brane is described by J=1, E=W and vice versa;
  has trivial open string vacuum

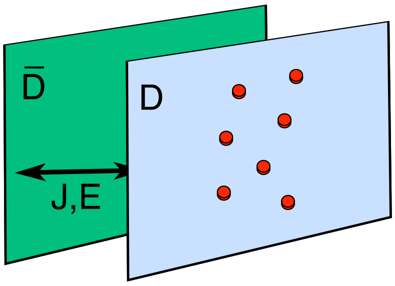
$$Q = \begin{pmatrix} & 1 \\ W & \end{pmatrix}$$

  We can thus always mod out such trivial brane/brane pairs,
  matrices are taken only up to such (1,W) pieces

Physical interpretation

- N... Chan-Paton labels of space-filling $D\bar{D}$ pairs

  Boundary potentials J,E form a tachyon profile that describes condensation to given B-type D-brane configuration in IR limit



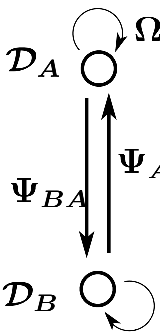
$$J(x,u) \;=\; \prod (x-u_i)$$

- Geometrically: Maps J,E are sections of certain bundles

  Ker J, Ker E encode bundle data of branes: $(r, c_1, \ldots; u)$

Open string spectrum

- Physical open string spectrum is determined by the cohomology of the BRST operator:



$$[Q_A, \Omega_A] \;=\; 0\,, \quad \Omega_A \neq [Q_A, \Lambda]$$

boundary preserving

$$Q_A \Psi_{AB} - (-)^f \Psi_{AB} Q_B \;=\; 0$$

boundary changing

$$\Omega_B \quad [Q_B, \Omega_B] \;=\; 0\,, \quad \Omega_B \neq [Q_B, \Lambda]$$

... all ingredients to form a nice category!

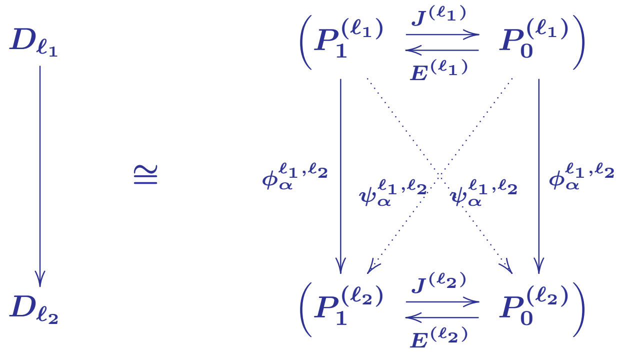
Kontsevich's category C_W

The LG model provides a concrete physical realization of a certain triangulated Z_2-graded category C_W : all maps have explicit LG representatives

- objects: "complexes" (~composites of D$\overline{\text{D}}$ branes):

$$D_\ell \;\cong\; \left(P_1^{(\ell)} \underset{E^{(\ell)}}{\overset{J^{(\ell)}}{\rightleftarrows}} P_0^{(\ell)} \right) , \qquad J^{(\ell)} E^{(\ell)} = W$$

- maps (boundary Q-cohomology):



Kontsevich's category C_W

Category of Matrix factorizations is isomorphic to D(Coh(M)), the derived category of coherent sheaves on M =
category of B-type D-branes!

[Orlov]

Simplest example: boundary A_{k+1} minimal models

- Bulk superpotential:

$$W_{LG}(x) \;=\; \tfrac{1}{k+2}\, x^{k+2}$$

D0-branes D_l are described by all the possible polynomial factorizations:

$$D_\ell : \quad J(x) = x^{\ell+1}, \quad E(x) = \tfrac{1}{k+2}\, x^{k-\ell+1}, \quad \ell = -1, 0, \ldots, [k/2]$$

(l >[k/2]: anti-branes)

This precisely matches results obtained in BCFT !

- Same is true for the open string spectrum, described by matrices that belong to the non-trivial cohomology of the BRST operator:

$$Q_\ell \;=\; \begin{pmatrix} & x^{\ell+1} \\ \tfrac{1}{k+2} x^{k-\ell+1} & \end{pmatrix} \qquad \Psi : \; \{Q_\ell, \Psi\} = 0, \quad \Psi \neq \{Q_\ell, \Lambda\}$$

Physical spectrum: Q-cohomology

- Boundary preserving physical fields $\mathcal{E} \sim \mathrm{Hom}(D_\ell, D_\ell)$:
  x, ω = even/odd generators of boundary ring

fields	deformation parameters	Q-exact
$\phi_i = \{1, x, \ldots, x^k\}$	$\{t_{k+2}, t_{k+1}, \ldots t_2\}$	$\partial_x W_{LG} \sim 0$
$\phi_a = \{1, x, \ldots, x^\ell\}$	$\{v_{(k+2)/2}, \ldots, v_{(k+2)/2-\ell}\}$	$\mathrm{gcd}(J, E) \sim 0$
$\psi_a = \omega \otimes \{1, x, \ldots, x^\ell\}$	$\{u_{\ell+1}, u_\ell, \ldots, u_1\}$	$\mathrm{gcd}(J, E) \sim 0$

- Boundary changing fields $\Psi_{\ell_1,\ell_2} \sim \mathrm{Ext}(D_{\ell_1}, D_{\ell_2})$ betw. D_{ℓ_1} and D_{ℓ_2}:

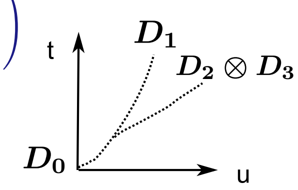
fields	parameters	Q-exact
$\phi_a^{\ell_1,\ell_2} = \beta^{\ell_1,\ell_2} \otimes \{1, x, \ldots, x^{\ell_{12}}\}$	$\{v_a^{[\ell_1,\ell_2]}\}$	$\mathrm{gcd}(J_i, E_i) \sim 0$
$\psi_a^{\ell_1,\ell_2} = \omega^{\ell_1,\ell_2} \otimes \{1, x, \ldots, x^{\ell_{12}}\}$	$\{u_a^{[\ell_1,\ell_2]}\}$	$\mathrm{gcd}(J_i, E_i) \sim 0$

$(\ell_{12} \equiv \min(\ell_1, \ell_2))$

Deforming the minimal models

Consider infinitesimal perturbations:

$$\begin{aligned} \delta W_{LG}(x) &= -\sum_{i=0}^{k} t_{k+2-i} x^i \\ \delta J(x) &= -\sum_{a=0}^{\ell} u_{\ell+1-a} x^a \\ \delta E(x) &= -x^{k-2\ell} \left(\sum_{a=0}^{\ell} u_{\ell+1-a} x^a \right) \end{aligned}$$



Generic effects:

- Spoils factorization, so SUSY will be broken; may be restored along sub-loci.
- Along those, branes can condense ("boundary flow"); open string spectrum truncates
- Starting from several branes, composites (``bound states") may be formed via tachyon condensation

"Bound State" formation via tachyon condensation

- Switch on boundary changing deformation of 2-brane system,

$$J(u) = \begin{pmatrix} J_{\ell_1} & u\Psi_{12} \\ 0 & J_{\ell_2} \end{pmatrix}$$

Rediagonalizing

$$U^{-1} \begin{pmatrix} J_{\ell_1} & u\Psi_{12} \\ 0 & J_{\ell_2} \end{pmatrix} V = \begin{pmatrix} J_{\ell_3} & 0 \\ 0 & J_{\ell_4} \end{pmatrix}$$

yields new factorization, ie, new brane(s)

"Bound State" formation via tachyon condensation

- Example: brane/anti-brane annihilation $D_\ell \oplus \bar{D}_\ell$

$Q_\ell \oplus Q_{k-\ell} \oplus u\Psi_{\ell,k-\ell} \;=$

$$\begin{pmatrix} & & x^{\ell+1} & u \\ & & & x^{k-\ell+1} \\ x^{k-\ell+1} & -u\,x^{k-2\ell} & & \\ & x^{\ell+1} & & \end{pmatrix}$$

This rotates to:

$$\begin{pmatrix} & & x^{k+2} & \\ & & & 1 \\ 1 & & & \\ & x^{k+2} & & \end{pmatrix}$$

which describes $D_{-1} \oplus D_{k+1}$ = two copies of the trivial brane

- In general, reproduce boundary flow patterns known from BCFT:

$$D_{\ell_1} \oplus D_{\ell_2} \stackrel{u_{12}\neq 0}{\longrightarrow} D_{\ell+j+1} \oplus D_{\ell-j-1}$$

Toy model for the "cone" construction

- Geometrical interpretation:

$$J(u) \;=\; \begin{pmatrix} J_{\ell_1} & u\Psi_{12} \\ 0 & J_{\ell_2} \end{pmatrix}$$

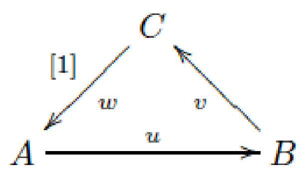
$u = 0$ direct sum of branes, reducible bundle

$u \neq 0$ "extension" of reducible bundle by Ψ

- Physical realization of the "cone" construction:

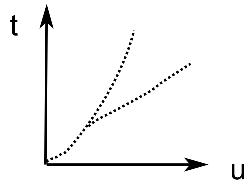
triangle: $D_{\ell_1} \stackrel{\Psi_{12}}{\longrightarrow} D_{\ell_2} \longrightarrow C(u) \longrightarrow D_{\ell_1}[1]$

cone: $C(u) = \left(P_1^{(\ell_1)} \oplus P_1^{(\ell_2)} \underset{E(u)}{\overset{J(u)}{\rightleftarrows}} P_0^{(\ell_1)} \oplus P_0^{(\ell_2)} \right)$



Deformation theory

- LG model provides prototype for dealing with off-shell physics, ie., effective potentials encoding obstructions



- Wanted: compute effective potential W whose critical locus reproduces SUSY deformations

- Consider perturbation

$$Q \;=\; Q_0 \;+\; \delta Q = Q_0 \;+\; u_i \Psi_i$$

Factorization will be generically spoiled

$$Q^2 - W \;=\; \underbrace{\{\, Q_0, u_i \Psi_i \,\}}_{=0} + u_i u_j \{ \Psi_i, \Psi_j \,\}$$

Massey products

correct in higher order by using an “inverse” BRST operator:

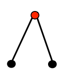
$$\delta Q \;=\; u_i \Psi_i \;-\; Q^+ \{\, u_i \Psi_i, u_j \Psi_j \,\} \qquad Q^+ : \mathcal{H}_{exact} \to \mathcal{H}_{unphys}$$

Problem shifted to next order: .... just keep on iterating

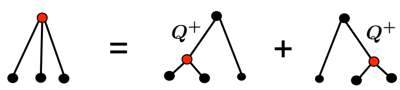
$$\delta Q \;=\; u_i \Psi_i \;-\; Q^+ \sum_m \lambda_m (\Psi^{\otimes m})$$

“Massey products”

$$\lambda_2(\Psi_1, \Psi_2) \;=\; \{\Psi_1, \Psi_2\}$$



$$\lambda_3(\Psi_1, \Psi_2, \Psi_3) \;=\; \lambda_2(\Psi_1, Q^+ \lambda_2(\Psi_2, \Psi_3)\} + \lambda_2(Q^+ \lambda_2(\Psi_1, \Psi_2), \Psi_3)$$



These are precisely the higher products

that solve the A∞ relations!

Graphical expansion = “homological perturbation theory”, string field theory

The obstruction potential

- however: iteration fails whenever $\lambda_m \in \mathrm{Coh} : \rightarrow \lambda_m \neq \{ Q, Q^+ * \}$

then deformation is obstructed at m-th order:

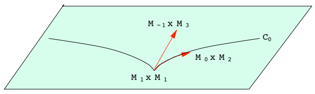
$$Q^2(u) - W \;=\; f_m(u)\lambda_m \;\neq\; 0$$

The obstructions can be integrated to an effective potential:

$$Q^2(u) - W \;=\; \sum \partial_{u_i} \mathcal{W}_{eff}(u)\lambda_m$$

> matrix factorization locus = critical locus of effective superpotential!

... allows to systematically map out vacuum manifold and study composite formation ("topol. tachyon condensation") along it



Example: minimal model A4 with a single brane D1

Factorization: $W = \frac{1}{5}x^5 \qquad Q_0 = \begin{pmatrix} & x^2 \\ \frac{1}{5}x^3 & \end{pmatrix}$

Cohomology: $\Phi_0 = 1\,, \quad \Phi_2 = x1$

$$\Psi_0 = \begin{pmatrix} & 1 \\ -\frac{1}{5}x & \end{pmatrix} \qquad \Psi_1 = x\Psi_0 = \begin{pmatrix} & x \\ -\frac{1}{5}x^2 & \end{pmatrix}$$

Second order Massey products:

$\lambda_2(\Psi_0, \Psi_0) = -\frac{1}{5}\Phi_1$ in cohomology, so: $f_2^{(1)} = -\frac{1}{5}{u_0}^2$

$$\lambda_2(\Psi_1, \Psi_1) = -\tfrac{2}{5}x^2\Phi_0$$

$$\lambda_2(\Psi_0, \Psi_1) = -\tfrac{1}{5}x^3\Phi_0$$

Choose: $Q^+\lambda_2(\Psi_1, \Psi_1) = \begin{pmatrix} & 1 \\ & \end{pmatrix}$

$Q^+\lambda_2(\Psi_0, \Psi_1) = \begin{pmatrix} & \\ \frac{2}{5} & \end{pmatrix}$ and go on with iteration

Example: minimal model A_4 with a single brane D_1

Non-zero third order Massey products:

$$\lambda_3(\Psi_1,\Psi_1,\Psi_0) = -\tfrac{1}{5}\Phi_1 \quad \text{in cohomology, so:} \quad f_3^{(1)} = -\tfrac{1}{5}{u_1}^2 u_0$$

$$\lambda_3(\Psi_1,\Psi_1,\Psi_1) = -\tfrac{1}{5}x^2\Phi_0$$

Choose: $Q^+\lambda_3(\Psi_1,\Psi_1,\Psi_1) = \begin{pmatrix} & \\ -\frac{1}{5} & \end{pmatrix}$ and go on with iteration

Non-zero fourth order Massey products are both in cohomology:

$$\lambda_4(\Psi_1,\Psi_1,\Psi_1,\Psi_1) = \tfrac{1}{5}\Phi_1 \qquad f_4^{(1)} = \tfrac{1}{5}{u_1}^4$$

$$\lambda_4(\Psi_1,\Psi_1,\Psi_1,\Psi_0) = -\Phi_0 \qquad f_4^{(0)} = -{u_1}^3 u_0$$

Non-zero fifth (and final) order Massey product is in cohomology:

$$\lambda_5(\Psi_1,\Psi_1,\Psi_1,\Psi_1,\Psi_1) = -\tfrac{3}{5}\Phi_0 \qquad f_5^{(0)} = -\tfrac{3}{5}{u_1}^5$$

Effective potential

Sum all contributions up:

$$\begin{aligned} f^{(0)} &= \tfrac{1}{5}(-u_0{u_1}^3 + 3{u_1}^5) \\ f^{(1)} &= \tfrac{1}{5}({u_0}^4 + u_0{u_1}^2 - {u_1}^4) \end{aligned}$$

- Deformed Q:

$$\begin{aligned} Q &= Q_0 + u_i\Psi_i - Q^+\sum_m \lambda_m(\Psi^{\otimes m}) \\ &= \begin{pmatrix} & x^2 - u_1 x - u_0 + {u_1}^2 \\ \frac{1}{5}(-x^3 - u_1x^2 - u_0x - 2u_0u_1 + {u_1}^3 & \end{pmatrix} \end{aligned}$$

squares into: $Q^2(u) - W = f^{(0)}\Phi_0 + f^{(1)}\Phi_1$

So factorization is preserved if $f^{(i)} = \partial_{u_i}\mathcal{W}_{eff}(u) = 0$

- Integrate relations to potential:

$$\mathcal{W}_{eff}(u) = \tfrac{1}{5}\Big(\tfrac{1}{3}{u_1}^6 - u_0{u_1}^4 + \tfrac{1}{2}{u_0}^2u_1^2 + \tfrac{1}{3}{u_0}^3\Big)$$

Tomorrow in Part II:

Include moduli,
combine with mirror symmetry

Application to elliptic curve