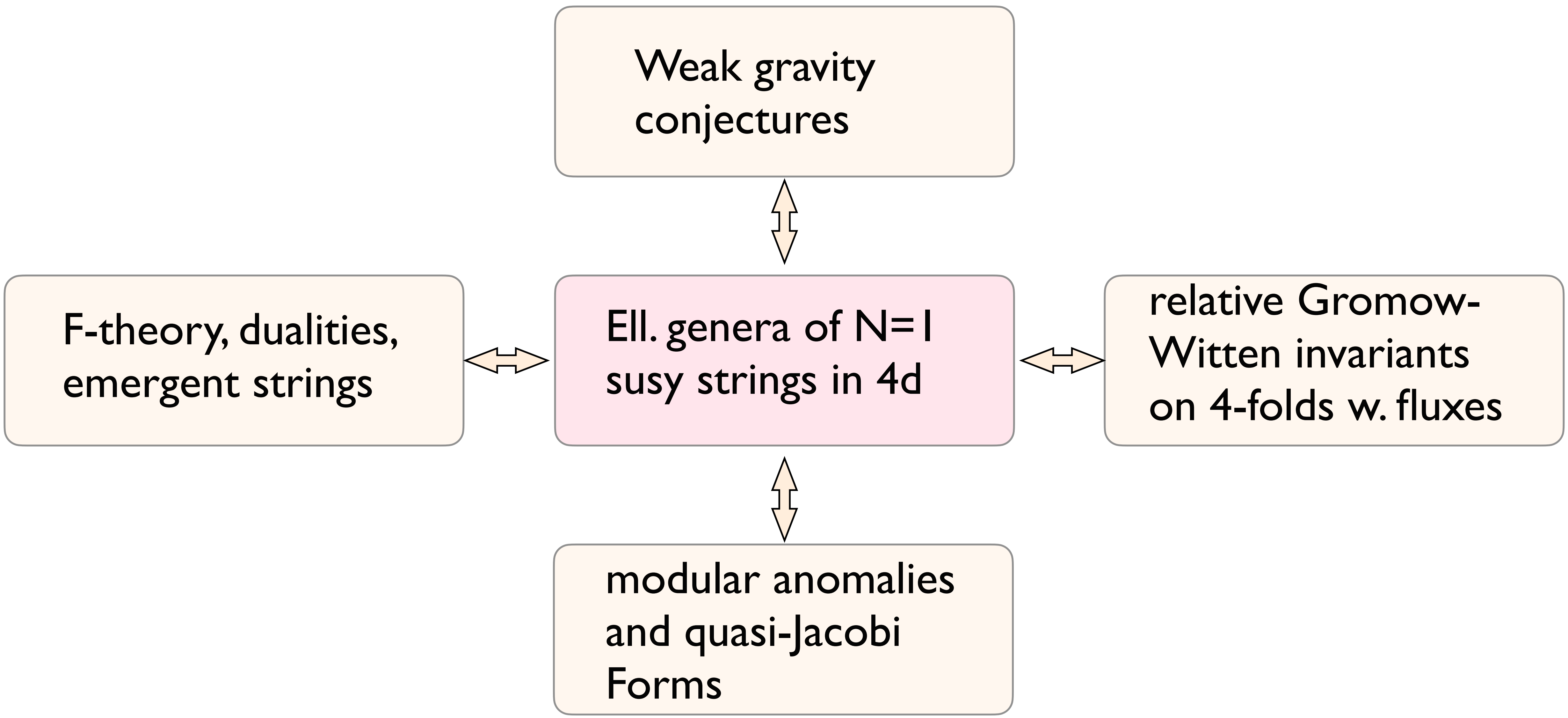
Fluxes, Holomorphic Anomalies and Elliptic Genera in Four Dimensions

W.Lerche, Paris, 6/2021



Works done with Seung-Joo Lee, Guglielmo Lockhart and Timo Weigand 2018-2020

Overview

- Motivation: Weak Gravity Conjectures
  Tensionless emergent strings

- Non-perturbative elliptic genera from geometry (F-theory)
  Examples in 6d
  Holomorphic anomalies

- Four dimensions: 4-folds with fluxes G
  Relative Gromow-Witten invariants N_G

- Modularity of flux-induced elliptic genera
  Quasi-Jacobi forms
  Novel kind of elliptic holomorphic anomalies

Main results

LLLW '20

- Mirror symmetry of CY 4-folds with U(1) gauge symmetry, fluxes G_4:

  determines relative Gromov-Witten invariants, $N_{[C0,G4]}(n,r)$

  ⟹ non-perturbative elliptic genera for 4d N=1 strings (eg. het. with NS5-branes)

$$Z^{ell}[C_0, G_4](\tau, z) = -q^{E_0} \sum_{n \geq 0} N_{C_0,G_4}(n,r) q^n \xi^r$$

- 4d elliptic genera have surprisingly rich features as compared to 6d:

$$Z^{ell}(\tau, z) \; = \; Z_{-1}(\tau, z) \; + \; \partial_z Z_{-2}(\tau, z)$$

  involve quasi-modular and quasi-Jacobi forms $E_2(\tau), E_1(\tau, z)$

  ⟹ novel elliptic holomorphic anomaly equations, expose hidden higher dimensional sector

- Physics applications: anomaly cancellation, WGC conjectures for chiral N=1 supersymmetric theories in 4d, elliptic holomorphic anomalies

Motivation: Weak Gravity Conjectures & Co

AH-MNV '07, ….
Review: P '20

Study mathematical underpinnig of (interrelated) WGC-type conjectures:

> As so often, physical consistency requirements turn out to be guaranteed by mathematical properties

- No global symmetries in QG ⇒ degeneration geometry of CY manifolds

  Infinite distances in moduli space: asymptotically massless towers (either KK-modes: decompactification, or tensionless strings)

- Superextremal states exist into which extremal black holes can decay

focus ⇒ GW invariants, modular properties of elliptic genera, quasi-Jacobi forms

Large distance limit in moduli space

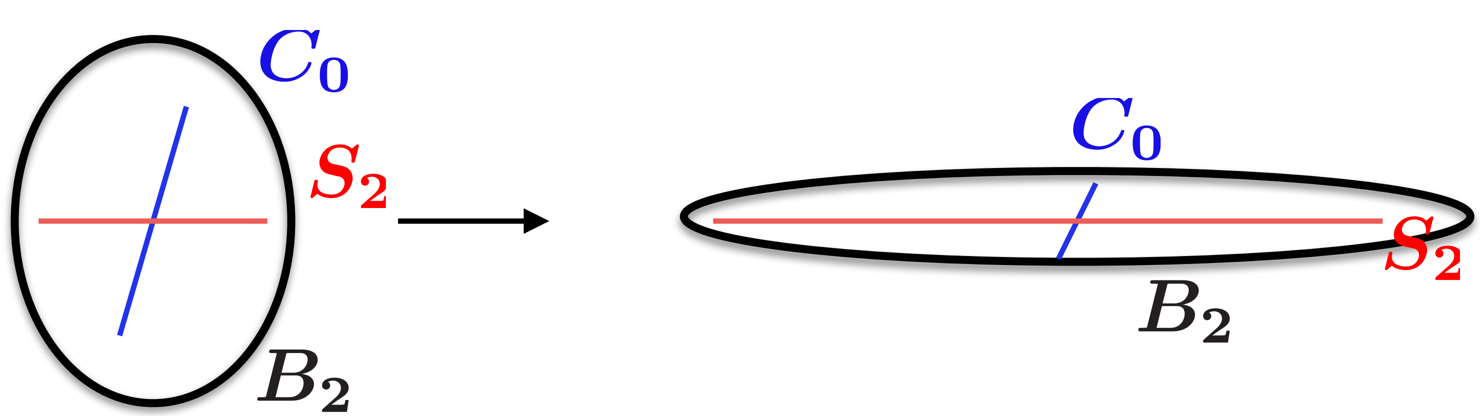
- What happens if we dial appropriate parameters to go to weak gauge coupling while not decoupling gravity?

- Relevant piece of local geometry: submanifold B_2

want to keep gravity: $M_{pl}^4 \sim \mathrm{Vol}(B_2) = \mathrm{const}$

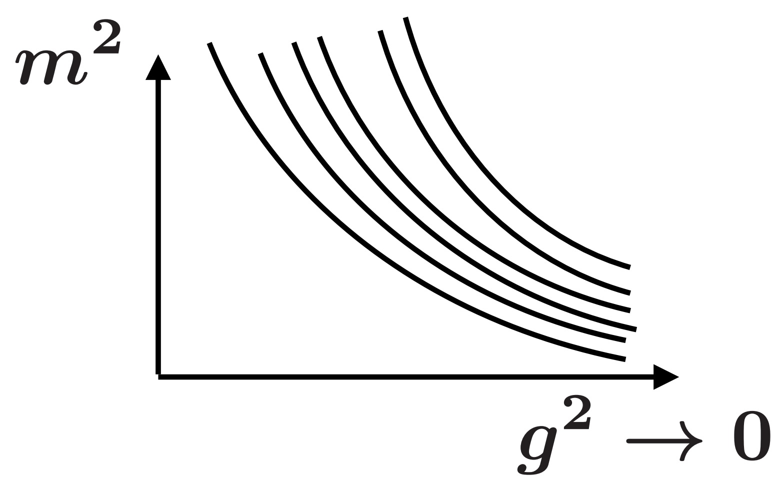
weak coupling: $g^2 \sim 1/\mathrm{Vol}(S_2) \to 0$

implies a certain unique dual curve shrinks: $\mathrm{Vol}(C_0) \to 0$



Asymptotically tensionless heterotic strings

- Upshot: vanishing gauge coupling implies that a solitonic string becomes tensionless… so the effective field theory approximation breaks down!



Infinite tower of charged particles with asymptotically vanishing mass gap, as posited by the WGC

AH-MNV '07 + many

- While this picture seems naive, there are powerful mathematical theorems behind the possible large distance degenerations of Calabi-Yau spaces that guarantee this outcome.

GPV '18, CGV '18, LLW '19, GRH '19, G,M…

- Can we possibly run into surprises of some sort?

Tensionless higher dimensional branes, or multiple strings, or potential so far unknown weakly coupled theories of quantum gravities do not appear as dominant d.o.f! (Emergent string conjecture)

LLW '19

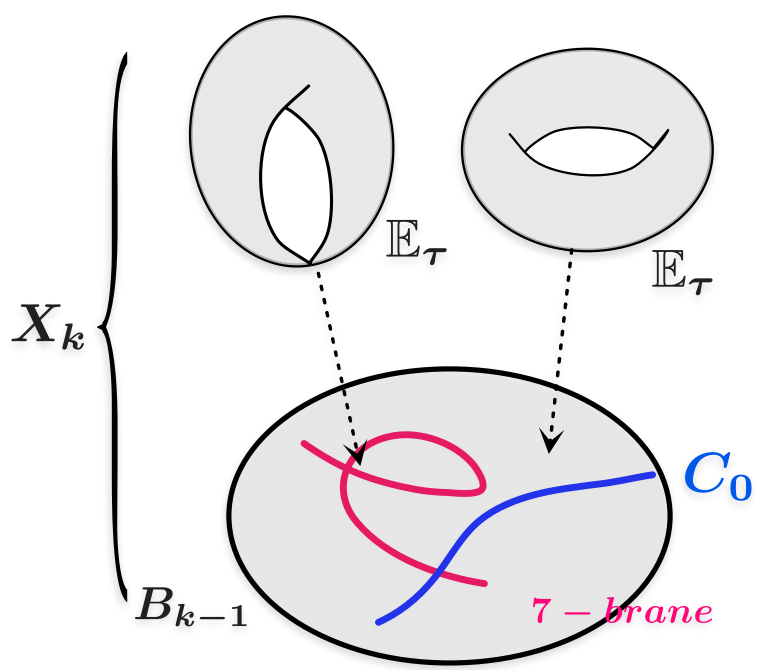
Setup: solitonic strings in F-theory

- Recap:

V '96, MV '96,…

Type IIB strings on a non-perturbative 7-brane geometry

… define formally a d=12-2k dim "F-theory" compactification on elliptically fibered Calabi-Yau k-folds, X_k



D3-brane wrapped around 2-cycle C_0 yields a solitonic string

What happens if C_0 shrinks to zero volume?

- Example in 6d: F-theory on 3-fold X_3

$C_0 \cdot C_0 < 0$: Shrink at finite distance: strongly coupled non-critical strings

$C_0 \cdot C_0 \geq 0$: Cannot shrink at finite distance in moduli space

$C_0 \cdot C_0 = 0$: Solitonic heterotic strings

Elliptic genus of heterotic strings

- Elliptic genus = loop space index = RR partition function SW '86, W '87

  Encodes protected, largely deformation invariant subsector of spectrum, and underlies Green-Schwarz anomaly cancellation

- Consider 2-dim (0,2)-sigma model, refined by a single left-moving U(1) charge Q(*)

$$Z^{ell}(\tau,z) = \mathrm{Tr}_{RR}\Big[(-1)^{F_R} F_R^{1/2(d-2)} q^{L_0} \bar{q}^{\bar{L}_0} \xi^{Q}\Big]$$

- Z^{ell} is naively a meromorphic function of $q \equiv e^{2\pi i \tau}, \quad \xi \equiv e^{2\pi i z}$ where τ is the complex structure of toroidal world-sheet, and z the U(1) field strength

  Generic expansion: $Z^{ell}(\tau,z) = -q^{E_0} \sum_{n\geq 0, r} N(n,r) q^n \xi^r$

  n= excitation level,
  r= U(1) charge ,
  N(n,r)= degeneracies

- It has distinguished modular transformation properties: for heterotic strings in d dimensions, it has modular weight $w = 1 - d/2$

$E_0 = -1/2 C_0 \cdot \bar{K}_{B_{n-1}} = -1$

(*)Zero modes cancelled; note Q here NOT R-symmetry!

Elliptic genus of the emergent 6d heterotic string

- Use duality with M-theory and mirror symmetry on elliptic X_3 to compute relative degeneracies $N_{C0}(n,r)$ via:

F-theory on $\mathbb{R}^{4,1} \times S^1$

BPS string on S^1:

Wrapping number w=1,
KK momentum n,
U(1) charge r

modular parameter τ,
background gauge field z

index degeneracies N(n,r)

⟺

M-theory on $\mathbb{R}^{4,1}$

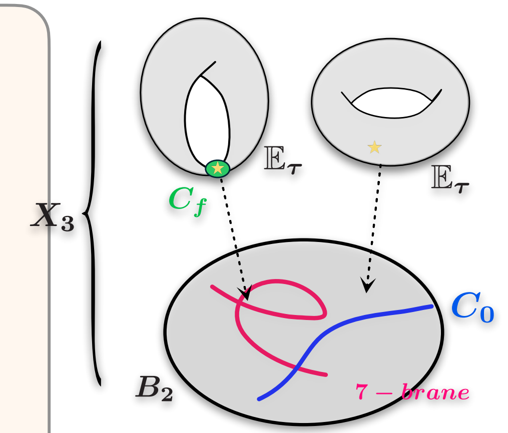
BPS particle in 5d:

M2 brane wrapped on

$$C_{n,r} = w\, C_0 + n\, \mathbb{E}_\tau + r\, C_f$$

τ = Kähler parameter of E_τ,
z = Kähler parameter of C_f

relative BPS invariants $N_{C_0}(n,r)$



KIMV '96,
HIKoLV '13,
HKILV '14,
……..

- For primitive C_0 (w=1), the 3-fold relative, genus zero BPS invariants $N_{C_0}(n,r)$ coincide with the GW invariants, and thus:

$$Z^{ell}(\tau, z) = -q^{E_0} \mathcal{F}_{C_0}(\tau, z)$$

6d het string ell genus   4d N=2 prepot

Elliptic genus as Jacobi form

(almost)

- Ell. genera as partition functions are expected to behave well under modular transf.

SW,W…

When refined by an extra U(1), they should be "Jacobi forms" KYY '93, EZ'95, DMZ '12, …

$$Z^{ell}(\tau, z) \;=\; \varphi_{w,m}(\tau, z)$$

Defining properties: modularity and double periodicity

$$\varphi_{w,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^w e^{2\pi i \frac{m\,c}{c\tau+d} z^2} \varphi_{w,m}(\tau, z)\,,$$

$$\varphi_{w,m}\left(\tau, z+\lambda\tau+\mu\right) = e^{-2\pi i m(z^2\tau+2\lambda z))}\varphi_{w,m}(\tau, z) \quad \lambda, \mu \in \mathbb{Z}$$

Het. strings: modular weight $w = 1 - d/2$ and index(*) $m = 1/2 C_0 \cdot b \in \mathbb{N}$

- Ring of relevant Jacobi forms with given w and m is finitely generated:

$$\mathcal{R}^{Jac} \;=\; \mathbb{Q}\Big[E_4, E_6, \varphi_{-2,1}, \varphi_{-1,2}, \varphi_{0,1}\Big]$$

(standard defs; see lit.)

⇒ Need to determine just a finite number of GW invariants N(n,r) to find exact ell genus!

(*)b="height pairing": geom. definition of U(1)

Example: elliptic genus of K3

Consider F-theory on CY_3 on
certain elliptic fibration over Hirzebruch surface F1 with extra U(1).

Shrinking C_0 leads to emergent heterotic string on K3 surface (index m=2):

$$\begin{aligned}
\mathcal{F}_{C_0}(\tau,z) &= -q\, Z^{ell,K3}_{-2,2}(\tau,z) \\
&= \frac{q}{\eta(\tau)^{24}}\Big(-\frac{1}{72}E_4(\tau)^2E_6(\tau)\varphi_{-2,1}(\tau,z)^2+\frac{7}{432}E_4(\tau)^3\varphi_{-2,1}(\tau,z)\varphi_{0,1}(\tau,z) \\
&\qquad +\frac{5}{432}E_6^2(\tau)\varphi_{-2,1}(\tau,z)\varphi_{0,1}(\tau,z)-\frac{1}{72}E_4(\tau)E_6(\tau)\varphi_{0,1}^2(\tau,z)\Big) \\
&= -2+\left(288+96\xi^{\pm 1}\right)q+\mathcal{O}(q^2)
\end{aligned}$$

For $z\to 0$ / $\xi\to 1$ this reproduces the well-known expression:

$$Z^{ell,K3}_{-2,2}(\tau,0)=2\frac{E_4E_6}{\eta^{24}}(\tau)=\frac{2}{q}-480-282888q+\ldots$$

20 x Euler number χ = 24

Spectral flow property of Jacobi forms

- Ell genus is a Jacobi form…as such it has automatically a theta-expansion:

EZ '95, DMZ'14

$$\varphi_{w,m}(\tau,z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\,\theta_{m,\ell}(\tau,z)$$

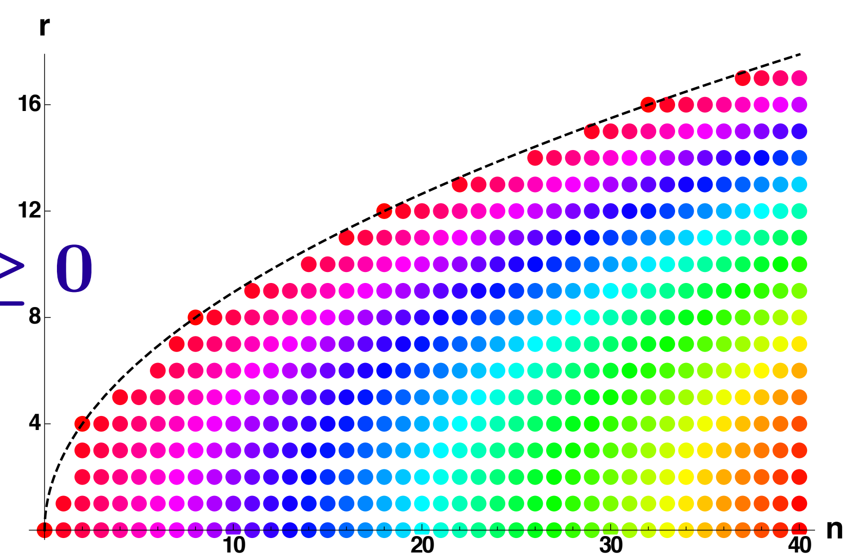
$$= \sum_{n\geq 0}\sum_{r^2\leq 4mn} c(n,r)q^n\xi^r$$

Theta-fct = partition function of a free 2d boson, with built-in relation between charge and excitation numbers:

$$\theta_{m,\ell}(\tau,z) = \sum_n q^{(\ell+2mn)^2/4m}\xi^{\ell+2mn}$$

States fall in "spectral flow" orbits characterized by discriminant: $\Delta = 4mn - r^2$

$$c(n,r) = C(\Delta,r)$$



- Distinguish:

⇨ Holomorphic Jacobi forms: $c(n,r) = 0 \text{ unless } \Delta \geq 0$

Weak Jacobi forms: $c(n,r) = 0 \text{ unless } n \geq 0$

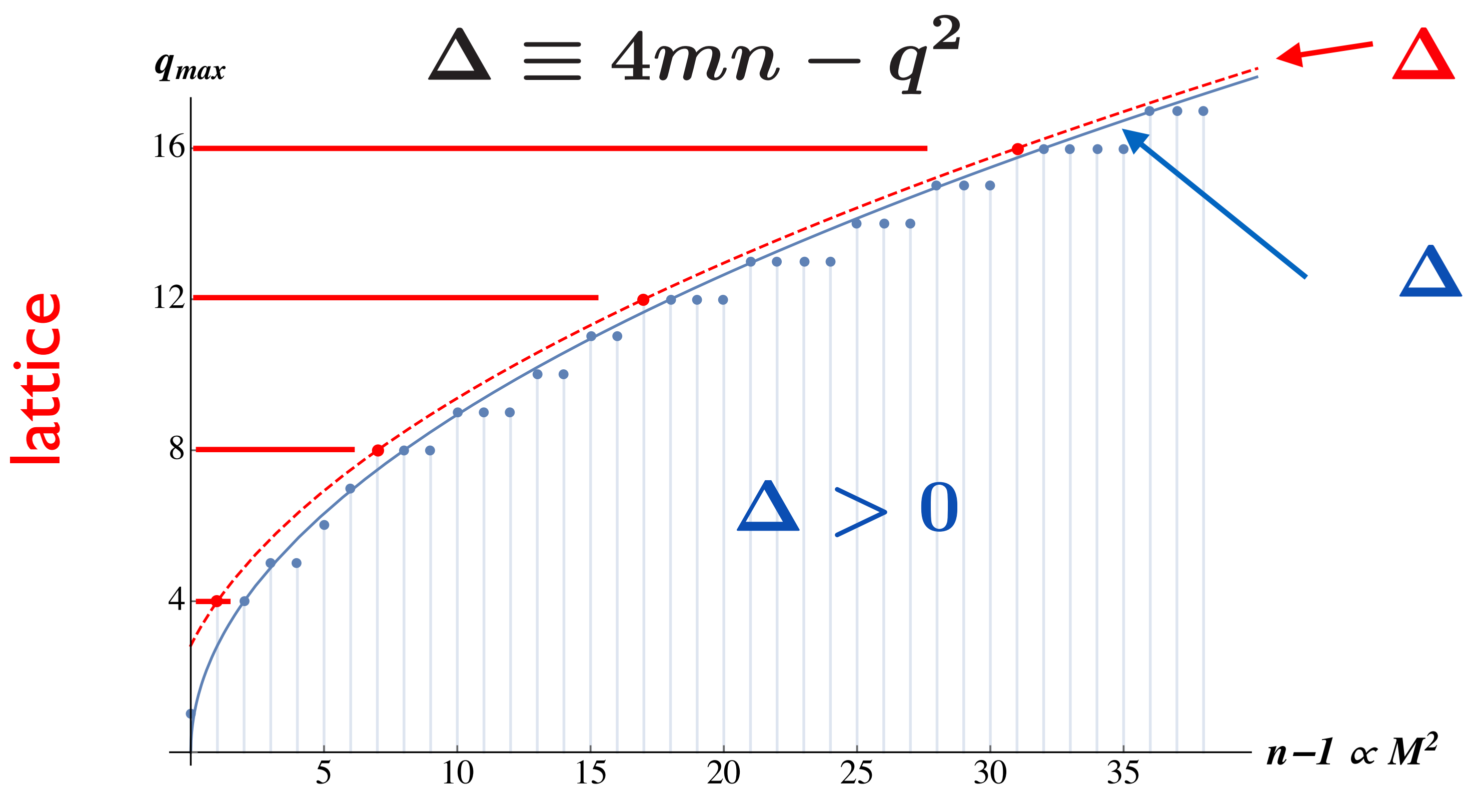
Jacobi cusp forms: $c(n,r) = 0 \text{ unless } \Delta > 0$

Superextremality from Jacobi Forms

- As basic WG conjecture states that there must exist (generally a lattice of) superextremal states in order for black holes be able to decay and so avoid massive remnants (related to "gravity must be the weakest force")

AH-MNV '07, ….

- The Jacobi property implies the following structure of the U(1) charge/mass string spectrum encoded in the elliptic genus:(*)



$\Delta = 0$ Maximal superextremal states, no smooth horizon BH

$\Delta = 4mn$ Extremal states, asymptote to black holes

The underlying mathematics of elliptic fibrations is consistent with expectations from quantum gravity!

- Note: holomorphic Jacobi forms here, no polar states with $\Delta < 0$; unlike spinning strings

(*)Refers to our example with m=2

Quasi-modularity from non-perturbative transitions

- Other example: consider F-theory on CY_3 which is a certain elliptic fibration over the del Pezzo surface dP_2 with extra U(1).

  Again leads to an emergent dual heterotic string on K3 surface, however without a fully perturbative world-sheet description (NS5 brane defect):

$$\begin{aligned}\mathcal{F}_{C_0}(\tau,z) &= -q\, Z^{ell,\tilde{K3}}(\tau,z)\\ &= \frac{q}{\eta^{24}}\Big(-\frac{23}{1728}E_4^2E_6\varphi_{-2,1}^2+\frac{1}{64}E_4^3\varphi_{-2,1}\varphi_{0,1}+\frac{19}{1728}E_6^2\varphi_{-2,1}\varphi_{0,1}-\frac{23}{1728}E_4E_6\varphi_{0,1}^2\\ &\qquad +E_2(-\frac{1}{1728}E_6^2\varphi_{-2,1}^2+\frac{1}{864}E_4E_6\varphi_{-2,1}\varphi_{0,1}-\frac{1}{1728}E_4^2\varphi_{0,1}^2)\Big)\end{aligned}$$

  as z→0 this now turns into

$$\begin{aligned} Z^{ell,\widetilde{K3}}(\tau,0) &= Z^{ell,K3}(\tau)+\frac{1}{12\eta^{24}}\Big(\underbrace{E_2{E_4}^2-E_4E_6}\Big)(\tau)\\ &= \frac{2}{q}-420-265968q+\ldots \end{aligned}$$

  small instanton/NS5 brane transition

- Modified index reflects extra massless tensor multiplet and hypermultiplets

$$2(\Delta n_T-\Delta n_H)=60$$

Holomorphic vs modular anomalies

- We see that in general the quasi-modular form E_2 appears, which displays a modular anomaly:

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau+d)$$

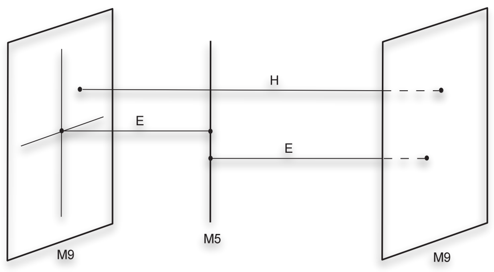
- As is well known, the lack of modularity can be repaired by considering the modular, but only``almost''-holomorphic variant:

SW '87

$$\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}\tau}$$

Physical interpretation: zero modes due to a non-compact branch in the geometry provide the non-holomorphic modular completion and render the physical partition function invariant.

MNVW '98,
HLV '14, ….



- The holomorphic anomaly detects the source of this phenomenon:

$$\frac{\partial}{\partial E_2} Z^{ell,\widetilde{K3}} = -\frac{2\pi i}{3}(\mathrm{Im}\tau)^2 \frac{\partial}{\partial \bar\tau} \widehat{Z}^{ell,\widetilde{K3}} = \frac{1}{12\eta^{24}}(E_4)^2(\tau)$$

Reflects that het. string = bound state of two "E-strings", each with $Z^{ell,E} \sim \frac{1}{\eta^{12}} E_4(\tau)$

Elliptic genera for 4d N=1 supersymmetric theories

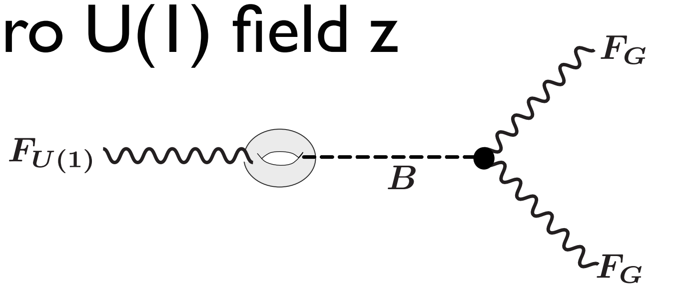
- Consider F-theory on 4-fold X_4 =CY_4, where a suitable curve C_0 shrinks

  For chiral theory: need non-zero background 4-flux $G_4 \in H^{2,2}(X_4)$

- (genus 0) Gromov-Witten invariants on CY_4 are intrinsically defined in relation to G_4

$$Z^{ell}_{-1,*}[C_0, G_4](\tau, z) = -q^{E_0} \sum_{n \geq 0, r} N_{C_{n,r};G_4} q^n \xi^r$$

- Elliptic genus has modular weight w= -1: vanishes unless nonzero U(1) field z



- We find:

  Intriguing interrelationships between flux backgrounds, modular properties of ell genus, and novel holomorphic anomalies

  Consistent with WGC, but more subtle

  [Modified anomaly cancellation mechanism reflecting "hidden" 6d structure]

  [Novel kinds of non-perturbative strings, like 4d E-strings]

Fluxes and Gromov-Witten Invariants on CY_4

- Genus g GW invariants on X_k count stable holomorphic maps $\Sigma_g \to C \subset H_2(X_k,\mathbb{Z})$ with n fixed punctures p_i, subject to certain incidence relations.

  The virtual dimension of moduli space is:

$$\dim_{virt,\mathbb{C}} \mathcal{M}_{g,n}(X_k, C) \;=\; (k-3)(1-g)+n$$

  Thus for X_4 the invariants for g=0 need at least one insertion.
  Loosely speaking: invariants are pinned to, and thus labelled by the flux.
  No invariants for g>1!

- Invariants $N_{C;G_4}$ are computable via mirror symmetry for 4-folds: M '96, KLRY '97, KP '07, ...

$$G_4 \in H^{2,2}(X_4) = H^{2,2}_{hor}(X_4) \oplus H^{2,2}_{vert}(X_4) \oplus H^{2,2}_{rest}(X_4)$$

Type IIA on X_4

$$\mathcal{F}_G = \int_X G_4^{hor} \wedge \Omega_X$$

$\Longleftrightarrow$

Type IIA on Y_4

$$\mathcal{F}_G = \sum_{\beta \in H_2(Y_4)} N_{\beta; G_4^{vert}} \, \mathrm{Li}_2(q_\beta)$$

[F: 2d N=2 pre-/superpotential]

Modularity of flux sectors

- F-theory on elliptic 4-folds X_4 imposes additional structure:
  - not all fluxes can be lifted from 2d Type IIA to 4d F-theory
  - partition functions will have different modular weights depending on the flux sector

HMY '15, CKS '17, LLLW '20

Choose suitable eigenbasis of fluxes adapted to modularity:

$$H^{2,2}_{vert}(X_4, \mathbb{R}) = H^{2,2}_{(0)}(X_4, \mathbb{R}) \cup H^{2,2}_{(-1)}(X_4, \mathbb{R}) \cup H^{2,2}_{(-2)}(X_4, \mathbb{R})$$

The subscript denotes the modular weight w of the partition function:

$$Z_{w,m}[G_4^{(w)}](\tau, z) = -q^{E_0} \sum N_{G_4^{(w)}}(n,r) q^n \xi^r, \quad G_4^{(w)} \in H^{2,2}_{(w)}(X_4, \mathbb{R})$$

- Only the "-1" fluxes can be straightforwardly lifted to 4d, where their partition functions then play the role of elliptic genus. The definition of $H^{2,2}_{(-1)}$ crucially involves the U(1) gauge symmetry.

The other fluxes just determine 2d superpotentials for Type IIA comp. on X_4

Anomalous modularity of 4d elliptic genus

- Computing ell genera, given $C_{n,r} = C_0 + n\,\mathbb{E}_\tau + r\,C_f$ and $G_4^{(-1)} \in H^{2,2}_{(-1)}(X_4,\mathbb{R})$

$$Z^{ell}_{-1,m}[C_0, G_4^{(-1)}](\tau,z) = -q^{E_0} \sum_{n\geq 0} N_{C_0,G_4^{(-1)}}(n,r) q^n \xi^r$$

one finds that in general

$$Z^{ell}_{-1,m}[C_0, G_4^{(-1)}](\tau,z) \;=\; \widetilde{Z}_{-1,m}[C_0, G_4^{(-1)}](\tau,z) + \xi\frac{\partial}{\partial\xi} Z_{-2,m}[C_0, G_4^{(-2)}](\tau,z)$$

spoils modularity and double periodicity!

- Derivatives break local symmetries:

$$\xi\partial_\xi\, \varphi_{w,m} = \frac{\tilde{\varphi}_{w,m+2}}{\varphi_{-1,2}} + 2m\, E_1\, \varphi_{w,m} \qquad \left(\xi\partial_\xi \equiv \frac{1}{2\pi i}\partial_z\right)$$

$$q\partial_q \varphi_{w,m} = \frac{\tilde{\varphi}_{w,m+1}}{\varphi_{-2,1}} + E_1 \frac{\tilde{\varphi}_{w,m+2}}{\varphi_{-1,2}} + \left(\frac{w}{12} E_2 + m\, {E_1}^2\right) \varphi_{w,m}$$

Induce connection terms: $E_2(\tau) = q\partial_q \log\, \eta^{24}(\tau)$, $\quad E_1(\tau,z) = \xi\partial_\xi \log\, \theta_1(\tau,z)$

$1/z$ pole of E_1 cancels!

Quasi-Jacobi vs. almost holomorphic Jacobi forms

- Thus we see that 4d ell genera lie in the ring

$$\varphi^{QJ}_{*,*} \in \mathcal{R}^{QJ} = \mathbb{Q}\Big[E_2, E_4, E_6; E_1, \varphi_{-1,2}, \varphi_{-2,1}, \varphi_{0,1}\Big] / \Big\{\varphi_{-1,2}, \varphi_{-2,1}\Big\}$$

Modularity and periodicity can be restored by substituting

$$E_2(\tau) \to \widehat{E}_2(\tau) = E_2(\tau) - 24\nu, \qquad \nu \equiv \frac{1}{8\pi \mathrm{Im}\tau}$$

$$E_1(\tau, z) \to \widehat{E}_1(\tau, z) = E_1(\tau, z) + \alpha, \qquad \alpha \equiv \frac{\mathrm{Im}z}{\mathrm{Im}\tau}$$

- This leads to "almost holo/meromorphic" Jacobi forms:

L'09, O'12, OP '17-19

$$\widehat{\Phi}(\tau, z) = \sum_{i,j \geq 0} \varphi^{(i,j)}(\tau, z) \left(\frac{1}{\mathrm{Im}\tau}\right)^i \left(\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right)^j$$

If $\widehat{\Phi}(\tau, z)$ transforms as Jacobi form, then $\varphi^{(0,0)}(\tau, z)$ is per def. quasi-Jacobi

⇨ The physical 4d elliptic genera are such almost holomorphic Jacobi forms

$$\widehat{Z}_{-1,m}(\hat{E}_1, \hat{E}_2, \ldots)$$

Elliptic holomorphic anomalies

- The novel feature is that anti-holomorphic derivatives act been different flux sectors, for example we have observed that

$$Z^{ell}_{-1,m}[C_0, G_4^{(-1)}] = \widetilde{Z}_{-1,m}[C_0, G_4^{(-1)}] + \xi \frac{\partial}{\partial \xi} Z_{-2,m}[C_0, G_4^{(-2)}]$$

A modular ell genus $\widehat{Z}_{-1,m}(\hat{E}_1, \hat{E}_2, \ldots)$ is obtained by replacing

$$\xi \partial \xi \to \frac{1}{2\pi i} \nabla_z = \frac{1}{2\pi i} \left(\partial z + 4\pi i m\, \alpha \right) \qquad \left(\alpha \equiv \frac{\mathrm{Im} z}{\mathrm{Im} \tau} \right)$$

and so we obtain the elliptic holomorphic anomaly equation:

$$\frac{d}{d\alpha} \widehat{Z}^{ell}_{-1,m}[C_0, G_4^{(-1)}] = \widehat{Z}_{-2,m}[C_0, G_4^{(-2)}]$$

- This and variants can be mathematically derived as general property of relative GW invariants of elliptic 4-folds, CY_4 OP'17-'19, LLLW '20

Eg. in 2d:

$$\frac{d}{d\nu} \widehat{Z}_{0,m}[C_0, G_4^{(0)}] = \widehat{Z}_{-2,m}[C_0, G_4^{(-2)}] \qquad \left(\nu \equiv \frac{1}{8\pi \mathrm{Im} \tau} \right)$$

Algebra of anomalies

$\mathrm{non-transversal\ flux\ in}\ H^{2,2}_{(0)}(Y_4,\mathbb{R}):$ $\quad Z_{0,m}(q,\xi)$

2d prepot

$\downarrow \partial_{E_1} \quad \uparrow \partial_z$

$\mathrm{transversal}\ U(1)\ \mathrm{flux\ in}\ H^{2,2}_{(-1)}(Y_4,\mathbb{R}):$ $\quad Z_{-1,m}(q,\xi)$ $\quad \downarrow \partial_{E_2} \quad \uparrow \partial_\tau$ (between $Z_{0,m}$ and $Z_{-2,m}$)

2d prepot,
4d elliptic genus

$\downarrow \partial_{E_1} \quad \uparrow \partial_z$

$\mathrm{non-transversal\ flux\ in}\ H^{2,2}_{(-2)}(Y_4,\mathbb{R}):$ $\quad Z_{-2,m}(q,\xi)$

2d prepot

[4d prepot
6d ell genus?]

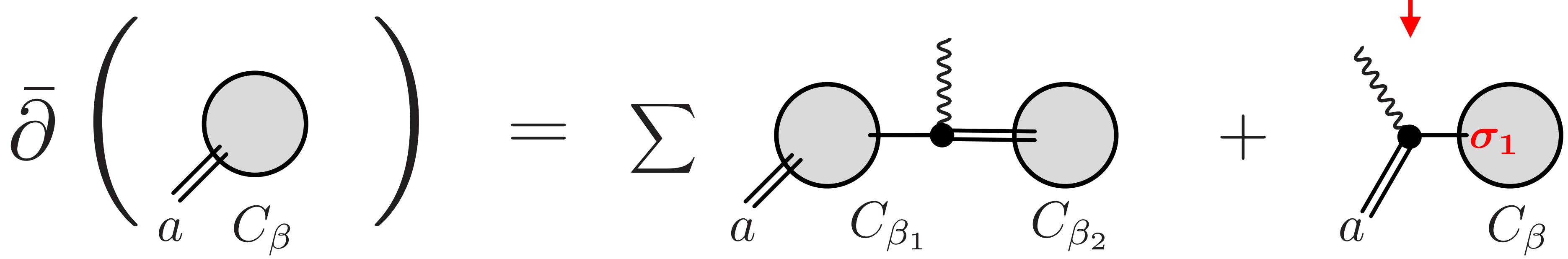
Holomorphic anomaly à la BCOV for 4-folds

BCOV '93

- So far this was a mathematical observation.
  It can also be derived in 2d TCFT on 4-folds with fluxes. LLLW '20

  The novel feature is a contact term of an anti-holom. operator insertion with the flux vertex operator. It gives rise to a gravitational descendant invariant:

$$-\frac{1}{2\pi i}\bar{\partial}_{\bar{i}}\mathcal{F}_{a|C_\beta} = \overline{C}_{\bar{i}}^{\,jb}\left(\sum_{\substack{C_{\beta_1}+C_{\beta_2}\\=C_\beta}}\mathcal{F}_{a;j|C_{\beta_1}}\mathcal{F}_{b|C_{\beta_2}} - \eta_{ab}\,\langle\int\sigma_1^{(2)}\phi_j\rangle_{C_\beta}\right)$$



splitting of reducible curve — new for 4-folds

- Encodes both modular and elliptic anomalies

$\left(\sigma_1^{(2)} \equiv \partial\bar{\partial}\varphi_{dil}\right)$

Physical interpretation in space-time ?

- As mentioned before, the well-known quadratic "splitting term" detects non-holom. zero modes in the binding of two E-strings into a heterotic string: $\partial_{\bar{\tau}} \hat{Z}(\tau) \sim \left(\frac{E_4}{\eta^{12}}\right)^2$

MNVW ' 98, HLV '14, ...

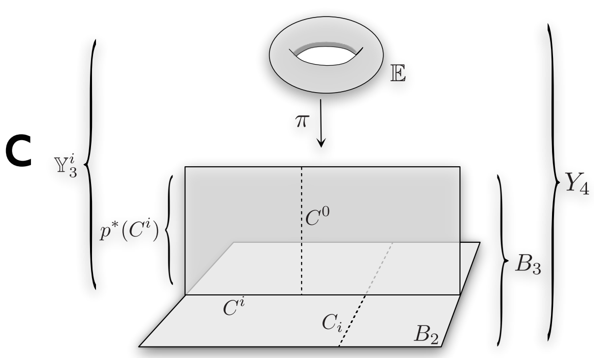
- Is there a similar meaning of the linear term that underlies the extra linear part of the holom. anomaly?

LLLW '20

$$\bar{\partial}_{\bar{\tau}} \mathcal{F}_{G^{(0)},C_\beta} = \sum_{C_{\beta_1}+C_{\beta_2}=C_\beta} \mathcal{F}_{G^{(0)},C_{\beta_1}} \mathcal{F}_{G^{(-2)},C_{\beta_2}} + \mathcal{F}_{G^{(-2)},C_\beta}$$

Conjecture: non-holom. zero modes arise when het. string meets an NS5 brane (dualized flux). This component of the moduli space is given by a certain 3-fold, $Y_3 \subset X_4$.

Indeed (for favorable geometries) the extra term the of holomorphic anomaly eqn. can be interpreted as elliptic genus $Z_{-2,m}[Y_3]$ of Y_3 . Its modular weight is w=-2, as is appropriate for a 6d theory.



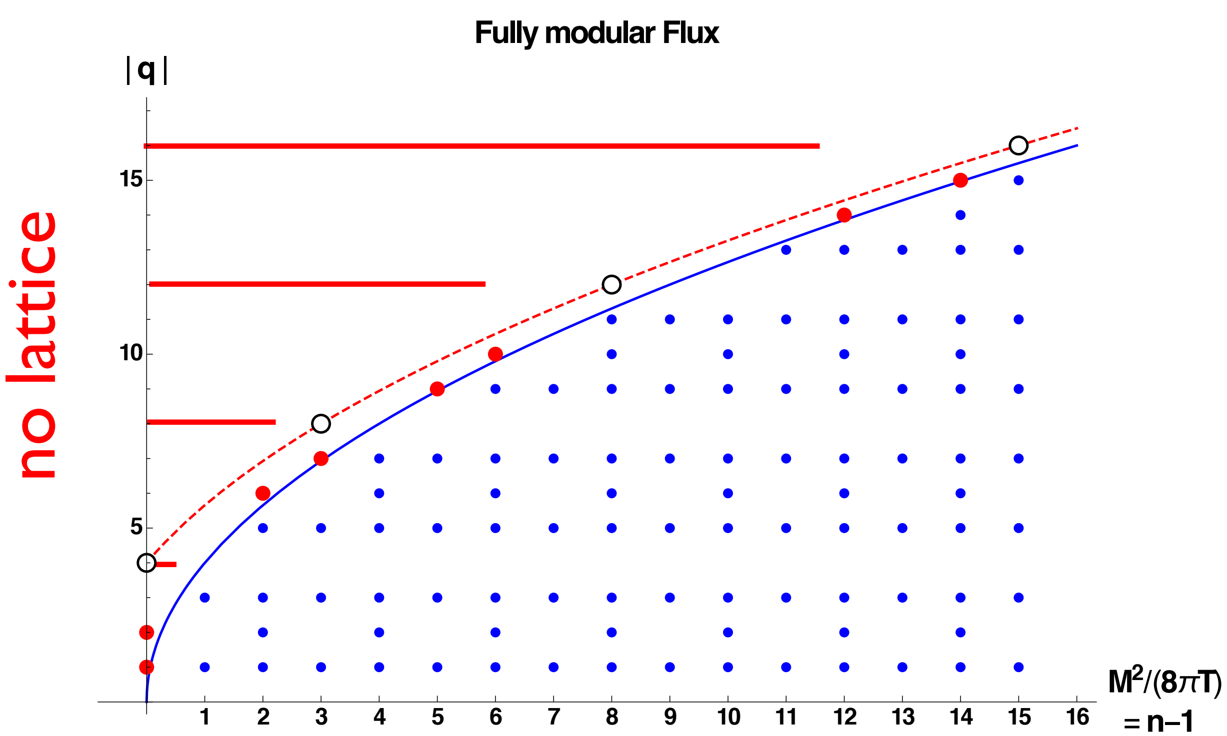
- Anomaly cancellation: there are extra massless fields originating from a "hidden" 6d sector, which modifies the naive perturbative GS mechanism in 4d. This non-perturbative modification precisely accounts for the extra derivative piece!

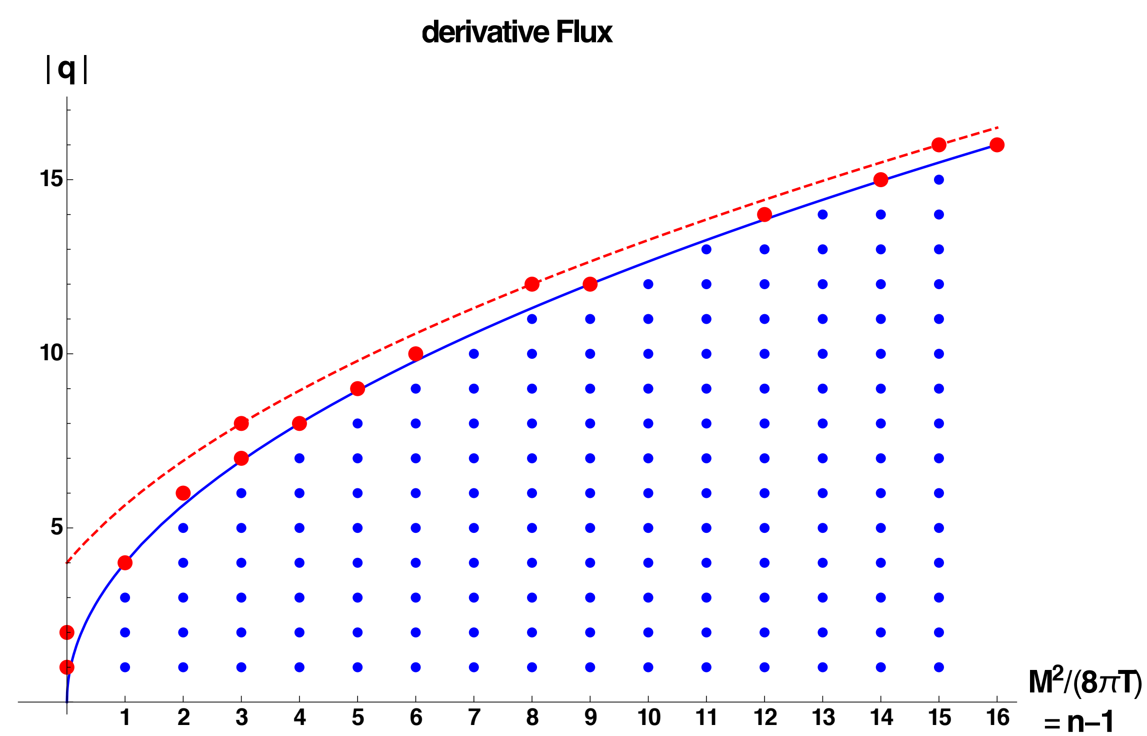
Back to WGC:

$$Z^{ell}_{-1,m} = \tilde{Z}_{-1,m} + \partial_z Z_{-2,m}$$

Odd modular weight: anti-symm in z; no maximally superextremal states, no fully populated charge lattice(*)

Restores maximally superextremal states, and completes charge lattice





(*)Caveats: possible cancellations in index; quantum corr

Summary

- Elliptic genera in 4d have surprisingly complex features:
  - U(1) is essential; weight -1 quasi-Jacobi forms
  - Non-perturbative versions can be obtained via duality to F-Theory on 4-folds with background 4-fluxes, in combination with mirror symmetry

    The degeneracies map to 4-fold GW invariants relative to curves C_0 and fluxes, G_4
  - Parts of the ell. genera are given by z-derivatives of partition functions corresponding to weight -2 flux sectors; formally 6d elliptic genera
  - The modular anomalies induced by the derivatives can be associated with a novel kind of elliptic holomorphic anomalies
- Physics applications: compatible with WGC, non-pert anomaly cancellation