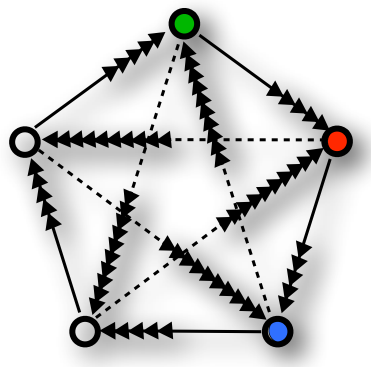
Matrix Factorizations and Homological Mirror Symmetry

Homological Mirror Symmetry

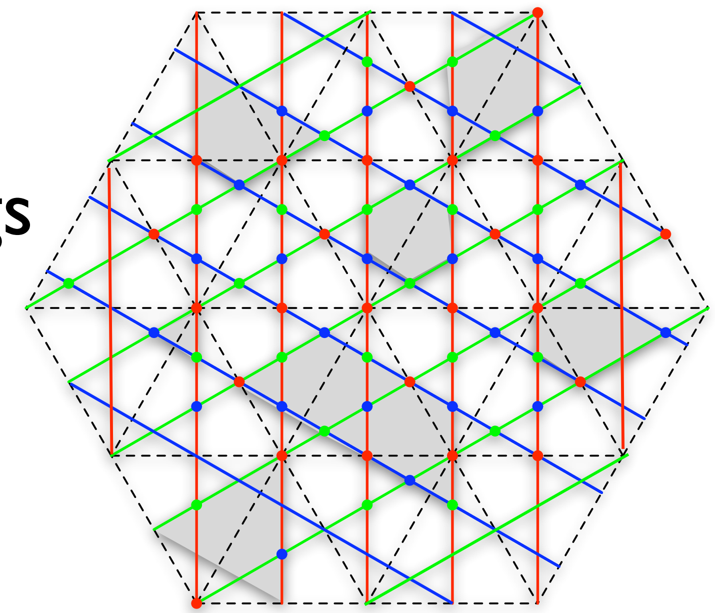
W.Lerche, Mittag-Leffler Inst, 7/2022

arXiv:1803.10333

- Motivation: quantum geometry of general D-brane configurations
- Recap: closed string mirror symmetry

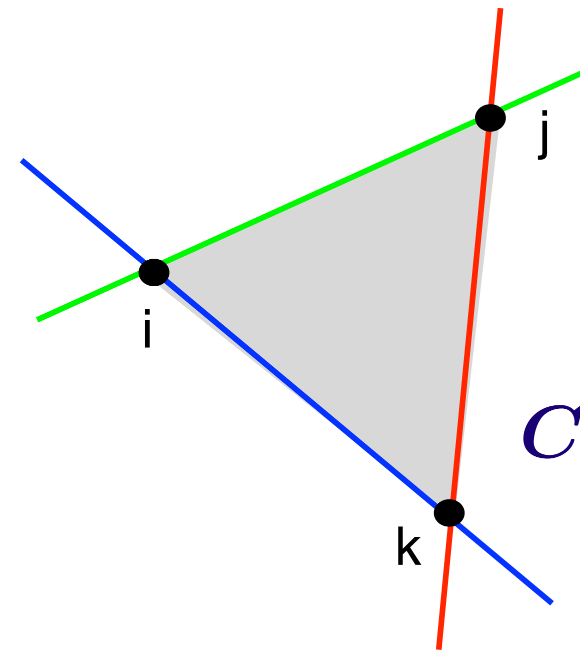


- LG models: contact terms vs. flat coordinates
- Open string = homological mirror symmetry
- Matrix factorizations and their deformations
- Open string mirror map from super-residue pairings
- Example: elliptic curve



Physics of intersecting brane geometries

- Phenomenological interest:
  - Chiral fermions
  - Exponentially suppressed Yukawa's



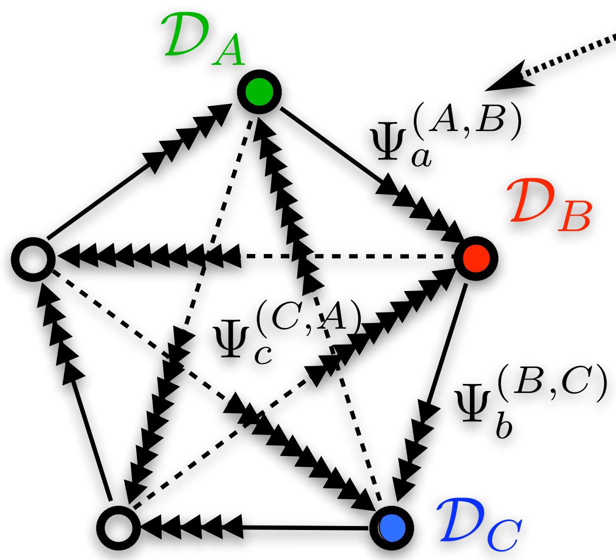
$$C_{ijk} \sim e^{-\mathbf{Area}}$$

- Open string mirror symmetry is by far not as well developed as for closed strings!

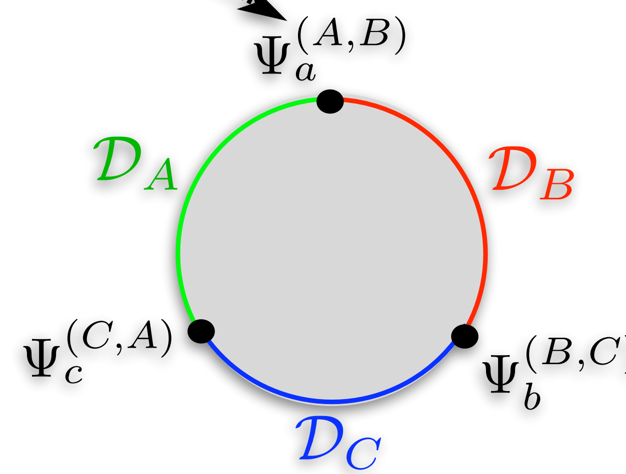
So far, mostly non-generic (toric/non-compact, non-intersecting) brane configurations were considered; almost nothing has ever been computed for intersecting branes eg. on Calabi-Yau threefolds!

Application: effective superpotential for quivers

boundary changing operator



Quiver diagram



Disk world sheet in TCFT

F-term superpotential ~ closed paths in quiver

$$\mathcal{W}_{eff}(T,u,t) = T_a T_b T_c \underbrace{\langle \Psi_a^{(A,B)} \Psi_b^{(B,C)} \Psi_c^{(C,A)} \rangle}_{C_{abc}(t,u)} + T_a T_b T_c T_d \underbrace{\langle \Psi_a^{(A,B)} \Psi_b^{(B,C)} \Psi_c^{(C,D)} \Psi_d^{(D,A)} \rangle}_{C_{abcd}(t,u)} + \ldots$$

space-time fields,
~ relevant ops

closed and open string
moduli $\sim \mathbf{const} + \mathcal{O}(e^{-t}, e^{-u})$

instanton corrections = open GW invariants: how to compute them?

Homological Mirror Symmetry

Kontsevich

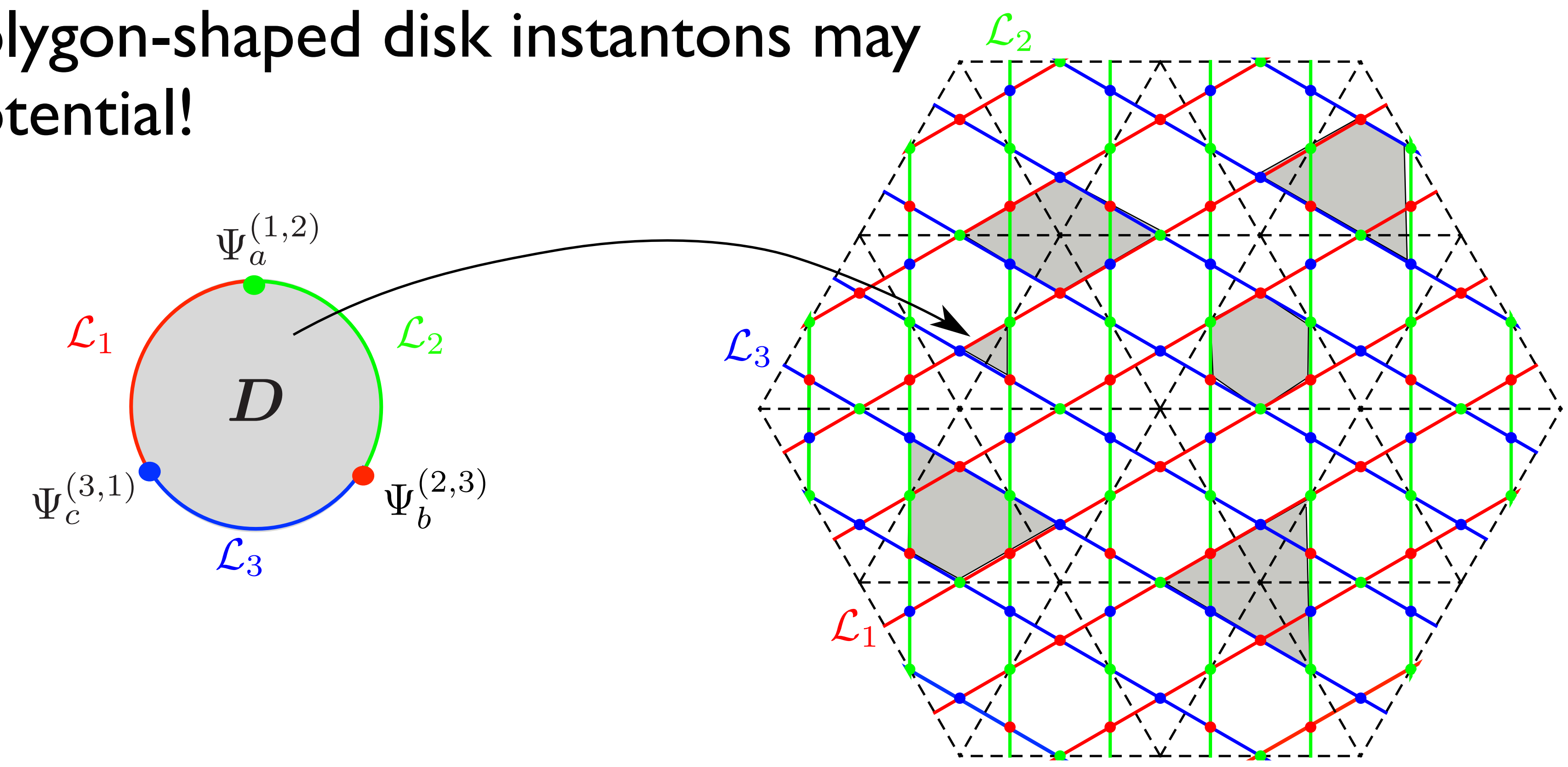
- Open string mirror symmetry becomes highly non-trivial for intersecting branes

  There is an infinitely richer diversity of open Gromov-Witten invariants, ie., world-sheet instantons.

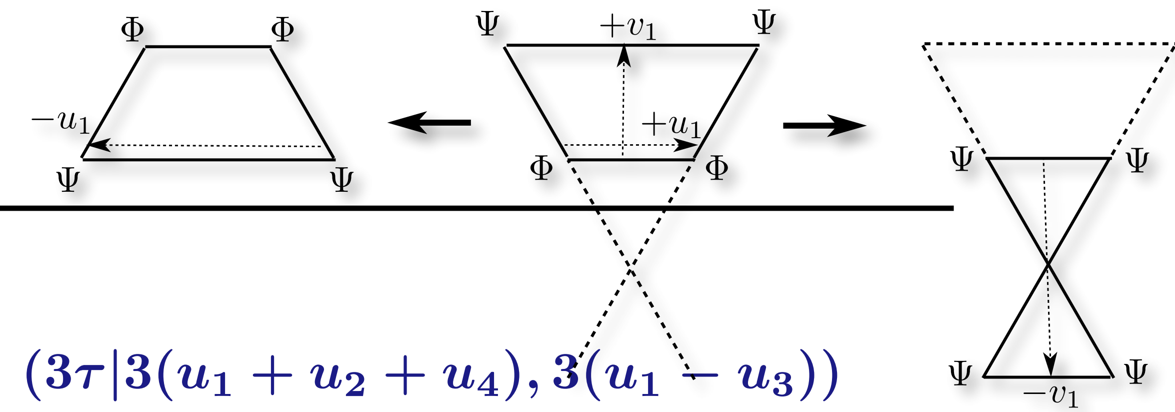
- Eg. the elliptic curve is almost trivial from the point of view of closed string instantons: $T_2 \to T_2$

- However in the open string sector with intersecting SL A-type branes, an arbitrary number of polygon-shaped disk instantons may contribute to the superpotential!

Polishchuk, Zaslow



Polygonal instantons



N=4: trapezoids

$$\mathcal{T}_{ab\bar{c}\bar{d}}(\tau,u_i) \;=\; \delta^{(3)}_{a+b,\bar{c}+\bar{d}}\,\Theta_{trap}\begin{bmatrix}[b-\bar{c}]_3\\ [\bar{d}-\bar{c}+3/2]_3\end{bmatrix}(3\tau|3(u_1+u_2+u_4),3(u_1-u_3))$$

$$\Theta_{trap}\begin{bmatrix}a\\ b\end{bmatrix}(3\tau|3u,3v) \;=\; {\sum_{m,n}}' q^{\frac{1}{6}(a+3n)(a+3n+2(b+3m))}e^{2\pi i\left((a+3n)(u-1/6)+(b+3m)v\right)}$$

N=4: parallelograms

$$\mathcal{P}_{a\bar{b}c\bar{d}}(\tau,u_i) \;=\; \delta^{(3)}_{a+c,\bar{b}+\bar{d}}\,\Theta_{para}\begin{bmatrix}[c-\bar{b}]_3\\ [\bar{d}-c]_3\end{bmatrix}(3\tau|3(u_1-u_3),3(u_4-u_2))$$

$$\Theta_{para}\begin{bmatrix}a\\ b\end{bmatrix}(3\tau|3u,3v) \equiv {\sum_{m,n}}' q^{\frac{1}{3}(a+3n)(b+3m)}e^{2\pi i\left((b+3m)u+(a+3n)v\right)}$$

N=5: pentagons

$$\wp_{a\bar{b}\bar{c}\bar{d}\bar{e}}(\tau,u_i) = \delta^{(3)}_{a,\bar{b}+\bar{c}+\bar{d}+\bar{e}}\,\Theta_{penta}\begin{bmatrix}[-b-c-d]_3\\ [e+c+d]_3\\ [c-d+\frac{3}{2}]_3\end{bmatrix}(3\tau|3(u_5-u_2),3(u_1-u_4),3(u_3+u_2+u_4))$$

$$\Theta_{penta}\begin{bmatrix}a\\ b\\ c\end{bmatrix}(3\tau|3u,3v,3w) \equiv {\sum_{m,n,k}}' q^{\frac{1}{3}(a_>+3(n+k))(b_>+3(m+k))-\frac{1}{6}(c+3k)^2}e^{2\pi i\left((a_>+3(n+k))u+(b_>+3(m+k))v+(c+3k)(w-1/6)\right)}$$

N=6: hexagons

$$\mathcal{H}_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}}(\tau,u_i) = \delta^{(3)}_{0,\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}}\Theta_{hexa}\begin{bmatrix}[-b-c-d]_3\\ [c+d+e]_3\\ [c-d+\frac{3}{2}]_3\\ [a-f+\frac{3}{2}]_3\end{bmatrix}(3\tau|3(u_5-u_2),3(u_1-u_4),3(u_3+u_2+u_4),3(-u_6-u_1-u_5))$$

$$\Theta_{hexa}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}(3\tau|3u,3v,3w,3z) \equiv {\sum_{m,n,k,l}}' q^{\frac{1}{3}(a+3n)(b+3m)-\frac{1}{6}(c+3k)^2-\frac{1}{6}(d+3l)^2}e^{2\pi i\left((a+3n)u+(b+3m)v+(c+3k)(w-1/6)+(d+3l)(z+1/6)\right)}$$

$$\sum_{m,n,k,l}{}' = \sum_{m,n\geq 0}^{\infty}\sum_{k\geq 0}^{<k_{max}}\sum_{l\geq 0}^{<l_{max}} - \sum_{m,n\leq -1}^{-\infty}\sum_{k\leq -1}^{>k_{min}}\sum_{l\leq -1}^{>l_{min}}$$

How to compute?

- The elliptic curve is flat, so it is easy to determine the areas by inspection, and sum them up by hand
  Polishchuk, Zaslow

  … but this is not what we ultimately want, because it does not easily generalize to higher dimensional Calabi-Yau n-folds!

- Rather we want to employ mirror symmetry, as familiar from the bulk, closed string theory.

- Recap ingredients of closed string mirror symmetry:

  - Pair of mirror Calabi-Yau's X,Y; $h_{21}(X)=h_{11}(Y)$
  - Variation of Hodge structures on X
  - Gauss-Manin flatness equations
  - Period integrals determining functional mirror map
  - Saito's higher residue pairings

Lightning recap: closed string mirror symmetry

Type IIA String on Calabi Yau Y ⟺ Type IIB String on Calabi Yau X

- Moduli space of N=2 vector SM: $\mathcal{QM}_K^{h_{1,1}}(Y,t) \simeq \mathcal{M}_{CS}^{h_{2,1}}(X,z)$

- 3-pt functions:

$$C_{klm} = \int_Y J_k \wedge J_l \wedge J_m + \sum_{d_1,\ldots,d_k} \frac{n^r_{d_1,\ldots,d_k} d_k\, d_l\, d_m}{1-\prod_{i=1}^k q_i^{d_i}} \prod_{i=1}^k q_i^{d_i} \longleftrightarrow \frac{p_{abc}(z)}{\prod \Delta(z)} \frac{\partial z_a}{\partial t_k} \frac{\partial z_b}{\partial t_l} \frac{\partial z_c}{\partial t_m}$$

A-model: deformed quantum geometry from world-sheet instantons = holom maps $P_1 \rightarrow Y$
$q = e^{-t}$

B-model: classical geometry

- Mirror map:

$$t_i := \int J_i^{1,1}(Y) + \ldots \longleftrightarrow \int_{\gamma_a^3} \Omega^{3,0}(X) =: \ln z_a(t) + \mathcal{O}(z)$$

flat coordinates on $\mathcal{QM}_K^{h_{1,1}}(Y)$ ⟺ Period integral = flat coo on $\mathcal{M}_{CS}^{h_{2,1}}(X)$

Phys: Superconformal B-twisted TCFT

All this has a concrete realisation in field theoretical models:

- Calabi-Yau defined by $X : W(x_i, z) = 0$

  W(x,z) is the superpotential of a N=(2,2) 2d Landau-Ginzburg model

  $\phi_i(x,t) = \partial_{t_i} W(x, z(t))$ forms a flat basis of the chiral ring

$$\langle \phi_k \phi_l \rangle = \mathbf{const.}$$

- In terms of these, all correlators are given in terms of residue integrals:

$$C_{klm}(t) \equiv \langle \phi_k \phi_l \phi_m e^{\int t_i \phi_i^{(2)}} \rangle = \oint \frac{1}{(dW(x,t))^N} \phi_k(x,t) \phi_l(x,t) \phi_m(x,t)$$

$$= \partial_{t_k} \partial_{t_l} \partial_{t_m} \mathcal{F}(t) \quad \text{integrability}$$

$$C_{klmn_1..n_r}(t) = \partial_{t_{n1}} ... \partial_{t_{nr}} C_{klm}(t)$$

"Special Geometry"

Math: Gauss-Manin system

- The period integrals satisfy certain flatness diff. equations that arise from the variation of Hodge structures.

  Essentially this boils down to a linear system of the form

$$\nabla \cdot \Pi \equiv \left(\delta_j^k \partial_{t_i} + (C_i)_j{}^k - (\Gamma_i)_j{}^k\right) \begin{pmatrix} \int \frac{1}{W} \\ \vdots \\ \int \frac{\phi^\lambda}{W^{\lambda+1}} \end{pmatrix}_k = 0$$

Yukawa's/ring OPE coeffs

Gauss-Manin connection

period vector Π

- $\Gamma = 0$ defines flat coordinates (and thus the mirror map): $z = z(t)$

  … as well as flat operator bases via $\phi_i(x, t) = \partial_{t_i} W(x, z(t))$

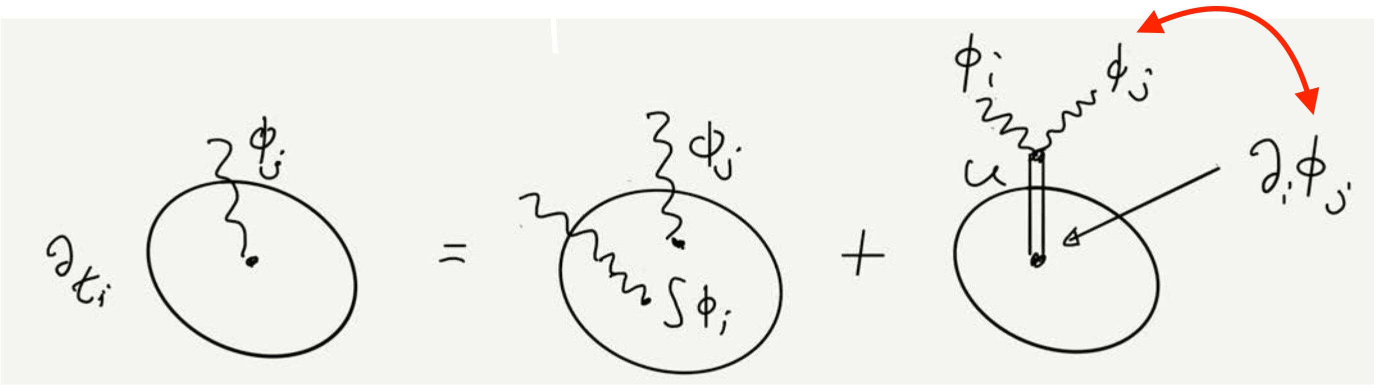
Phys: Contact terms versus flat coordinates

- The Gauss Manin eqn. encodes contact terms:

$$0 = \Gamma = \partial_{t_i}\phi_j - U(\phi_i\phi_j)$$

where U plays the role of the closed string propagator

$$U(\mathcal{O}(x,z)) \equiv d_{x_k}\left(\frac{\mathcal{O}(x,z)}{d_{x_k}W(x,z)}\right)_+ \sim \frac{G_0\bar{G}_0}{L_0}\mathcal{O} \qquad \mathcal{H}_E \to \mathcal{H}$$



- Functional dependence reflects renormalization by iteratively integrating out massive fields:

$$\phi(t) = \phi(0) + t\,U(\phi\phi) + 1/2\,t^2\,U(\phi\,U(\phi\phi)) + \cdots = \partial_t W(x,z(t))$$

Summing up all nested trees in one swoop!

L_∞ products

Math: Saito's higher residue pairings, K[u](-,-)

- Reformulate by avoiding period integrals while emphasizing contact terms:

  Localize path integral with insertion $e^{-\lambda(L_0+uU)}$ for $\lambda\to\infty$

  produces residue pairings $K[u](\phi_k,\phi_l) \equiv \sum_{\ell\geq 0} u^\ell K^{(\ell)}(\phi_k,\phi_l)$

  where u is a spectral parameter that counts the number of contact terms:

$$K^{(\ell)}(\varphi_k,\varphi_l) = \oint \frac{dx}{(dW)^N} \sum_{n=0}^{\ell} (-1)^{\ell-n} \overbrace{U(U(...U(}^{n}\varphi_k)..) \overbrace{U(U(...U(}^{\ell-n}\varphi_l)..)$$

- In terms of these, the Gauss-Manin eqs can be written compactly:

$$K^{(0)}(\varphi_k,\varphi_l) = \eta_{kl} = \text{const}, \qquad K^{(\ell>0)}(\varphi_k,\varphi_l) = 0, \qquad \text{(Siegel gauge)}$$

$$K[u](\nabla_t\varphi_a,\varphi_b) = K[u](\varphi_a,\nabla_t\varphi_b) = 0, \qquad \nabla_t \equiv \partial_t - \frac{\partial_t W}{u}$$

From closed to open strings…

- Mirror symmetry between A- and B-models

  ⟹ Homological Mirror Symmetry between categories of A- and B-type branes

  Kontsevich, … many…

- Hodge theory of CY-spaces

  ⟹ Non-comm. Hodge theory on A∞ categories

  Kontsevich, Soibelman, Katzarkov, Pantev, Sheridan….

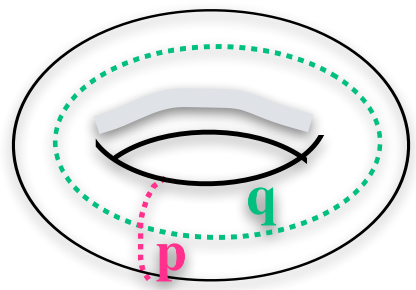
  Involves cyclic chain complexes, their (co)homologies,
  diverse Hochschild and Connes differentials, (b and B),
  a "Getzler connection" and a semi-infinite extension involving the
  spectral parameter u, with differential $d=b+uB$.

- Math lit focuses on rederiving Hodge-theoretic ("bulk") mirror symmetry from categorial one, but less on genuine open string invariants

See eg Ganatra,Perutz, Sheridan; Caldararu;Shklyarov; Ju…..

Phys: Homological Mirror Symmetry and D-branes

A-Model on Y



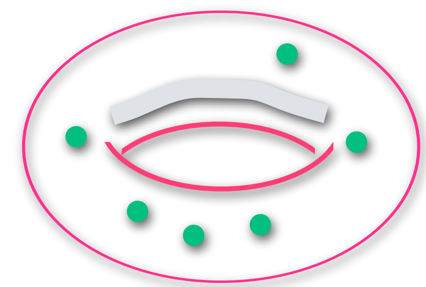
D1 branes on (p,q) cycles

"Fukaya category" of lagrangian cycles on Y

$Fuk(Y)$

mirror symmetry

B-Model on X



$(N_2,N_0) = (r,c_1)$ of gauge bundle

"Bounded derived category" of coherent sheaves on X

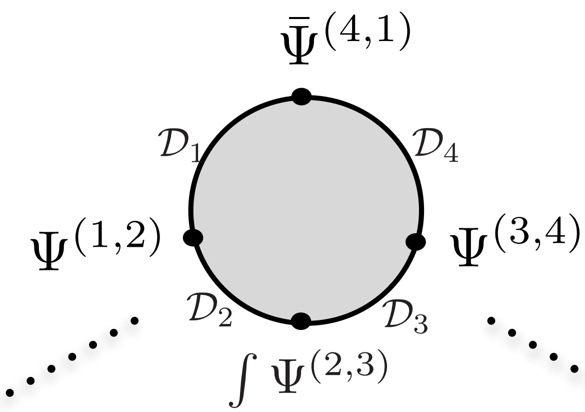
$D^b(Coh(X))$

However there is much more to this than just quantum numbers (K-theory), or isomorphisms between categories:

> Infinitely many open string correlation functions which encode enumerative invariants!

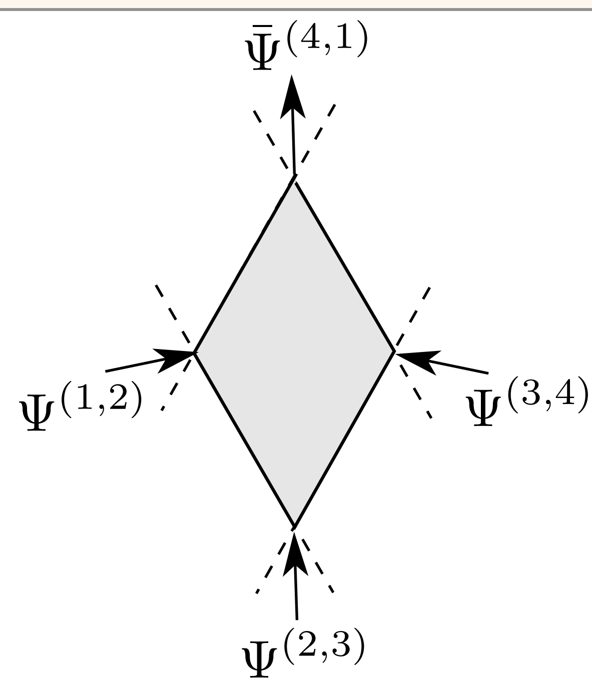
Mirror symmetry between A∞ products

$m_3(\Psi^{\otimes 3})$

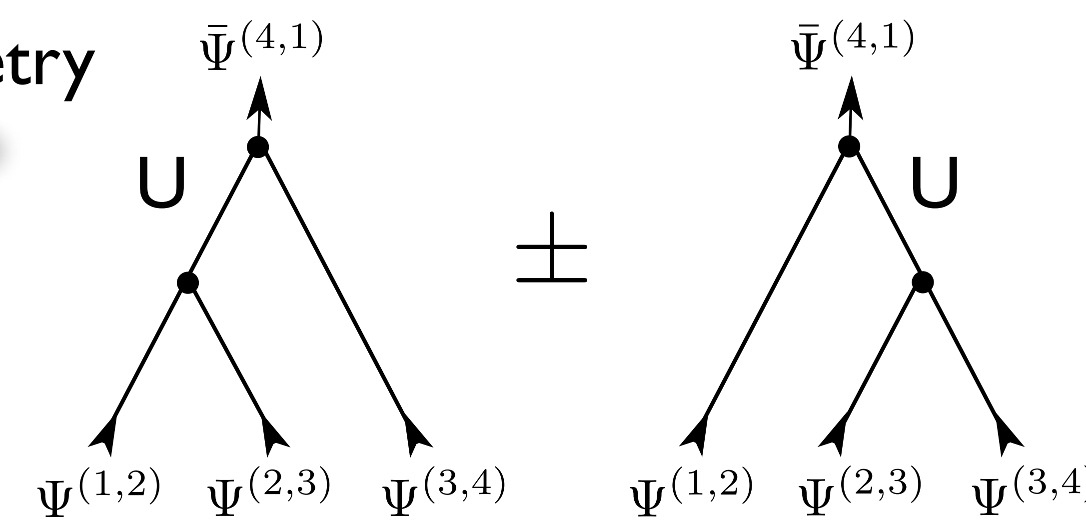


A-Model
localizes on holomorphic maps:
world-sheet instantons $D \to Y$

B-Model
localizes on constant maps:
nested trees



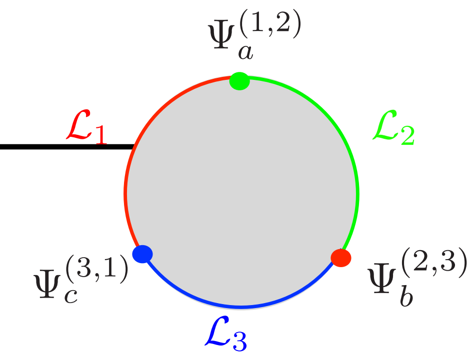
mirror symmetry



quantum Fukaya product $m_3 \sim e^{-S_{inst}}$

classical Massey product

Open string correlators and A∞ products



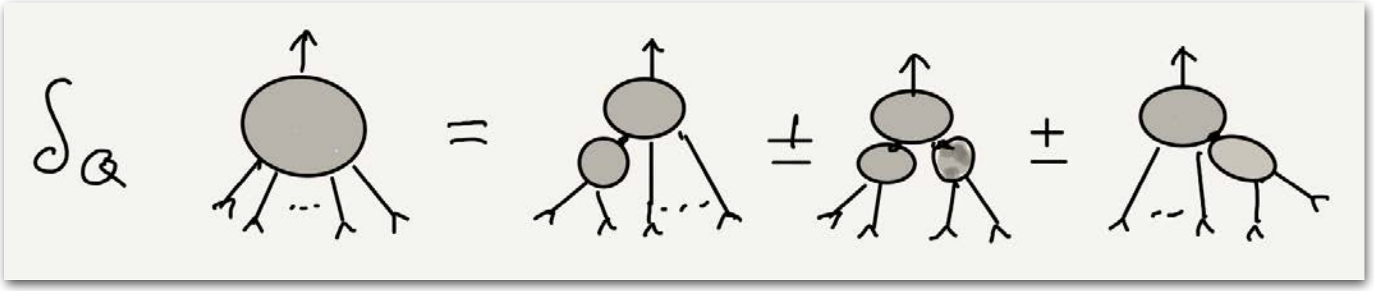
$$C_{a_0,a_1,\ldots,a_k} = \langle \Psi_{a_0} \Psi_{a_1} P \int \Psi^{(1)}_{a_2} \ldots \int \Psi^{(1)}_{a_{k-1}} \Psi_{a_k} \rangle$$

$$= \langle\langle \Psi_{a_0} , m_k(\Psi_{a_1} \oplus \ldots \oplus \Psi_{a_k} \rangle\rangle$$

- Multilinear, non-comm. maps $m_k : \Psi^{\otimes k} \to \Psi$

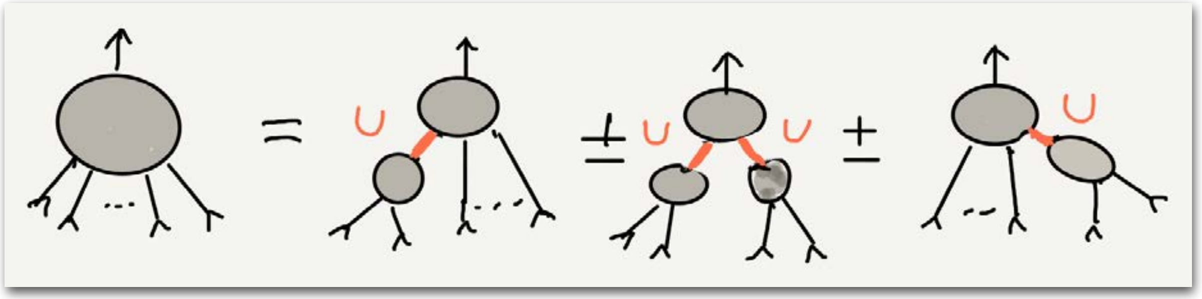
$$m_0 = 0, \quad m_1 = Q, \quad m_2 = \bullet$$

satisfy A∞ relations = Ward identities from disk factorization:



$$m_1 \cdot m_4(1,2,3,4) = m_3(m_2(1,2),3,4) \pm m_2(m_2(1,2),m_2(3,4)) \pm m_3(1,2,m_2(3,4))$$

- Can be recursively solved in closed form:



U = Q[-1]
open string propagator

$$m_4(1,2,3,4) = m_3(U \cdot m_2(1,2),3,4) \pm m_2(U \cdot m_2(1,2), U \cdot m_2(3,4)) \pm m_3(1,2,U \cdot m_2(3,4))$$

Where are the open enumerative invariants?

- That's all fine.. but where is the functional complexity (open GW invariants) concretely coming from?

  So how to tie open A- and B-models together quantitatively, ie, obtain transcendental functions encoding enumerative invariants ?

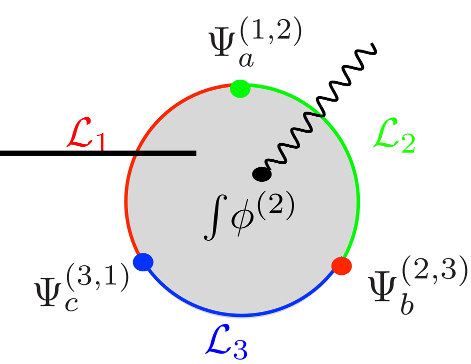
  …in analogy to closed string mirror map = period map: $t(z) \longleftrightarrow z(t)$

- We will consider deformations induced by closed string moduli t only, so

$$Fuk(Y)(t) \longleftrightarrow D^b(Coh(X))(z)$$

We need to extend this algebraic framework by an appropriate flat deformation structure, manifested in certain flatness diff eqs which determine flat operator bases.

Deformed A∞ Products



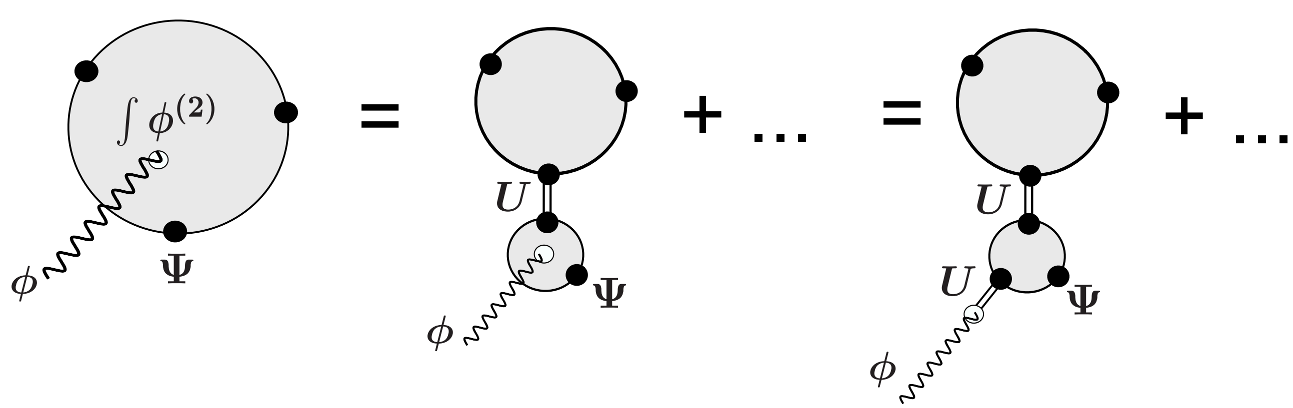
- We are interested in the dependence on bulk deformations t

$$C_{a_0,a_1,\dots,a_k}(t) = \langle \Psi_{a_0} \Psi_{a_1} P \int \Psi^{(1)}_{a_2} \dots \int \Psi^{(1)}_{a_{k-1}} \Psi_{a_k} e^{-t_k \int \phi^{(2)}_k} \rangle$$

$$= \langle\langle \Psi_{a_0}, m_k^t(\Psi_{a_1} \oplus \dots \oplus \Psi_{a_k}\rangle\rangle$$

- Deformed multilinear products satisfy "weak" A∞ relations where $m_0 \neq 0$

- Form extended structure: "open/closed homotopy algebra"

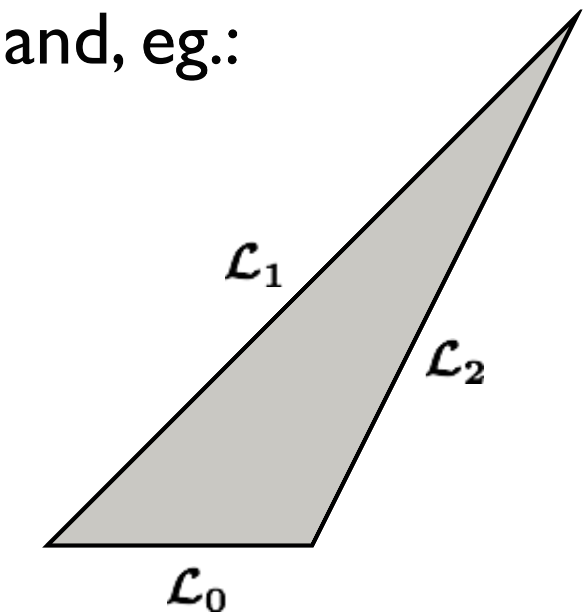


- How to sum up t-dependence to all orders explicitly?

HMS for the elliptic curve

Polishchuk, Zaslow

- The elliptic curve is flat, so it is easy to determine the areas by inspection, and sum them up by hand, eg.:



- Fukaya product

$$m_2 : \quad \mathrm{Hom}^*(\mathcal{L}_0, \mathcal{L}_1) \otimes \mathrm{Hom}^*(\mathcal{L}_1, \mathcal{L}_2) \to \mathrm{Hom}^*(\mathcal{L}_0, \mathcal{L}_2)$$

realized by theta-functions which are sections of the Hom's

$$\begin{aligned} \Theta[0,0](\tau,u) \cdot \Theta[0,0](\tau,u) = \ & \Theta[0,0](2\tau,u)\Theta[0,0](2\tau,2u) \\ & + \Theta[1/2,0](2\tau,u)\Theta[1/2,0](2\tau,2u) \end{aligned}$$

- Boils down to addition formulae of theta functions

… looks like an OPE, but these Θ's are not really field operators!

Phys: B-type, boundary LG models: matrix factorizations

Kapustin, Li
BHLS

- Consider 2d LG model with superpotential:

$$\int_\Sigma d^2z d\theta^+ d\theta^- W_{LG}(x,t) + cc.$$

(W(x,t)=0 describes CY X)

- If there is a boundary, B-type SUSY variations induce a "Warner"-term. This can be cancelled by boundary dof. whose BRST operator satisfies:

$$Q(x,t,u)_{2n\times 2n} \cdot Q(x,t,u)_{2n\times 2n} \;=\; W_{LG}(x,t)\, 1_{2n\times 2n}$$

- The matrices live in the Chan-Paton space and can have arbitrarily high dimension, 2n.

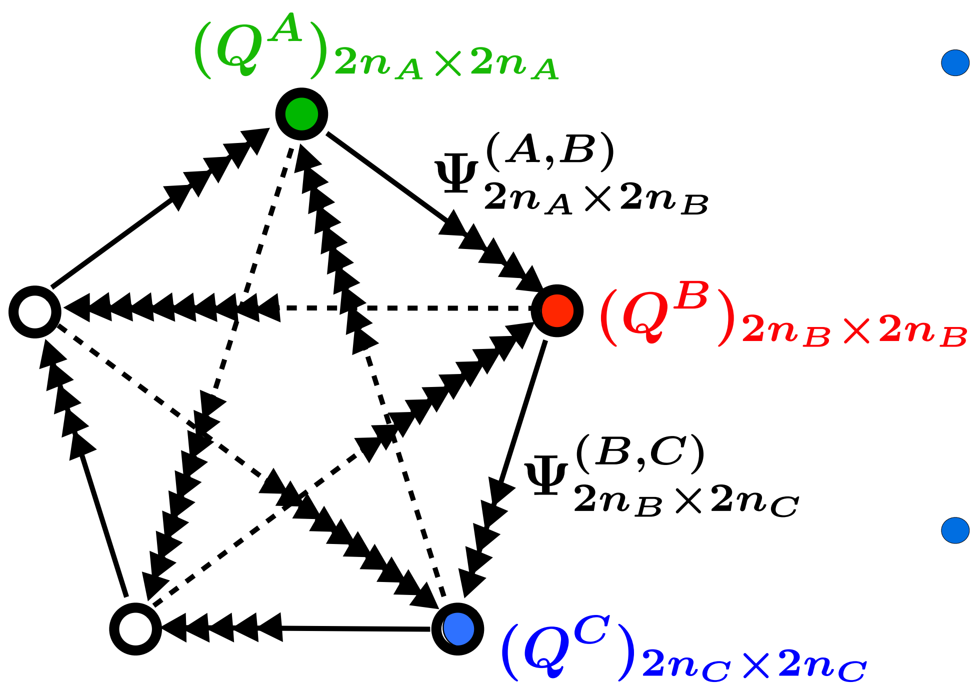
  The precise form encodes the brane geometry and depends on K-charges and possible deformation moduli t,u.

- The set of all matrix factorizations of W describes all possible B-type boundary conditions!

Math: The category of matrix factorizations

Math. Theorem: Kontsevich, Orlov

Cat(MF(W,X)) ~ D^b(Coh(X)), Category of coherent sheaves on X



- objects = chain complexes

$$\mathcal{P} = \left(P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \right) \qquad Q = \begin{pmatrix} 0 & p_0 \\ p_1 & 0 \end{pmatrix}$$

$p_{0,1}$ ~"tachyons" $p_0 p_1 = p_1 p_0 = W 1$

- morphisms = boundary changing operators

$$(\Psi_a^{(A,B)})_{2n_A\times 2n_B} \in \mathrm{Ext}^1(X;\mathcal{D}^A,\mathcal{D}^B)$$

- non-triv. cohomology $\Psi^{(A,B)}: \; d\cdot\Psi^{(A,B)} = 0, \quad \Psi_a^{(A,B)} \neq d\cdot *$

  where $d\cdot\Psi^{(A,B)} \equiv Q_A \Psi^{(A,B)} \pm \Psi^{(A,B)} Q_B$

- (non-comm.) composition maps $\Psi_a^{(A,B)} \cdot \Psi_b^{(B,C)} = C_{ab}{}^c\, \Phi_c^{(A,C)}$

  (contain as components analogs of theta-function identities)

Phys: Correlators from matrix factorizations

- Easy part:
  Construct representatives $\Psi \in \ker d/\mathrm{Im}\, d$ and recursively compute m$_k$:

$$C_{a_0,a_1,\ldots a_k}(t) \; = \; \langle\langle \Psi_{a_0}, m_k^t(\Psi_{a_1} \oplus \ldots \Psi_{a_k} \oplus)\rangle\rangle$$

with inner product = Kapustin-Li supertrace residue pairing

Kapustin, Li;
Lazaroiu, Herbst

$$\langle\langle A,\, B\rangle\rangle \; = \; \oint \mathrm{str}\left(\left(\frac{d_i Q}{d_i W}\right)^{\otimes N} A \cdot B\right)$$

- Can always choose representatives such that the two-point fct is const:

$$\langle\langle \Psi_a^{(A,B)},\, \Phi_b^{(B,A)}\rangle\rangle = \delta_{ab}$$

- Difficult part: what is the proper flat, renormalized operator basis?

$\Psi_a \to g_a(t)\Psi_a,\; \Phi_a \to g_a(t)^{-1}\,\Phi_a$ A priori freedom of rescaling....

...leaves corrs undetermined, eg: $\langle\langle \Psi, \Psi\Psi\rangle\rangle \sim g(t)^3$

Math: The boundary-bulk (or open-closed) map OC

- Generalization to non-commutative Hodge-Theory has been a major theme in math literature. Getzler; Kontsevich, Soibelman, Pantev, Katzarkov, Sheridan, ,….

- Usually one considers the open-closed map, eg.

$$OC(-) = \mathrm{str}[dQ^n \cdot -] : \qquad HH^*(CC_\bullet) \to \mathrm{Jac}\, W(X)$$

$$CC_\bullet = \bigoplus_k \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) \otimes \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \cdots \otimes \mathrm{Hom}(\mathcal{L}_k, \mathcal{L}_0)$$

…and thereby maps the open string sector (Hochschild complex) to the closed string sector with pairing: $\langle \alpha, \beta \rangle_{\partial D} \to \langle OC(\alpha), OC(\beta) \rangle_D$

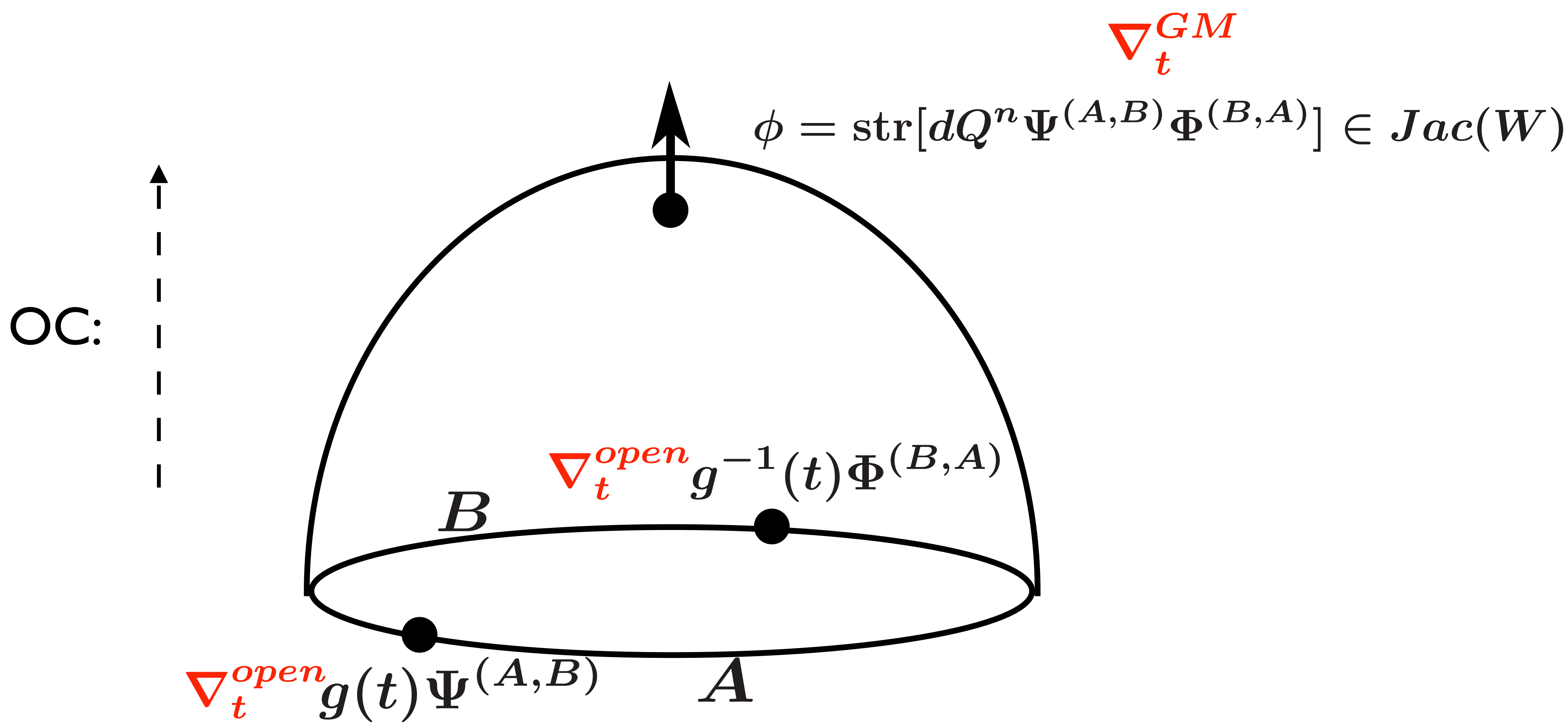
- This is different to what we want to do!
    - The open-closed map OC is non-vanishing only on cyclic chains of operators, and in particular on single boundary changing operators:

$$OC(\Psi^{A,B}) \equiv 0, \qquad \text{if}\, A \neq B$$

Our desired open Hodge theory must thus involve more data than just the isomorphism of the Hochschild cohomology HH*(CC) with the bulk cohomology!

The curse of the forgetful map OC

- Under OC, the relative normalization factor g(t) cancels out, and thus cannot be determined in this way:



- So need a boundary connection acting individually on the matrix-valued boundary changing operators!

Analog of Gauss-Manin connection at the boundary?

- There is a non-commutative version of the Gauss-Manin connection, the "Getzler" connection, but unclear to me if this is the full story, since

$$OC(\nabla_t^{Getz} \cdot -) \; \sim \; \nabla_t^{GM} OC(-)$$

  It acts on cyclic chains only and involves the degree-2 spectral parameter u which is an intrinsic bulk quantity (counting bulk propagators/contact terms)

- We go a physically inspired route:

  Crucial ingredients:

  - Generalization of Saito's residue pairings K to matrix factorizations
  - Coupled bulk-boundary deformation problem
  - Mixed bulk-boundary contact terms
  - Construct intrinsic boundary connection directly acting on matrices

Higher supertrace residue pairings

- Construct higher Kapustin-Li pairings to systematically capture contact terms

Shklyarov, uses OC

$$K^{(0)}_{KL}(\Psi_a, \Phi_b) = \oint \mathrm{str}\left(\left(\frac{d_i Q}{d_i W}\right)^{\otimes N} \Psi_a \cdot \Psi_b\right)$$
$$\overset{!}{=} \ \delta_{ab} \ = \ \mathrm{const}$$

$$K^{(1)}_{KL}(\Psi_a, \Phi_b) = \frac{(-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n} (-1)^{k(|\Psi_a|+1)} \sum_{i_*=1}^{n} \epsilon_{i_1 \ldots i_n} \times$$
$$2 \oint \mathrm{str}\Bigg[\left(\frac{d_{i_1} Q}{d_{i_1} W} \cdots \frac{d_{i_{k-1}} Q}{d_{i_{k-1}} W} \frac{d_k \Psi_a}{d_k W} \frac{d_{i_{k+1}} Q}{d_{i_{k+1}} W} \cdots \frac{d_{i_n} Q}{d_{i_n} W} \Phi_b\right)$$
$$- \left(\frac{d_{i_1} Q}{d_{i_1} W} \cdots \frac{d_{i_k} Q}{d_{i_k} W} \Psi_a \frac{d_{i_{k+1}} Q}{d_{i_{k+1}} W} \cdots \frac{d_{i_{n-1}} Q}{d_{i_{n-1}} W} \frac{d_n \Phi_b}{d_n W}\right)\Bigg]$$

Instead of a commuting spectral parameter u of degree 2, which counts insertions of the bulk propagator $U \sim d\left(\frac{*}{dW}\right)_+$

we (formally!) have an anti-commuting parameter ξ of degree 1, which counts insertions of the odd boundary propagator $U_\partial \sim$ "$\frac{1}{Q}$"

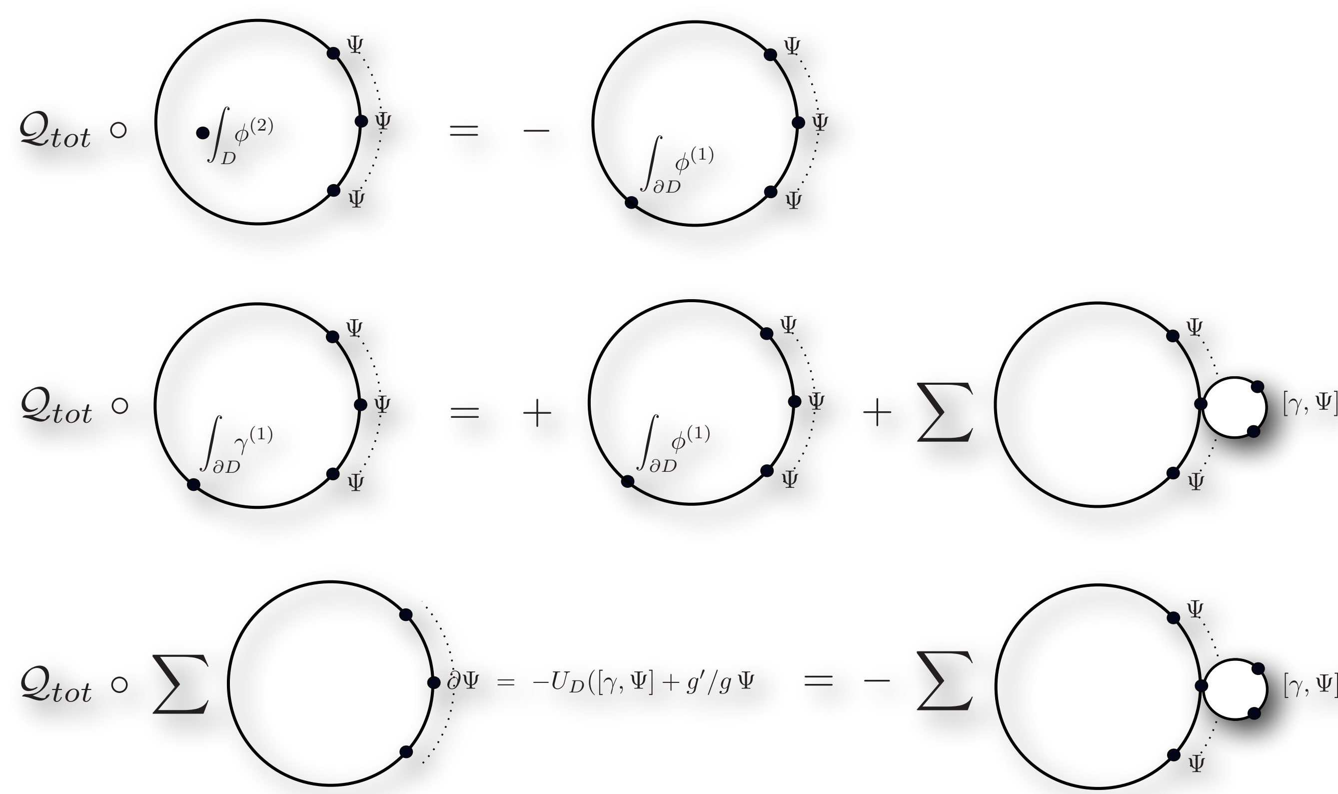
Coupled bulk-boundary deformation problem

- Due to bulk-boundary contact terms, the bulk perturbation $\phi = \partial_t W$ must be accompanied by a "Warner" boundary counter term $\gamma = \partial_t Q$

$$\delta S = t\left(\int_D \phi^{(2)}\mathbf{1} - \int_{\partial D}\gamma^{(1)}\right) \qquad \{Q(t),\gamma(t)\} = \phi(t)|_{\partial D}\mathbf{1}$$

This combo perturbation preserves $Q(t)^2 = W(t)\mathbf{1}$ so is unobstructed. It is the natural Q-invariant pairing in relative (co-)homology of disk.

- What matters are the contact terms of γ with the other boundary ops ψ:



Finally, flatness equations for matrix factorizations

- Taking all together, we propose "relative bulk-boundary" diffeqs. which play the role of the Gauss-Manin eqs familiar from standard bulk mirror symmetry:

$$K^{(0)}_{KL}(\nabla_t\Psi_a,\Phi_b) = K^{(0)}_{KL}(\partial_t\Psi_a,\Phi_b)+K^{(1)}_{KL}(\Psi_a,\gamma\cdot\Phi_b)-\frac{1}{2}K^{(0)}_{KL}(\sum_i \frac{d_i\phi}{d_iW};\Psi_a,\Phi_b)$$

$$\stackrel{!}{=} 0$$

boundary-boundary ct   boundary-bulk ct

Morally:

$$\nabla_t = \left(\partial t - \frac{\partial_t W}{u}\right)\Big|_{\text{bulk}} - \left(\partial t - \frac{\partial_t Q}{\zeta}\right)\Big|_{\text{boundary}}$$

These eqs supposedly determine the proper flat boundary changing representatives ψ(t) incl. moduli dependent renormalisation factors

When combined with the recursive A_∞ structure, the latter should eventually determine the t-moduli dependence of all correlation functions!

("splitting" ambiguities?)

Example: branes on cubic elliptic curve T_2

- Simplest 1-dim Calabi-Yau: elliptic curve

complex struct modulus

$$T_2: \quad W(x, z(t)) \equiv \frac{1}{3}({x_1}^3 + {x_2}^3 + {x_3}^3) - z(t) x_1 x_2 x_3 = 0$$

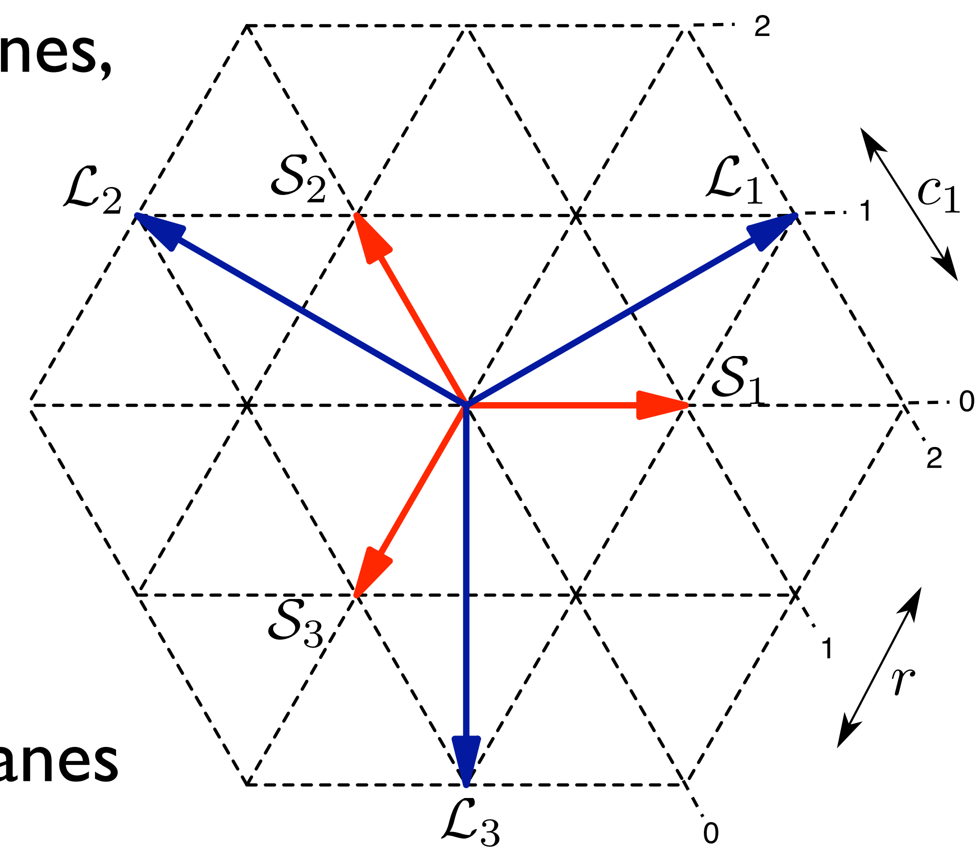
Mirror map: $t(z) = i/\sqrt{3}\,\frac{{}_2F_1(1/3, 2/3, 1; 1 - 1/z^3)}{{}_2F_1(1/3, 2/3, 1; 1/z^3)}$

- B-type D-branes are composites of D2, D0 branes, characterized by

$$(N_2, N_0; u) \;=\; (\mathrm{rank}(V), c_1(V); u)$$

- We will consider the ``long-diagonal'' branes with charges

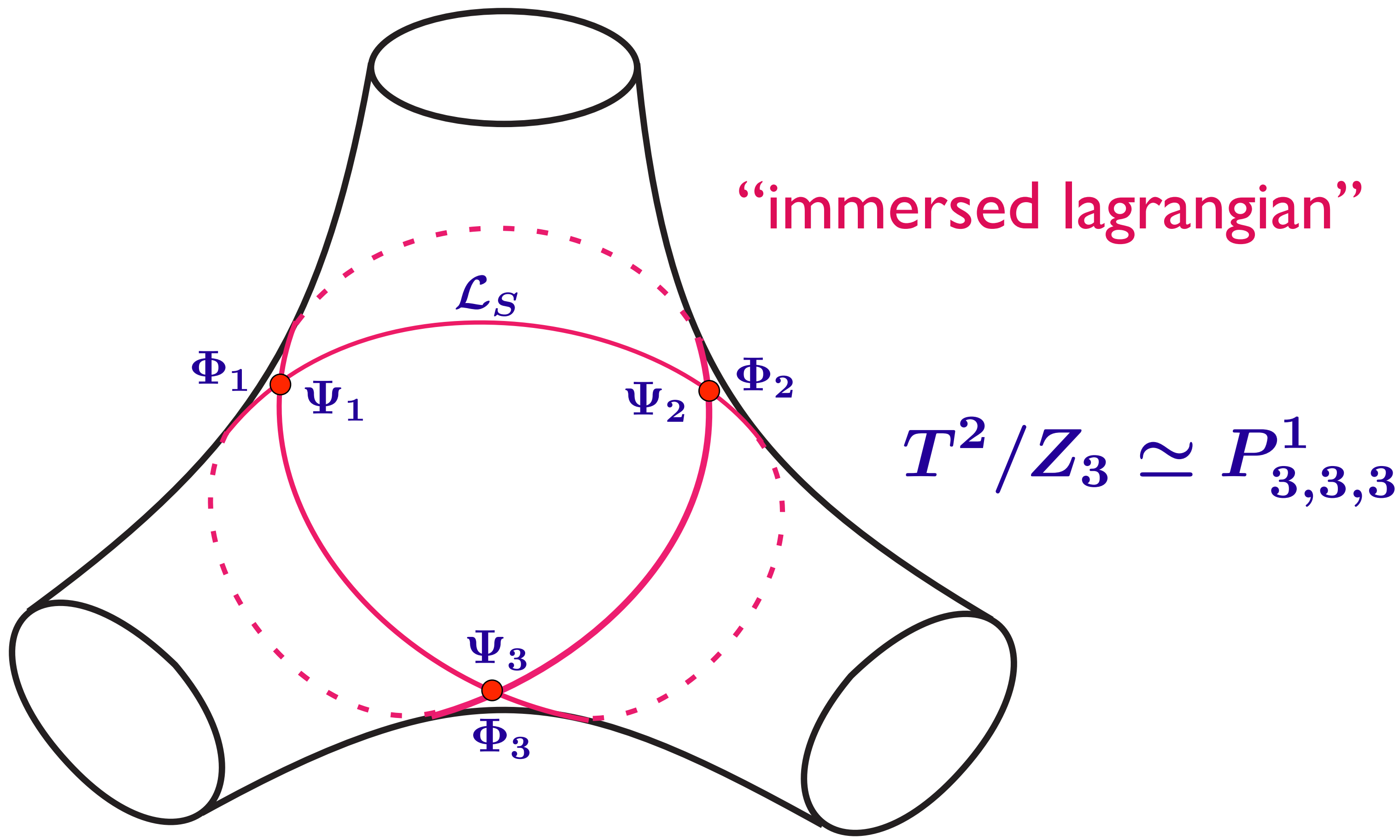
$$(N_2, N_0)_{\mathcal{L}_A} = \{(-1, 0), (-1, 3), (2, -3)\}$$



picture of mirror A-branes

Seidel lagrangian

- Actually the LG model describes the orbisphere T_2/Z_3 (or pair of pants), where the 3 branes map into one single, triply self-intersecting brane



- Need to go to equivariant matrix factorization to describe branes on T_2; in practice only labels change

Matrix factorization corr. to Seidel lagrangrian

Given by 8x8 matrix: $Q = \begin{pmatrix} 0 & p_0 \\ p_1 & 0 \end{pmatrix}$ satisfying $Q^2 = W(x, z(t))\, 1$

with

$$p_0 = \begin{pmatrix} \frac{x_1}{3} & \frac{x_2}{3} & \frac{x_3}{3} & 0 \\ x_2^2 - x_1x_3z(t) & x_2x_3z(t) - x_1^2 & 0 & \frac{x_3}{3} \\ x_3^2 - x_1x_2z(t) & 0 & x_2x_3z(t) - x_1^2 & -\frac{x_2}{3} \\ 0 & x_3^2 - x_1x_2z(t) & x_1x_3z(t) - x_2^2 & \frac{x_1}{3} \end{pmatrix}$$

$$p_1 = \begin{pmatrix} x_1^2 - x_2x_3z(t) & \frac{x_2}{3} & \frac{x_3}{3} & 0 \\ x_2^2 - x_1x_3z(t) & -\frac{x_1}{3} & 0 & \frac{x_3}{3} \\ x_3^2 - x_1x_2z(t) & 0 & -\frac{x_1}{3} & -\frac{x_2}{3} \\ 0 & x_3^2 - x_1x_2z(t) & x_1x_3z(t) - x_2^2 & x_1^2 - x_2x_3z(t) \end{pmatrix}$$

This realizes the "homological mirror functor" of Cho,Hong, Lau, Oh…

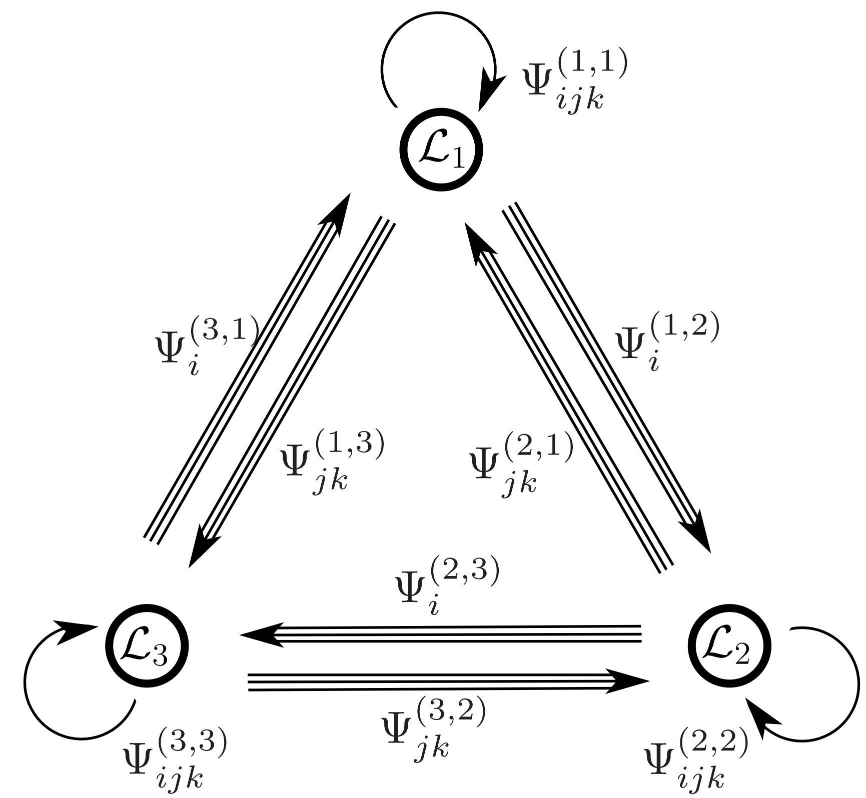
Open string BRST cohomology

- Solving for the BRST cohomology yields explicit moduli dependent matrix valued morphisms, eg.

$$\Psi_1^{(A,A+1)} = g(t)\begin{pmatrix} 0 & q_0 \\ q_1 & 0 \end{pmatrix}$$

$$q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{2}x_3 z(t) & 3x_1 & 0 & 0 \\ \frac{3}{2}x_2 z(t) & 0 & 3x_1 & 0 \\ 0 & \frac{3}{2}x_2 z(t) & -\frac{3}{2}x_3 z(t) & 1 \end{pmatrix}$$

$$q_1 = \begin{pmatrix} -3x_1 & 0 & 0 & 0 \\ \frac{3}{2}x_3 z(t) & -1 & 0 & 0 \\ \frac{3}{2}x_2 z(t) & 0 & -1 & 0 \\ 0 & \frac{3}{2}x_2 z(t) & -\frac{3}{2}x_3 z(t) & -3x_1 \end{pmatrix}$$



$$[Q, \Psi_a^{(*,*)}] = 0$$

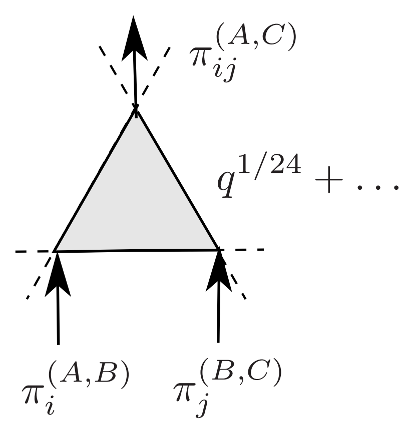
- Again, the issue is to determine the flattening, moduli dependent renormalisation factor g(t)

Solving the proposed "relative bulk-boundary" diffeqs yields

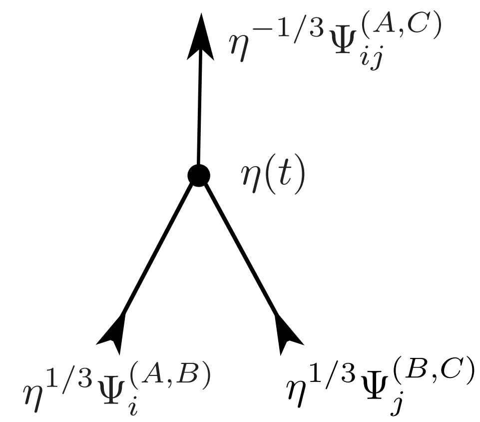
$$g(t) = \eta(q)^{1/3}, \quad q = e^{2\pi i t}$$

A-model instantons

- This defines via open string mirror symmetry the quantum Fukaya product m_2:



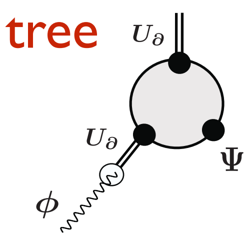
A-Model



B-Model

In B-model, the functional complexity is entirely due to the flattening renormalization factor g(t)!

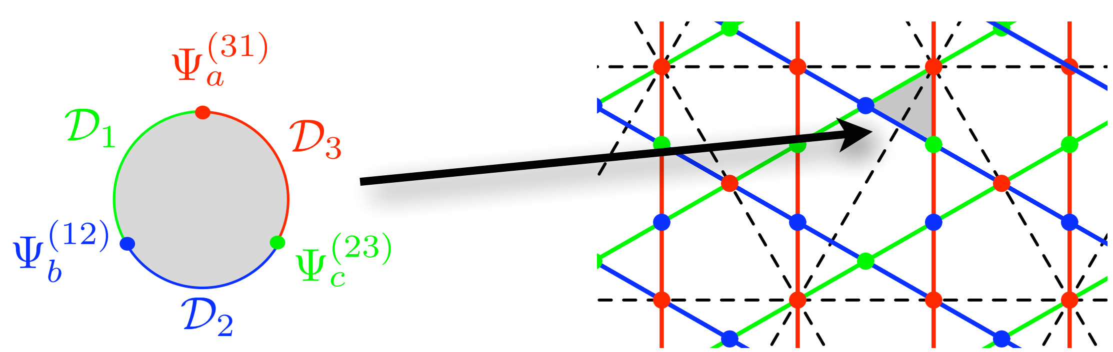
It sums up infinitely many tree diagrams



- Phys. interpretation in A-model: 3-point function counts disk instantons

$$C_{abc}(t) \;=\; \langle\langle \Psi_a^{(1,2)}, m_2(\Psi_b^{(2,3)}\Psi_c^{(3,1)})\rangle\rangle = \epsilon_{abc}\,\eta(q)$$

$$\eta(q) \equiv q^{1/24}\prod_{n>0}(1-q^n)$$



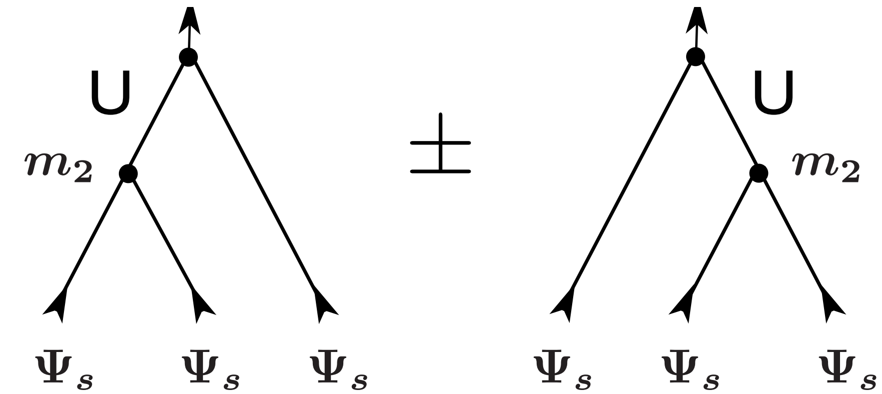
minimal area:
1/24 of fundamental domain

Higher order B-model correlators: 4 pt function

- Define "boundary chain"

$$\Psi_s = -1/3 \sum s_i \Psi_i$$

Compute m3 via nested trees and propagators:



$$m_3(\Psi_s, \Psi_s, \Psi_s,) = \frac{\eta(t)}{\zeta(t)} W(s,t)1$$

$$\zeta(t) = \sqrt{\frac{z'(t)}{z^3(t)-1}}$$

fundamental period

- $m_3(\Psi^{\otimes 3}) \sim W1\ldots$ = Maurer-Cartan equ,
  means that Seidel lagrangian on $P^1_{3,3,3}$ is "weakly obstructed"
  FOOO

- Matches results on the A-model side Cho, Hong, Lau, Oh…

Summary and Outlook

- math: Cat of matrix factorizations $\longleftrightarrow$ D(Coh(M))

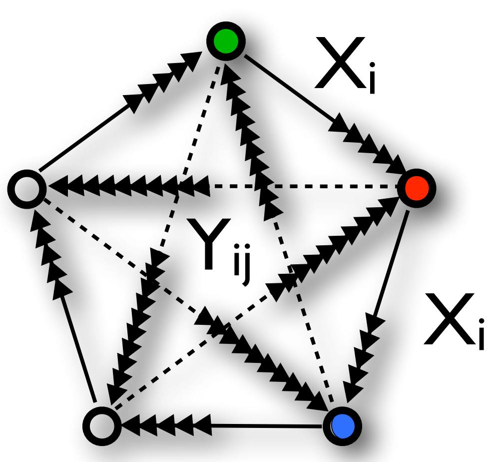
  phys: Boundary B-type TCFT $\longleftrightarrow$ B-type D-branes

- Field theoretical LG model allows to explicitly compute non-trivial correlation functions also for intersecting branes

- Main issue: find suitable Gauss-Manin type differential eqs that determine the proper flat operator bases

  Main tool: matrix analogs for higher residue pairings

- Generalization to M = CY 3-folds, eg. for quintic?



$$\mathcal{W}_{eff} = C_{XXY}(t)\,\mathrm{Tr}XXY + C_{XXYXXY}(t)\,\mathrm{Tr}(XXY)^2 + \ldots$$

t... Kähler modulus

... expect infinitely many new results in enumerative geometry