Local Mirror Symmetry and Rigid Limits of N=2 Supersymmetric String Theories

W.Lerche 1997

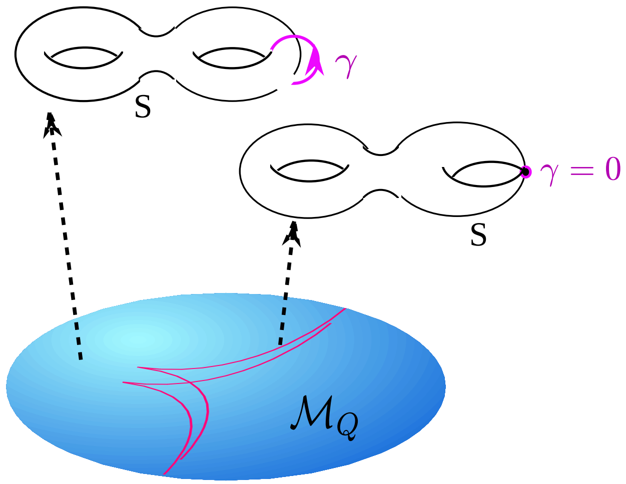
2 main parts:

- **Recovering SW theory from string duality**
  - heterotic-type II string duality
  - K3-fibrations
  - local geometry and mirror symmetry
  - SW curves, blow-up in moduli space
- **Novel rigid limits**
  - closed monodromy sub-problems
  - vanishing 4-cycles and tensionless, non-critical strings
  - stringy effective actions

Recap: Generalities of N=2 Gauge Theory

- The theory is solved in terms of an auxiliary Riemann surface S; for SU(n), it is of genus $g = n - 1$.
- Over singular regions in the moduli space, this surface degenerates in that certain 1-cycles γ shrink to zero.

  The corresponding periods $\{\phi(u), \phi_D(u)\} \sim \oint_\gamma \lambda$ vanish, signalling the appearence of extra massless states ("monopoles").
- The shrinking 1-cycles can be associated with certain non-critical strings, whose wrappings create BPS states in space-time

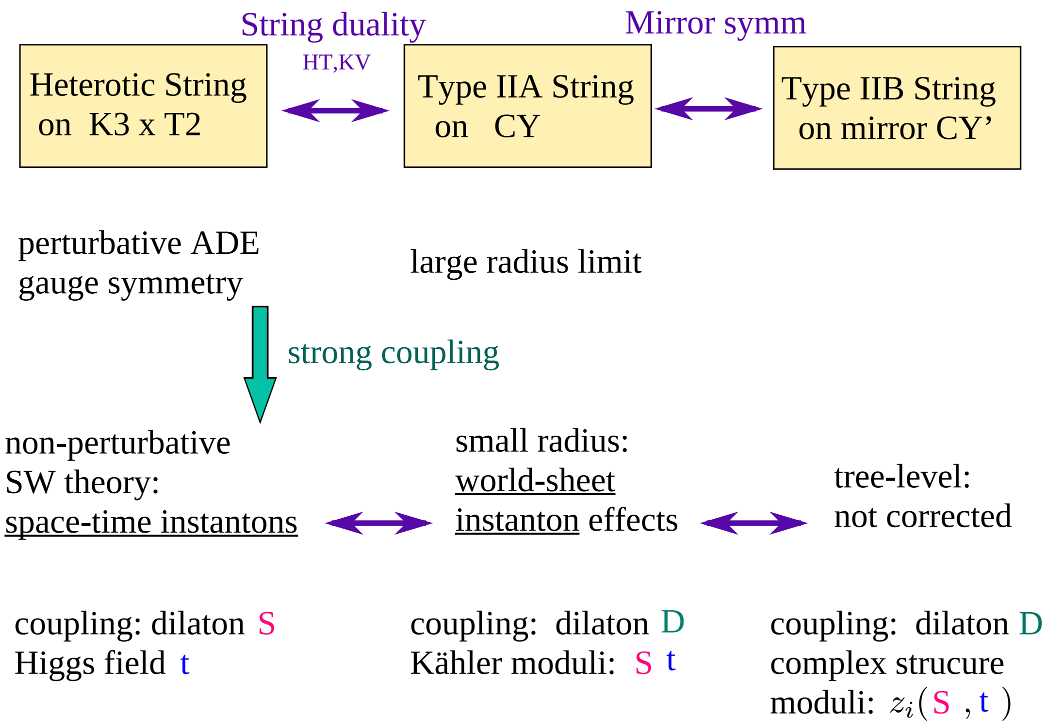


- Local coordinate patches on $\mathcal{M}_Q$ describe different local approximations in terms of different, weakly coupled physical degrees of freedom; perturbative physics looks different in the various patches.

> Extend these ideas to obtain novel kinds of exactly solvable N=2 supersymmetric (field ?) theories, as rigid limits of known string theories ....

Heterotic-Type II String Duality

(N=2 SUSY in d=4)



String duality

HT,KV

Mirror symm

Heterotic String on K3 x T2 ↔ Type IIA String on CY ↔ Type IIB String on mirror CY’

perturbative ADE gauge symmetry

large radius limit

strong coupling

non-perturbative SW theory: space-time instantons ↔ small radius: world-sheet instanton effects ↔ tree-level: not corrected

coupling: dilaton S
Higgs field t

coupling: dilaton D
Kähler moduli: S t

coupling: dilaton D
complex strucure moduli: z_i(S , t)

Common moduli space:

heterotic coupling

type II coupling

$$\mathcal{M} \;=\; \mathcal{M}_V(\, S \,,\, t \,,\ldots) \;\otimes \mathcal{M}_H(D,\ldots)$$

$\mathbb{C}$ structure

$\mathbb{Q}$ structure

vector multiplets
exact in type IIB

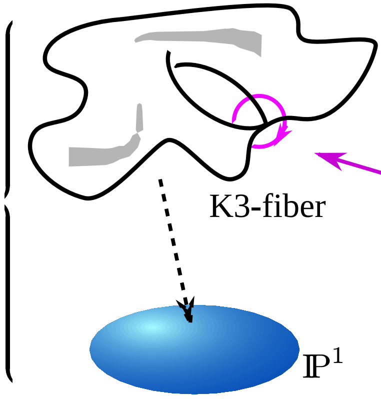
hypermultiplets
exact in heterotic string

Use tree-level IIB on mirror CY’ to derive SW effective action, plus gravitational corrections !

K3-Fibrations

Important: this duality works (naively) only for
a special class of Calabi-Yau compactifications
on the type II side .....
CY's must be **K3 fibrations**

..... essentially because the basic duality
is in 6 dimensions, between type IIA
on K3 and heterotic string on T4 (HT)



the total space
is a Calabi-Yau
3-fold

the vanishing 2-cycles
give ADE gauge symmetries
in 6d

gauge particles arise from
wrapping IIA 2-branes
around these 2-cycles

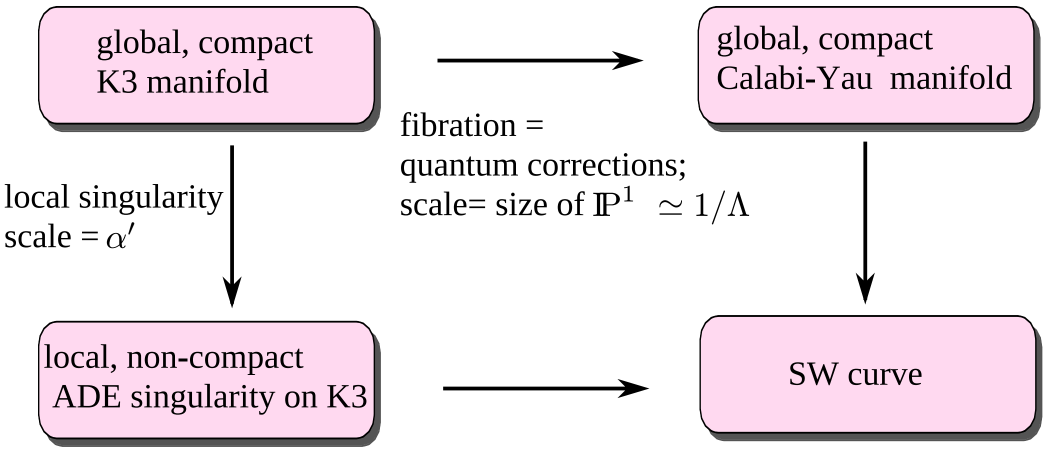
base = complex plane

The non-perturbative corrections in 4d
(a la SW) arise from
brane wrappings of the base $\mathbb{P}^1$

large $\mathbb{P}^1$ = weak coupling = instanton
effects suppressed

$\mathbb{P}^1 \simeq 1/\Lambda$

General Scheme



global, compact
K3 manifold

global, compact
Calabi-Yau manifold

fibration =
quantum corrections;
scale= size of $\mathbb{P}^1 \simeq 1/\Lambda$

local singularity
scale = α'

local, non-compact
ADE singularity on K3

SW curve

All what is important for a rigid, field theory limit is
the **local** singularity structure of the
compactification manifold !

Universality:

ADE classification of "simple singularities"

↔

ADE type of gauge theories

Local ADE Singularities on the K3 Surface

- Near a A_1 ~ SU(2) type of singularity, K3 looks locally like:

$$W_{K3} = \epsilon\,[x^2 + y^2 + z^2 + \mu] + \mathcal{O}(\epsilon^2)$$

which is a 2-sphere (vanishing 2-cycle if $\mu \to 0$)

- More generally, near any ADE type of singularity, K3 looks locally like a non-compact "ALE space of ADE type":

$$W_{K3} = \epsilon\,[P_{ALE}^{(ADE)}(x,y,z)] + \mathcal{O}(\epsilon^2)$$

For SU(n),

$$P_{ALE}^{(A_{n-1})}(x,y,z) = \underbrace{x^n - \sum_{k=0}^{n} \mu_k\, x^{n-k-2}}_{} + y^2 + z^2 \qquad H_2(K3,\ \mathbb{Z}) \cong \Gamma_{20}$$

"Simple singularity" of type A_{n-1} ;
moduli parameters μ_k~ Casimirs of SU(n)

For SU(n), this looks like n-1 intersecting 2-spheres, whose intersection form is nothing but the A_{n-1} Cartan matrix

- The vanishing 2-cycles of an ALE space generate the corresponding ADE root lattice, and the global embedding of this lattice into the full lattice of all the 2-cycles of K3 mirrors the embdding of a root lattice into the Narain lattice $\Gamma_{20,4}$ on the heterotic side:

$$\begin{array}{lccc} \text{local ...} & H_2(ALE,\ \mathbb{Z}) & \cong & \Gamma_R(ADE) \\ & \cap & & \cap \\ \text{global ...} & H_2(K3,\ \mathbb{Z}) & \cong & \Gamma^{\text{Narain}}_{20,4} \\ & \text{type IIA} & & \text{heterotic} \end{array}$$

Non-perturbative, dual
Type IIA string version
of the heterotic Frenkel-Kac
mechanism ....

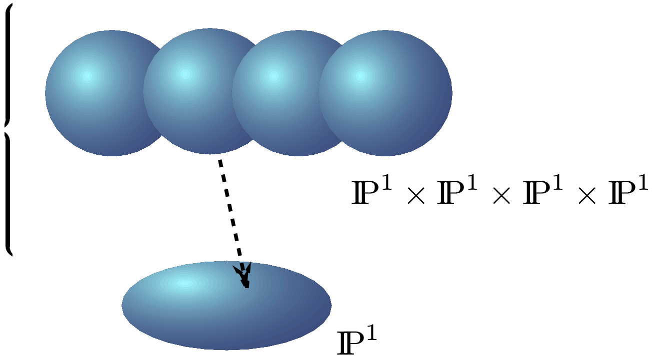
Fibrations of ADE Singularities

- We have seen that the relevant local geometry of the K3 is given (for SU(n)) by n-1 intersecting vanishing 2-spheres

Relevant in 4 dimensions is however not the K3, but a CY 3-fold obtained as fibration of K3 over $\mathbb{P}^1$.

Therefore the relevant local geometry for N=2 quantum Yang-Mills theory is simply a fibration of these 2-spheres over $\mathbb{P}^1$:

the total space
is a non-compact
CY 3-fold



- D2 brane wrappings around the fiber give the perturbative contributions to the eff lagrangian:

$$\mathcal{F} \sim \sum (a \cdot \alpha_i)^2 \log(a \cdot \alpha_i)$$

The non-perturbative corrections in 4d (a la SW) arise from brane wrappings of the base $\mathbb{P}^1$

large $\mathbb{P}^1$ = weak coupling = instanton effects suppressed

- This geometry is in fact exactly the one of the corresponding Seiberg-Witten curve ! [KLMVW]

SW Curves from Local Mirror Symmetry

- In the framework of toric geometry (which is a toolkit especially suited to describe Calabi-Yau manifolds), it is extremely easy to obtain the SW curves directly from the local geometrical data.
- One describes it by the following "mori cone vectors" (KKV):

$$
\begin{array}{rl}
v_b = (1,1,-2,\;\; 0,\;\; 0,\;\; 0,\;\; 0,\ldots,\;\; 0,\;\; 0,\;\; 0,\;\; 0) & \text{base } \mathbb{P}^1 \\
\left.\begin{array}{r}
v_{f_1} = (0,0,\;\; 1,-2,\;\; 1,\;\; 0,\;\; 0,\ldots,\;\; 0,\;\; 0,\;\; 0,\;\; 0) \\
v_{f_2} = (0,0,\;\; 0,\;\; 1,-2,\;\; 1,\;\; 0,\ldots,\;\; 0,\;\; 0,\;\; 0,\;\; 0) \\
v_{f_3} = (0,0,\;\; 0,\;\; 0,\;\; 1,-2,\;\; 1,\ldots,\;\; 0,\;\; 0,\;\; 0,\;\; 0) \\
\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \\
v_{f_n-1} = (0,0,\;\; 0,\;\; 0,\;\; 0,\;\; 0,\;\; 0,\ldots,\;\; 0,\;\; 1,-2,\;\; 1)
\end{array}\right\} & \begin{array}{l}\text{sphere}\\ \text{tree as}\\ \text{fiber}\end{array}
\end{array}
$$

Each vector corresponds to an equation of the form $y_{i-1}y_{i+1} = {y_i}^2$ and encodes the correct intersection/fibration properties of the various $\mathbb{P}^1$

- The whole system is solved by $\{y_i\} = \{z, \frac{1}{z}, 1, x, x^2, \ldots, x^n\}$

This amounts to "local mirror symmetry" of the above local geometry.

That is, the IIB mirror on the IIA string side is given by the algebraic curve $W = \sum a_i y_i = 0$, which can be cast in the form:

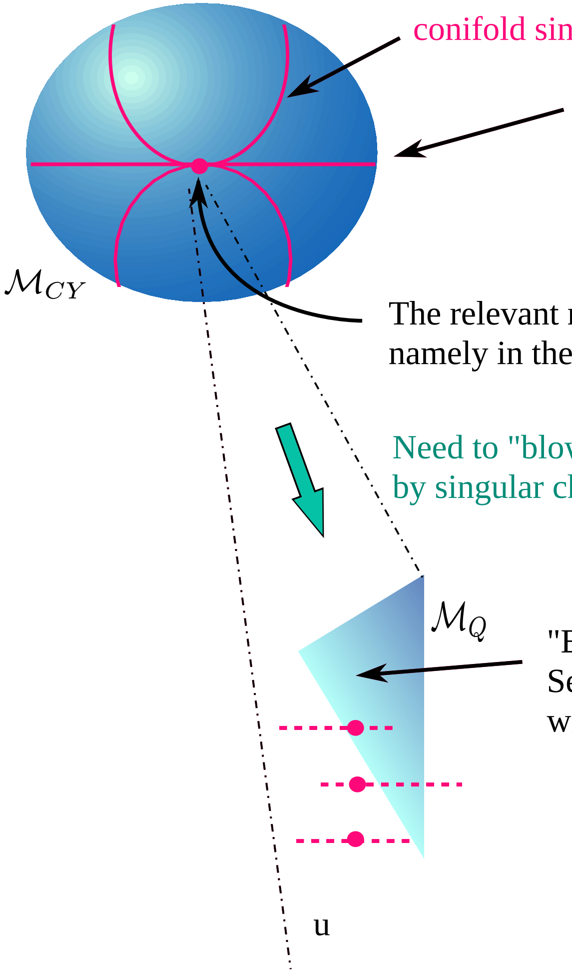
$$
W_{SW} = z + \frac{\Lambda^{2n}}{z} + x^n - \sum_{k=0}^{n} \mu_k \, x^{n-k-2} = 0
$$

This is exactly the Seiberg-Witten curve for the pure SU(n) gauge theory !

(also: periods OK)

Embedding of the SW Moduli Space

The moduli space of the non-compact CY 3-fold looks roughly



conifold singularity (~ SW monopole)

large base space limit
(weak coupling: exp(-S) = 0)

The relevant region is located in one point, namely in the intersection of the singular loci

Need to "blow up" the point of tangency by singular change of variables

"Exceptional Divisor":
Seiberg-Witten moduli space
with singularities at $u = \{\pm\Lambda^2, \infty\}$

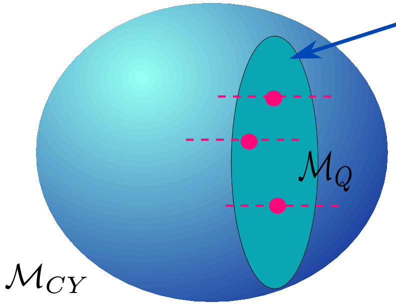
The Seiberg-Witten theory represents a "closed monodromy subproblem":

That is, one can consistently single out 2 periods (a,aD) from the CY periods (X^A, F_A) that do not mix with the other periods under the monodromies in the u-plane, $M_{+\Lambda^2}, M_{+\Lambda^2}, M_\infty$

Other Globally Consistent Sub-Theories ?

Conjecture: all possible consistent "rigid" truncations of a string compactification are given by closed sub-monodromy problems, that allow to single out a subset of the periods.

Any such theory appears as consistent as the SW theory itself, since it is on the same footing.



Any such closed subproblem **defines** a canonical physical, non-perturbatively consistent sub-theory !

However, this does not necessarily always give a QFT that we already know, like SYM theory with matter. There can be novel things like non-critical strings or membranes..... !

Clearly such closed subproblems should always be associated with a distinguished geometrical structure.

Consider here:
$\mathcal{M}_Q$ ~ size of a "del Pezzo" 4-cycle B_n in a Calabi-Yau

del Pezzo types B_n = ($\mathbb{P}^2$ blown up at n points;
n=0,1,....8)

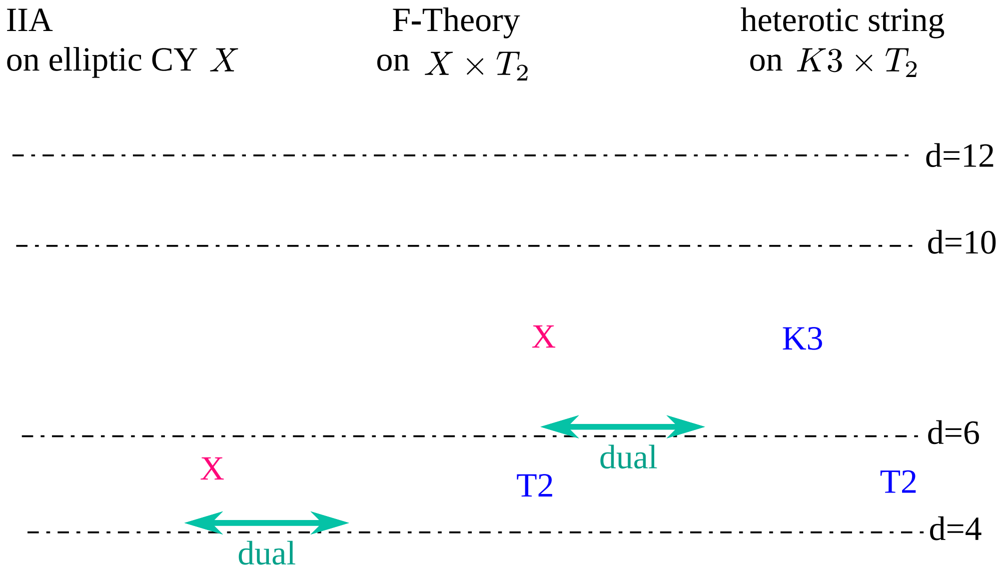
..... have canonical En symmetry

The Physics of Vanishing 4-cycles ?

- Related to still another kind of non-critical strings in 6 dimensions:

Before we had (0,2) susy of strings wrapping around SW curves: "Type II"
Here we have chiral (0,1) non-crit. strings with E8 symmetry: "Heterotic"
$\left(\text{“}\sqrt{E_8 \times E_8 \text{ het. String}}\text{”} \right)$

- **Scheme:**



IIA on elliptic CY X

F-Theory on $X \times T_2$

heterotic string on $K3 \times T_2$

If del-Pezzo 4-cycle B_n vanishes: 5-brane in F-theory leads to tensionless En string in d=6.

- If we consider type IIA string on the same degenerating CY X , we obtain the torus compactification of the En string to d=4.

This gives an effective U(1) N=2 SUSY theory in d=4 , analogous to the Seiberg-Witten theory:

➡ what are its properties, the effective action ?

Intrinsic Formulation (B8)

- Embed 4-cycle B8 into non-compact Calabi-Yau 3-fold:

  mori vector: $v_{B_8} = \{\underbrace{-6,\ 3,\ 2,\ 1,\ 1,}_{B_8}\ \underbrace{-1}_{\text{normal bundle}}\}$

  According to our rules, this immediately gives:

$$W_{B_8} = \underbrace{\frac{1}{w^6}}_{\text{2d gravity}} + \underbrace{y^2 + x^3 + z_1{}^6 + z_2{}^6}_{B_8\ \text{singularity}} - \psi\,(xywz_1z_2)$$

- This non-compact 3-fold is the IIB local mirror of the B8 4-cycle in the embedding Calabi-Yau.

  It contains all relevant information in its periods.

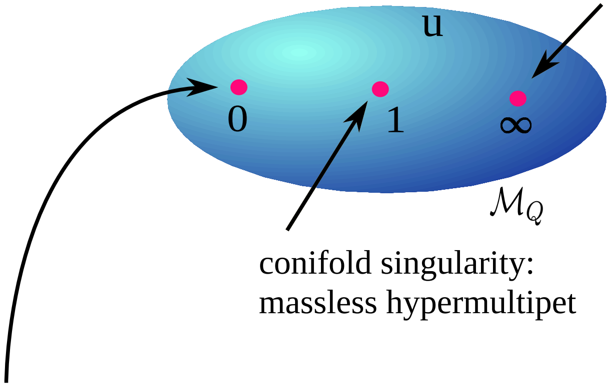
- The Euler number is $\chi = 2\,h_{E_8}$, $h_{E_8} = 30$ (Coxeter number)

  Similar for the other Bn~ En; these are sub-cases

- This is actually a 1-modulus sub-theory of a larger theory ; in principle, there are 8 more moduli (E8 Wilson lines) that could be switched on to break E8 to a subgroup (G,GMS..)

Properties of the Moduli Space

- $\mathcal{M}_Q$ parametrized by $u \equiv \psi^6 \simeq e^{2\pi i a} + \text{instanton corr}$
- Only 3 independent periods $\Pi = \Big(1, a(u), a_D(u)\Big)$ ~ (0,2,4)-cycles
- Only 3 singularities in moduli space:



large complex structure limit of CY

conifold singularity: massless hypermultipet

show: both 2- and 4-cycles (a and aD) vanish at origin.
conformal point with ∞ many massless excitations,
electric and magnetically charged

- Note: $\mathcal{M}_Q$ is a global slice through the CY moduli space, that includes the large complex structure point. This is in contrast to SW theory that is concentrated in one point, as we have seen.

Properties of the Periods

$$\Pi \;\equiv\; \Big(1, a(u), a_D(u)\Big) \;=\; \int_\Gamma \Omega\,, \qquad\qquad \Omega \;\equiv\; \psi\,\frac{x\,dw\,dy\,dz_1\,dz_2}{W_{B_8}}$$

$\nwarrow$ 3-cycle mirrors of (0,2,4)-cycles

The periods are killed by the Picard-Fuchs differential operator $\mathcal{L}$:

$$\mathcal{L}\cdot\Pi \;=\; 0\,, \qquad\qquad \mathcal{L} \equiv \mathcal{L}_{E_8}\cdot\Theta \qquad \Theta \equiv u\frac{\partial}{\partial u}$$

which contains the hypergeometric operator

$$\mathcal{L}_{E_8} \;\equiv\; \Theta^2 - 12u(6\Theta+5)(6\Theta+1)$$

This just happens to be the PF operator for the elliptic curve

$$\boxed{P_{\tilde{E}_8} \;=\; y^2 + x^3 + z^6 - \psi\,xyz \;=\; 0}$$

(" $\tilde{E}_8$ elliptic singularity".....)

$$\mathcal{L}_{E_8}\cdot\Theta\cdot\Pi \;=\; 0$$

So this means that $\Theta\cdot\Pi \;\equiv u\dfrac{\partial}{\partial u}(1, a(u), a_D(u)) \;=\; (0, \underbrace{\varpi, \varpi_D}_{\text{Ordinary torus periods}})$

This is very similar to the usual SW periods that obey

$$\frac{\partial}{\partial u}(a(u), a_D(u)) \;=\; (\varpi, \varpi_D)$$

Suggests to relate the del Pezzo periods Π to SW type periods of the above torus; for this, need to find a suitable meromorphic differential λ.

Geometric Reduction to Elliptic Curve

$$\Pi \;\equiv\; \Big(1, a(u), a_D(u)\Big) \;=\; \int_\Gamma \Omega\,, \qquad\qquad \Omega \;\equiv\; \psi\,\frac{x\,dw\,dy\,dz_1\,dz_2}{W_{B_8}}$$

Evaluate explicitly the 3-cycle integrals (technical):

$$\Pi \;\equiv\; \int_\Gamma \Omega \quad\longrightarrow\quad \int_{\alpha,\beta}\lambda \qquad \text{where}$$

$$\lambda \;=\; \frac{1}{2}\log\left[\frac{\sqrt{1+x^3+\frac{1}{4}\psi^2x^2}+\frac{1}{2}\psi\,x}{\sqrt{1+x^3+\frac{1}{4}\psi^2x^2}-\frac{1}{2}\psi\,x}\right]\frac{dx}{x}$$

Can now explicitly obtain the periods $(1, a, a_D)$ in terms of hypergeometric functions:

$$(1, a, a_D) \;=\; \int\frac{du}{u}(\varpi, \varpi_D) + \text{const.} \qquad\qquad \longleftarrow \text{ from } \int\lambda$$

where the ordinary torus periods are

$$\varpi_D(u) \;=\; \frac{3^{1/4}}{4\pi^{3/2}i}\Big(\xi F_0(u) + \frac{1}{\xi}F_1(u)\Big)$$

$$\varpi(u) \;=\; \frac{3^{1/4}}{4\pi^{3/2}i}\;\Big(\rho\xi F_0(u) + \frac{1}{\rho\xi}F_1(u)\Big)\;,$$

$$\text{with}\quad \xi \;\equiv\; -\frac{i\,3^{1/4}}{2^{2/3}\,\pi^{3/2}}\Gamma(1/3)^3\;, \quad \rho \equiv e^{2\pi i/3}$$

$$\text{and}\quad F_0(u) \;=\; u^{1/6}{}_2F_1\Big(\frac{1}{6},\frac{1}{6},\frac{1}{3};u\Big)\;,$$

$$F_1(u) \;=\; u^{5/6}{}_2F_1\Big(\frac{5}{6},\frac{5}{6},\frac{5}{3};u\Big)\;.$$

What we have achieved is a formulation of the del Pezzo theory in terms of an "ordinary" Seiberg-Witten curve, with a however very complicated differential.

Effective Actions near the Singularities

Just like for ordinary SW theory, we can now write down an effective action for each local coordinate patch, centered at any given singularity:

$$\left(\text{prepotential } \mathcal{F}(a) \equiv \int da\, a_D(a)\right)$$

- u=0: conformal point

$$\mathcal{F}_0 \;=\; \frac{1}{2}\tau_0\, a^2 + \mathcal{O}(a^6)\ , \qquad\qquad \tau_0 = e^{2pii/3}$$

no logarithm: vanishing beta-function

- u=1: conifold point

$$a_D(u) \simeq \text{const} + \mathcal{O}(u-1)\ , \quad a(u) \simeq (u-1)\log[u-1] + \mathcal{O}(u-1)$$

just like for ordinary SW theory at the monopole point

- u=∞ : large complex structure limit

$$\Pi \;\equiv\; \Big(1, a(u), a_D(u)\Big) \;\simeq\; \Big(1, \log[u], \log[u]^2\Big)$$

$$\mathcal{F}_\infty \;=\; \frac{1}{6}a^3 + \frac{1}{4}a^2 - \frac{5}{12}a + \text{const} + \frac{1}{(2\pi i)^3}\underbrace{\sum_{\ell=1}^{\infty} n_\ell\, \mathrm{Li}_3(e^{2\pi i \ell a})}$$

Familiar world-sheet instanton expansion (nl=252,-9252,.......)

This is very different as compared to the usual rigid SW gauge theory, where

$$\mathcal{F}_\infty \;\simeq\; a^2\log[a^2] + \sum c_\ell \left(\frac{1}{a}\right)^{4\ell}$$

It is intrinsically stringy, but we have here a rigid theory where gravity is already switched off !

Physical Interpretation / Conclusion

- We have hit in the CY moduli space a sub-space that forms a closed monodromy sub-problem and that thus **defines** a consistent physical sub-theory. What is the physics of this theory ?

- This rigid sub-theory displays intrinsic stringy behavior, in terms of world-sheet instantons near infinity.
  The constant period can be interpreted in terms of KK excitations.

  Thus this sub-theory indeed behaves much like a compactified d=6 non-critical E8-type of string !

- The correct identification of the modulus is

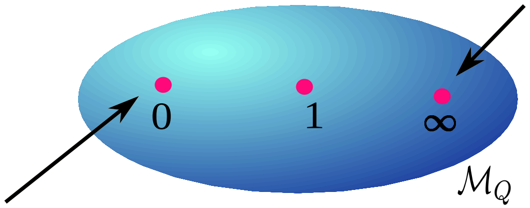
$$a \;\simeq\; B + iR_5R_6\Phi$$

non-crit. string tension

T2 radii

But the compactification radii can be absorbed in an "effective" string tension, and there are no net geometrical parameters to vary !

- It might be that the moduli space we found is intrinsically related to the d=6 non-crit string itself, and not necessarily to its 4d compactification ... ?



large complex structure limit: stringy expansion

$$\mathcal{F}_\infty \;\simeq\; \sum c_\ell\, e^{2\pi i \ell a}$$

Conformal point:
infinitely many massless excitations