Open String TFT on the Elliptic Curve

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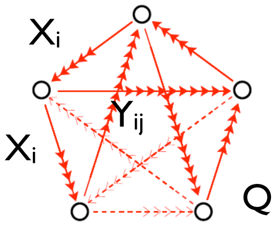
hep-th/0512208, with S. Govindarajan, H.Jockers, N.Warner
hep-th/0603085, with M. Herbst, D. Nemeschansky

Prior important work by:
Kontsevich, Kapustin/Li, Zaslow/Polishchuk

Overview

- Why is this interesting to study ?
  ...completely solvable toy model
- Non-trivial for intersecting brane configurations
- Playground for studying "homological mirror symmetry" between categories of A- and B-type of topological D-branes
- Practical application: compute correlation functions, effective superpotential including world-sheet instanton corrections



Quiver diagram

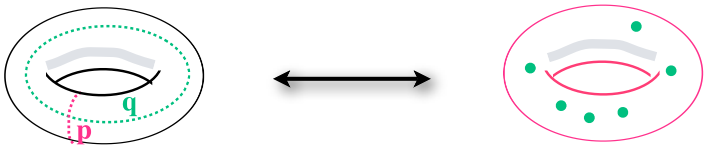
$$\mathcal{W}_{eff} = C(t)\,\mathrm{Tr}XXY + B(t)\,\mathrm{Tr}(XXY)^2 + \ldots$$

$$\sim \mathrm{const} + \mathcal{O}(e^{-t})$$

Homological mirror symmetry

[Kontsevich]

A-Model B-Model



D1 branes on (p,q) cycles (N2,N0) = (r,c_1) of "gauge bundle"

... Fukaya category of lagrangian cycles

... derived category of coherent sheaves

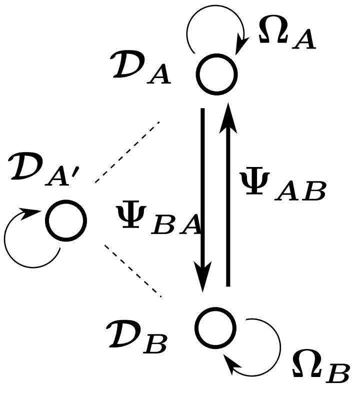
- There is much more to categories than just some diagrams with dots and arrows:

Besides RR charges (p,q)=(r,c_1), it also keeps information about the positions u of the branes:
(irreducible) bundles on T2 are fully characterized by (r,c_1; u)

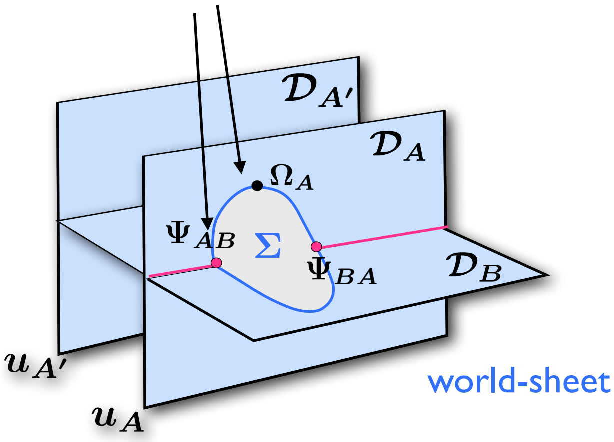
These open string "moduli" are an important ingredient in $\mathcal{W}_{eff}(\tau, u)$

The category of topological D-branes

- objects: $\mathcal{D}$ ⟷ boundary conditions, D-branes
- morphisms (maps): Ω, Ψ ⟷ boundary preserving/changing open string vertex operators

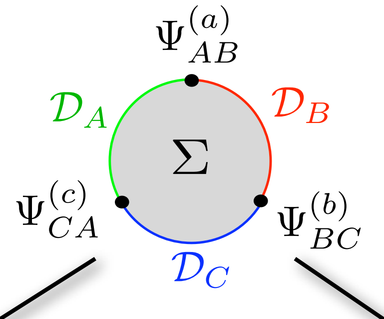


Quiver diagram



TFT correlation functions

Disk amplitude for intersecting branes $C_{abc}(\tau; u_1, u_2, u_3) = \langle \Psi^{(a)}_{AB} \Psi^{(b)}_{BC} \Psi^{(c)}_{CA} \rangle$

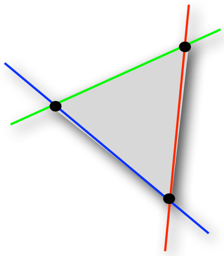


A-Model

localizes on holomorphic maps:
world-sheet instantons $\Sigma \longrightarrow T_2$

B-Model

localizes on constant maps:
classical only



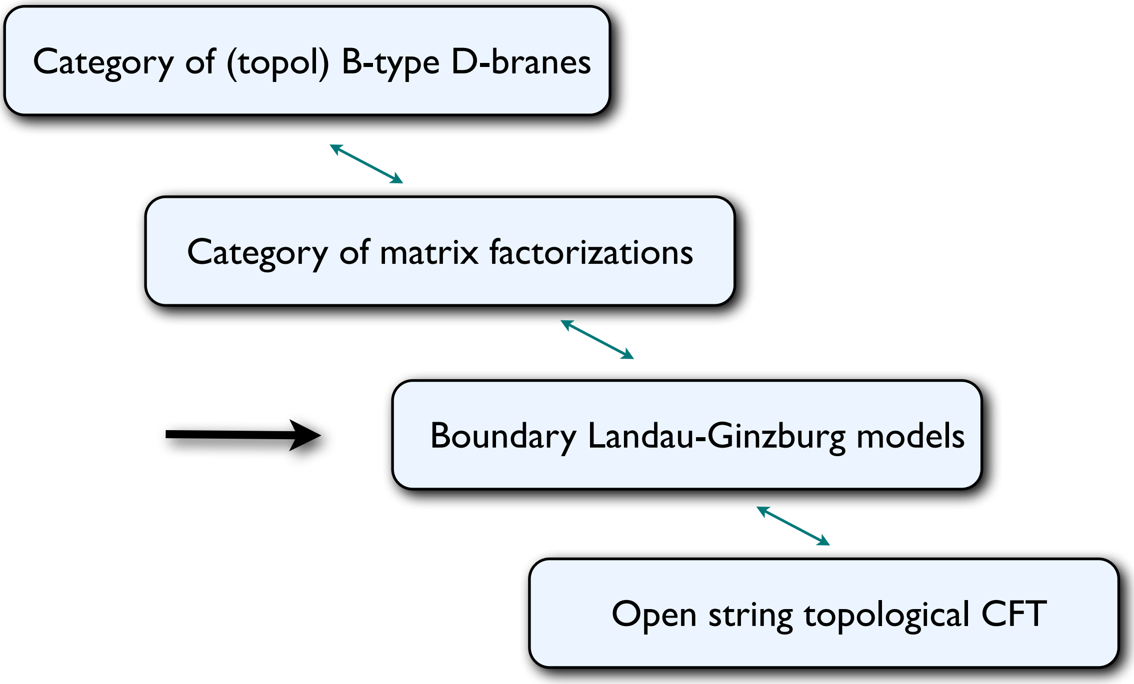
“Fukaya products” ◀┈┈┈▶ “Massey products”

$C_{abc} \sim e^{-S_{\mathrm{Inst}}}$

$S_{\mathrm{Inst.}} \sim \mathbf{Area}$

mirror symmetry

The topological B-Model



Category of (topol) B-type D-branes

Category of matrix factorizations

Boundary Landau-Ginzburg models

Open string topological CFT

Landau-Ginzburg description of B-type D-branes

[K,O,K-L,B-H-L-S]

- Consider bulk LG model with superpotential:

$$\int_\Sigma d^2z d\theta^+ d\theta^-\, W_{LG}(\Phi) + \text{cc.}$$

B-type SUSY variations induce boundary (“Warner”)-term:

$$\int_\Sigma d^2z d\theta^+ d\theta^- (\bar{Q}_+ + \bar{Q}_-)\, W_{LG} = \int_\Sigma d^2z d\theta^+ d\theta^- (\theta^+\partial_+ + \theta^-\partial_-) W_{LG}$$
$$= \int_{\partial\Sigma} dx d\theta\, W_{LG}$$

- Restore SUSY by adding boundary fermions $\Pi = (\pi + \theta^+ \ell)$

(... not quite chiral: $\bar{D}\,\Pi \;=\; E(\Phi)|_{\partial\Sigma}$)

via a boundary potential: $\delta S \;=\; \int_{\partial\Sigma} dx d\theta\, \Pi\, J(\Phi)$

Condition for SUSY: $\boxed{J(\Phi)E(\Phi) \;=\; W_{LG}(\Phi)}$

Matrix factorizations

- Generalization for n LG fields: need N=2^n boundary fermions, and

$$J_{N\times N} \cdot E_{N\times N} \;=\; E_{N\times N} \cdot J_{N\times N} = \; W_{LG}\, 1_{N\times N}$$

- Physical interpretation: N... Chan-Paton labels of space-filling D$\overline{\text{D}}$ pairs

  Boundary potentials J,E are tachyon profiles that describe condensation to given B-type D-brane configuration [Kapustin-Li, Lazaroiu]

- Physical open string spectrum: determined by the cohomolgy of the BRST operator:

$$\mathcal{Q} \;=\; \bar{\partial} + Q_\partial$$
$$Q_\partial \;=\; \pi\, J + \bar{\pi}\, E \;=\; \begin{pmatrix} & J \\ E & \end{pmatrix}$$
$$1/2\, Q_\partial \cdot Q_\partial \;=\; W_{LG}\, 1_{2N\times 2N}$$

Kontsevich's triangulated category C_W

The LG model provides a concrete physical realization of a certain Z_2-graded "twisted" category C_W : all quantities have explicit LG representatives

- objects: composites out of $D\overline{D}$ pairs:

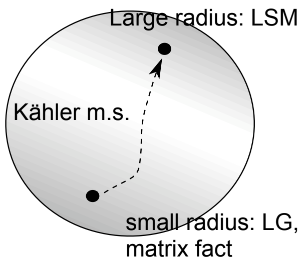
$$M_A \cong \left(P_1^{(A)} \underset{E^{(A)}}{\overset{J^{(A)}}{\rightleftarrows}} P_0^{(A)} \right), \qquad J^{(A)} E^{(A)} = W$$

- morphisms (boundary Q-cohomology):

$$\begin{tikzcd}
M_A \arrow[dd] & & P_1^{(A)} \arrow[r, shift left, "J^{(A)}"] \arrow[dd, "\phi_\alpha^{A,B}"'] \arrow[ddr, dotted, "\psi_\alpha^{A,B}"] & P_0^{(A)} \arrow[l, shift left, "E^{(A)}"] \arrow[dd, "\phi_\alpha^{A,B}"] \arrow[ddl, dotted, "\psi_\alpha^{A,B}"'] \\
& \cong & & \\
M_B & & P_1^{(B)} \arrow[r, shift left, "J^{(B)}"] & P_0^{(B)} \arrow[l, shift left, "E^{(B)}"]
\end{tikzcd}$$

Large radius linear sigma model

[Herbst/Hori/Page, in prep]



"fold out" acyclic complex:

$$\left(\cdots \quad P_1^{(A)} \xrightarrow{J^{(A)}} P_0^{(A)} \xrightarrow{E^{(A)}} P_1^{(A)} \xrightarrow{J^{(A)}} P_0^{(A)} \xrightarrow{E^{(A)}} \cdots \right)$$

Category of Matrix factorizations is isomorphic to D(Coh(M)) = category of B-type D-branes. [Orlov]

Maps J,E: sections of certain bundles [GHLW]

Ker J, Ker E encode bundle data of branes: $(r, c_{1,\ldots}; u)$

Bound State formation via tachyon condensation

- Boundary changing tachyon profile:

$$J_{AB}(u_A, u_B, T) \;=\; \begin{pmatrix} J_A(u_A) & T\Psi_{AB}(u_A, u_B) \\ 0 & J_B(u_B) \end{pmatrix}$$

For non-zero tachyon field T, this corresponds to a new
matrix factorization, describing a ``bound state”
(non-trivial bundle extension)

- Boundary RG flow: physical realization of the “cone” construction:

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"', "w"] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

triangle: $M_A \xrightarrow{\Psi_{AB}} M_B \longrightarrow C \longrightarrow M_A[1]$

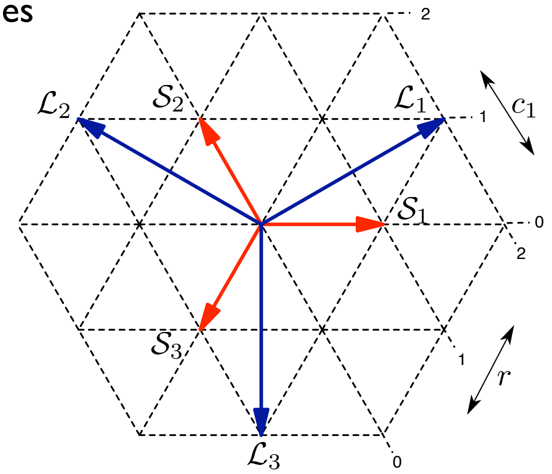
cone: $C = \left(P_1^{(A)} \oplus P_0^{(B)} \underset{E_{AB}}{\overset{J_{AB}}{\rightleftarrows}} P_0^{(A)} \oplus P_1^{(B)} \right)$

D-branes on the elliptic curve, B-Model

- Simplest Calabi-Yau: the cubic curve

$$T_2: \quad W_{LG} \;\equiv\; {x_1}^3 + {x_2}^3 + {x_3}^3 + a\, x_1 x_2 x_3 \;=\; 0$$

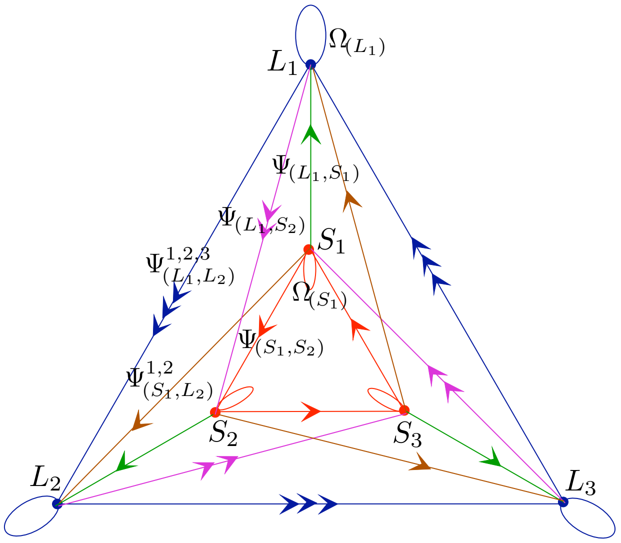
- Charges of simplest branes



The “short diagonals” S are related to 2x2 factorizations, while the “long diagonals” L are described by 3x3 (4x4) factorizations

D-branes on the elliptic curve,   B-Model

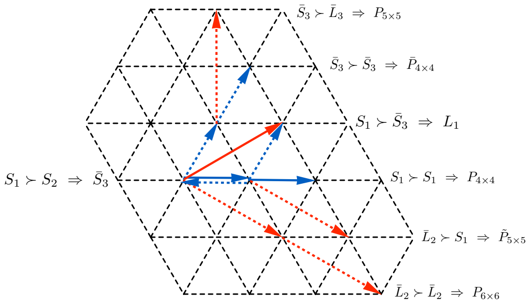
- Quiver diagram of open string spectrum



Generating the charge lattice

- One can systematically apply the cone construction, and generate matrix factorization corresponding to branes with arbitrary RR charges (rank(V),c₁(V)) as composites out from a generating set

[GHLW]

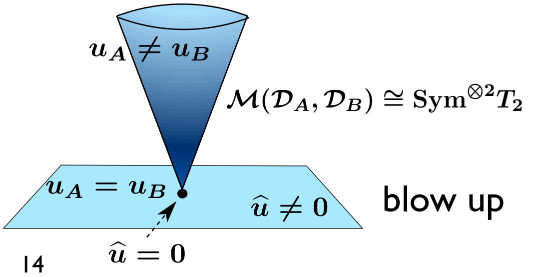


$$M_A \xrightarrow{\Psi^{(C)}_{AB}} M_B \longrightarrow C \longrightarrow M_A[1]$$

particular choice of tachyon determines bound state

- There is more to it than just adding RR charges, due to the moduli dependence of the matrices...   interesting phenomena e.g.,

bound states at threshold



blow up

3x3 matrix factorization

- Simplest are the factorizations corresponding to the long diagonals L$_i$ (i=1,2,3) [HW]

$$J_i \;=\; \begin{pmatrix} \alpha_1^{(i)}x_1 & \alpha_2^{(i)}x_3 & \alpha_3^{(i)}x_2 \\ \alpha_3^{(i)}x_3 & \alpha_1^{(i)}x_2 & \alpha_2^{(i)}x_1 \\ \alpha_2^{(i)}x_2 & \alpha_3^{(i)}x_1 & \alpha_1^{(i)}x_3 \end{pmatrix}$$

$$E_i \;=\; \begin{pmatrix} \frac{1}{\alpha_1^{(i)}}{x_1}^2-\frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}}x_2x_3 & \frac{1}{\alpha_3^{(i)}}{x_3}^2-\frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}}x_1x_2 & \frac{1}{\alpha_2^{(i)}}{x_2}^2-\frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}}x_1x_3 \\ \frac{1}{\alpha_2^{(i)}}{x_3}^2-\frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}}x_1x_2 & \frac{1}{\alpha_1^{(i)}}{x_2}^2-\frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}}x_1x_3 & \frac{1}{\alpha_3^{(i)}}{x_1}^2-\frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}}x_2x_3 \\ \frac{1}{\alpha_3^{(i)}}{x_2}^2-\frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}}x_1x_3 & \frac{1}{\alpha_2^{(i)}}{x_1}^2-\frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}}x_2x_3 & \frac{1}{\alpha_1^{(i)}}{x_3}^2-\frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}}x_1x_2 \end{pmatrix}$$

These satisfy $J_iE_i = E_iJ_i \;=\; W_{LG}\mathbf{1}$
if the parameters satisfy the cubic equation themselves:

$$W_{LG}(\alpha_i) \;\equiv\; {\alpha_1}^3+{\alpha_2}^3+{\alpha_3}^3+a(\tau)\,\alpha_1\alpha_2\alpha_3 \;=\; 0$$

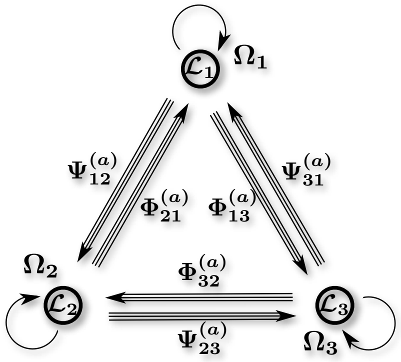
Thus the parameters parametrize the (jacobian) torus and can be represented by theta-sections:

$$\alpha_\ell^{(i)} \sim \Theta\left[\tfrac{1-\ell}{3} - \tfrac{1}{2} - \tfrac{1}{2}\,\middle|\, 3u_i, 3\tau\right]$$

$u,\ \tau$ ...flat coordinates of open/closed moduli space (natural in mirror A-model)

Open string BRST cohomology

Solving for the BRST cohomology yields explicit t,u-moduli dependent, "flat" matrix valued maps, eg (a=1,2,3):



- q=1 marginal operators corr. to brane locations

$$\mathrm{Ext}(\mathcal{L}_A,\mathcal{L}_A):\quad \Omega_A \;=\; \partial_{u_A}Q(u_A)$$

- q=1/3 tachyon operators

$$\mathrm{Ext}(\mathcal{L}_A,\mathcal{L}_B):\quad \Psi_{AB}^{(a)} \;=\; \begin{pmatrix} 0 & F_{AB}^{(a)} \\ G_{AB}^{(a)} & 0 \end{pmatrix}$$

with eg, $F_{12}^{(1)} = \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & \zeta_2 \\ 0 & \zeta_3 & 0 \end{pmatrix}$ $G_{12}^{(1)} = \begin{pmatrix} \frac{\zeta_1}{\alpha_1^{(1)}\alpha_1^{(2)}}x_1 & \frac{\zeta_3}{\alpha_1^{(1)}\alpha_2^{(2)}}x_2 & \frac{\zeta_2}{\alpha_1^{(1)}\alpha_3^{(2)}}x_3 \\ \frac{\zeta_2}{\alpha_1^{(2)}\alpha_3^{(1)}}x_2 & \frac{\zeta_1}{\alpha_2^{(2)}\alpha_3^{(1)}}x_3 & \frac{\zeta_3}{\alpha_3^{(1)}\alpha_3^{(2)}}x_1 \\ \frac{\zeta_3}{\alpha_1^{(2)}\alpha_2^{(1)}}x_3 & \frac{\zeta_2}{\alpha_2^{(1)}\alpha_2^{(2)}}x_1 & \frac{\zeta_1}{\alpha_2^{(1)}\alpha_3^{(2)}}x_2 \end{pmatrix}$

and $\zeta_\ell \sim \Theta\left[\tfrac{1-\ell}{3} - \tfrac{1}{2} - \tfrac{1}{2}\,\middle|\, 3u_2 - 3u_1, 3\tau\right]$

Superpotential on brane intersection

- Compute 3-point disk correlators = Yukawa couplings in LG framework

$$\mathcal{W}_{\text{eff}} \sim C_{abc}(u_i,\tau)\, T_{13}^{(a)} T_{32}^{(b)} T_{21}^{(c)} + \dots$$

$$\begin{aligned} C_{abc}(u_1,u_2,u_3) &= \big\langle \Psi_{13}^{(a)}(u_1,u_3)\Psi_{32}^{(b)}(u_3,u_2)\Psi_{21}^{(c)}(u_2,u_1)\big\rangle \\ &= \frac{1}{2\pi i}\oint \mathrm{Str}\Big[(\frac{dQ}{dW})^{\otimes\wedge 3}\Psi_{13}^{(a)}\Psi_{32}^{(b)}\Psi_{21}^{(c)}\Big] \end{aligned}$$

- Final result: theta functions

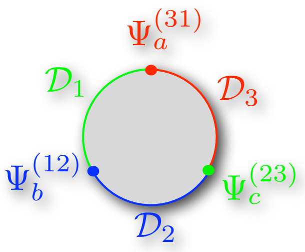
$$\begin{aligned} C_{111}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2} q^{3{\xi_2}^2/2} \sum_m q^{3m^2/2} e^{6\pi i m\xi} \\ C_{123}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2} q^{3{\xi_2}^2/2} \sum_m q^{3(m+1/3)^2/2} e^{6\pi i(m+1/3)\xi} \\ C_{132}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2} q^{3{\xi_2}^2/2} \sum_m q^{3(m-1/3)^2/2} e^{6\pi i(m-1/3)\xi} \end{aligned}$$

$(\xi \equiv u_1 + u_2 + u_3 = \xi_1 + \tau\xi_2)$

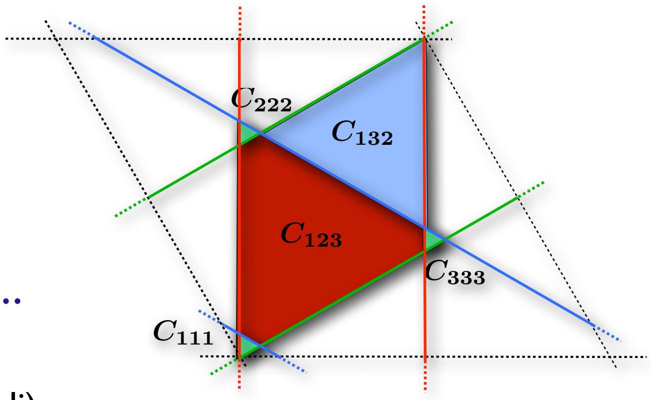
(Polishchuk,Cremades et al)

The topological A-Model: instantons

- Interpretation of q-series: In A model mirror language, these are contributions from triangular disk instantons whose world-sheets are bounded by the three D1-branes:



$$\Sigma \longrightarrow T_2$$



$$C_{abc} \sim e^{-S_{\text{inst}}} \sim q^{\Delta_{abc}} + \dots..$$

(the u-dependence corresponds
to position and Wilson line moduli)

- B-model: unclear how to compute higher N-point functions with N>3!

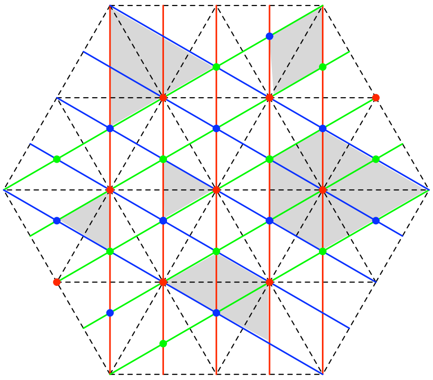
Complete effective potential (long diag branes)

$$\mathcal{W}_{\rm eff}(\tau,u,T) \;=\; \sum_{N=1}^{6} T^{(a_1)}\underset{\uparrow}{\ldots} T^{(a_N)}\, C^{(N)}_{a_1,\ldots,a_N}(\tau,u_1,\underset{\uparrow}{\ldots},u_N)$$

topol. tachyons

moduli

- Generically, N-point functions get contributions from N-gonal instantons
- General structure:
  indefinite theta-functions summing over all lattice translates, positive areas



$$\sum_{m,n}{}' q^{mn} \equiv \left(\sum_{m,n>0} - \sum_{m,n<0}\right) q^{mn}$$

Polygons and instantons

N=4: trapezoids

$$\mathcal{T}_{ab\bar{c}\bar{d}}(\tau,u_i) \;=\; \delta^{(3)}_{a+b,\bar{c}+\bar{d}}\,\Theta_{trap}\begin{bmatrix}[b-\bar{c}]_3\\ [\bar{d}-\bar{c}+3/2]_3\end{bmatrix}(3\tau|3(u_1+u_2+u_4),3(u_1-u_3))$$

$$\Theta_{trap}\begin{bmatrix}a\\ b\end{bmatrix}(3\tau|3u,3v) \;=\; \sum_{m,n}{}' q^{\frac{1}{6}(a+3n)(a+3n+2(b+3m))}e^{2\pi i\left((a+3n)(u-1/6)+(b+3m)v\right)}$$

N=4: parallelograms

$$\mathcal{P}_{a\bar{b}c\bar{d}}(\tau,u_i) \;=\; \delta^{(3)}_{a+c,\bar{b}+\bar{d}}\,\Theta_{para}\begin{bmatrix}[c-\bar{b}]_3\\ [\bar{d}-c]_3\end{bmatrix}(3\tau|3(u_1-u_3),3(u_4-u_2))$$

$$\Theta_{para}\begin{bmatrix}a\\ b\end{bmatrix}(3\tau|3u,3v) \equiv \sum_{m,n}{}' q^{\frac{1}{3}(a+3n)(b+3m)}e^{2\pi i\left((b+3m)u+(a+3n)v\right)}$$

N=5: pentagons

$$\wp_{a\bar{b}\bar{c}\bar{d}\bar{e}}(\tau,u_i) = \delta^{(3)}_{a,\bar{b}+\bar{c}+\bar{d}+\bar{e}}\,\Theta_{penta}\begin{bmatrix}[-b-c-d]_3\\ [e+c+d]_3\\ [c-d+\frac{3}{2}]_3\end{bmatrix}(3\tau|3(u_5-u_2),3(u_1-u_4),3(u_3+u_2+u_4))$$

$$\Theta_{penta}\begin{bmatrix}a\\ b\\ c\end{bmatrix}(3\tau|3u,3v,3w) \equiv \sum_{m,n,k}{}' q^{\frac{1}{3}(a_>+3(n+k))(b_>+3(m+k))-\frac{1}{6}(c+3k)^2}e^{2\pi i\left((a_>+3(n+k))u+(b_>+3(m+k))v+(c+3k)(w-1/6)\right)}$$

N=6: hexagons

$$\mathcal{H}_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}}(\tau,u_i) = \delta^{(3)}_{0,\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}}\Theta_{hexa}\begin{bmatrix}[-b-c-d]_3\\ [c+d+e]_3\\ [c-d+\frac{3}{2}]_3\\ [a-f+\frac{3}{2}]_3\end{bmatrix}(3\tau|3(u_5-u_2),3(u_1-u_4),3(u_3+u_2+u_4),3(-u_6-u_1-u_5))$$

$$\Theta_{hexa}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}(3\tau|3u,3v,3w,3z) \equiv \sum_{m,n,k,l}{}' q^{\frac{1}{3}(a+3n)(b+3m)-\frac{1}{6}(c+3k)^2-\frac{1}{6}(d+3l)^2}e^{2\pi i\left((a+3n)u+(b+3m)v+(c+3k)(w-1/6)+(d+3l)(z+1/6)\right)}$$

$$\sum_{m,n,k,l}{}' = \sum_{m,n\geq 0}^{\infty}\sum_{k\geq 0}^{<k_{max}}\sum_{l\geq 0}^{<l_{max}} - \sum_{m,n\leq -1}^{-\infty}\sum_{k\leq -1}^{>k_{min}}\sum_{l\leq -1}^{>l_{min}}$$

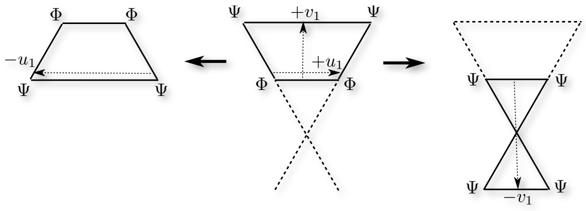
Global properties of open string moduli space

- Indefinite theta-fcts: singularities due to colliding branes

  eg., rewrite trapezoidal function in terms of Appel function:

$$\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau|3u,3v) \;=\; e^{2\pi i v b}\sum_{n\in Z}\frac{q^{\frac{1}{6}(a+3n)(a+2b+3n)}e^{2\pi i(a+3n)(u-1/6)}}{1-q^{a+3n}e^{6\pi i v}}$$

- analytic continuation



Area becomes negative:
resum instantons in terms
of different geometry

“instanton flop”

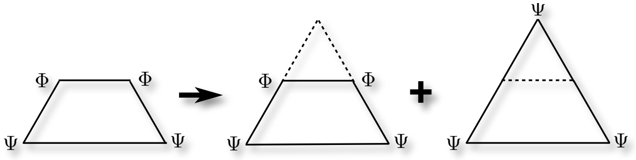
Global properties of open string moduli space

- monodromy

$$\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3(u\pm\tau),\,3v) \;=\; e^{\mp 6\pi i v}\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3u,\,3v)$$

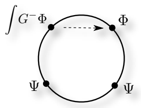
$$\mp\, e^{-2\pi i(u-\frac{1}{6})(b-\frac{3}{2}\pm\frac{3}{2})}e^{2\pi i v(b-\frac{3}{2}\mp\frac{3}{2})}q^{-\frac{1}{6}(b-\frac{3}{2}\pm\frac{3}{2})^2}\Theta\left[\begin{array}{c} a+b \\ -3/2 \end{array}\right](3\tau\,|\,3u)$$

induces “homotopy transformation”,
modular anomaly of eff action
(compensate by non-lin field redef)



$$\mathcal{T}_{ab\bar{c}\bar{d}} \;\rightarrow\; \mathcal{T}_{ab\bar{c}\bar{d}} + f_{\bar{c}\bar{d}}{}^{e}\Delta_{abe}$$

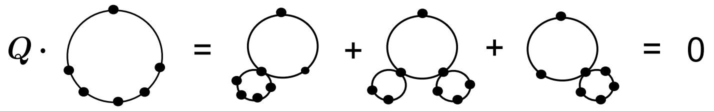
contact term



Open/closed top. string consistency conditions

- How can we be sure that these expressions are correct ?

  Make use of Q-closedness and factorization constraints



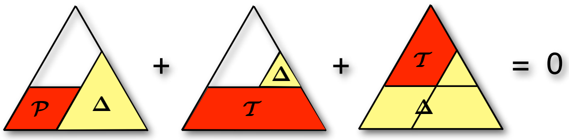
These lead to “A∞ relations” for correlators

$$\sum_{\substack{k,j=0\\k\leq j}}^{m}(-1)^{\tilde a_1+\ldots+\tilde a_k} r_{m-j+k}(\psi_{a_1}\ldots\psi_{a_k}, r_{j-k}(\psi_{a_{k+1}}\ldots\psi_{a_j}), \psi_{a_{j+1}}\ldots\psi_{a_m}) = 0$$

$$r_m(\Psi_{a_1}\ldots\Psi_{a_m}) \;\equiv \Psi_{a_0} C^{a_0}_{a_1\ldots a_m}$$

....here: simple interpretation in terms of instanton geometry:

$$\sum \mathcal{P}_{a_1\bar c a_4 \bar a_5}\;\Delta_{c a_2 a_3} + \sum \mathcal{T}_{a_1 a_2 \bar c \bar a_5}\;\Delta_{c a_3 a_4} + \sum \Delta_{a_1 a_2 c}\;\mathcal{T}_{\bar c a_3 a_4 \bar a_5} = 0$$

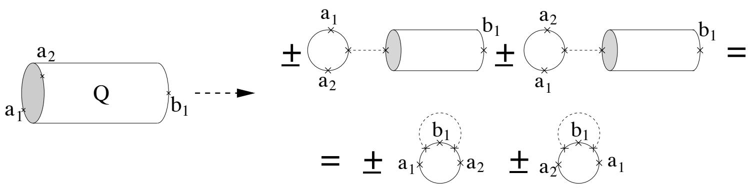


(compatible with homotopy transf)

Quantum A∞ relations for the annulus

[Herbst]

- There are analogous factorization relations in higher genus, eg:



$$\sum_{c,d}((-)^{\tilde a_1+\tilde d\tilde a_2}\;C^{0,1}_{a_1 c a_2}\;\eta^{cd}\;C^{0,2}_{d|\;b_1} + (-)^{\tilde a_1+\tilde a_2}\;C^{0,1}_{a_1 a_2 c}\;\eta^{cd}\;C^{0,2}_{d|\;b_1})$$

$$= \sum_{c,d}((-)^{\tilde a_1+\tilde b_1(\tilde d+\tilde a_2)}\eta^{cd}C^{0,1}_{a_1 c b_1 d a_2} + (-)^{\tilde a_1+\tilde a_2+\tilde b_1\tilde d}\eta^{cd}C^{0,1}_{a_1 a_2 c b_1 d})$$

- In concrete case, it boils down to an identity between disk and and annulus correlators:

$$\partial_{u_3}\mathcal{A}_{\Omega|}\;\cdot\;\equiv\;\partial_{u_3}\sum_{3\,n\neq 0,m}{}' q^{nm}e^{6\pi i n(u_1-u_3)}$$

$$= \sum_{c=1}\partial_{u_3}\mathcal{P}_{a\bar c c\bar a}$$

This maps disk and annulus instantons into each other!