$N = 2$ Supersymmetric Integrable Models from Affine Toda Theories

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We show how a class of $N = 2$ supersymmetric, quantum integrable theories in two dimensional space-time can be obtained by an appropriate choice of coupling in quantum Toda theories. The theories thus constructed correspond to the most relevant, supersymmetric perturbation of the $N = 2$ supersymmetric coset models based on G/H' where G/H' is a hermitian, symmetric space and G is level one. These perturbed conformal theories posses non-trivial conserved currents that can be constructed via a Miura transformation of the bosonic fields of the Toda theory. We show how the spectrum of these Toda theories is related to that of the associated Landau-Ginzburg model.

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1. Introduction.

It has long been known that Toda
example, [1] [2]). These theories have a I

$$\mathcal{L} \;=\; \frac{1}{2}(\partial\phi)^2 \;-$$

where ϕ represents a vector of bosonic fi
the α_j are some particular, constant vect
usually taken to be the simple roots of
simple roots of the affine Lie algebra, $\hat{\mathcal{G}}$, a
general Toda theories that are integrable,
Lie algebras or affine Lie algebras. Some
be associated to Lie algebras divided by
paper we will restrict ourselves to Toda
algebra $\mathcal{G}$ or to some affine Lie algebra $\hat{\mathcal{G}}$.
that a Toda theory will always be associ
ℓ, and we will always refer to their affine

Our primary purpose in this paper is
affine Toda field theories and integrable
mal field theories. It was apparent fron
a relationship between the $\hat{A}_\ell$ Toda the
symmetric, discrete series perturbed by t
Here we will establish this connection dir
a direct connection between Toda theor
metric perturbations of the $N = 2$ supe
hermitian, symmetric space G/H', where
most relevant perturbations of these mo
connection with Toda theories we can ea
of motion. The details of this will be dis

In section 3 we investigate some of
of perturbations of the N=2 discrete se
modular invariant minimal models to af

of motion for the most relevant perturbations of these minimal models. Upon orbifoldization these type D minimal models become type A minimal models, and the most relevant perturbation of the type D theory becomes the next-to-most relevant perturbation of the type A theory. Some of the integrals of motion of the type D theory survive this orbifoldization and so we see that the next-to-most relevant perturbation of the minimal models is also integrable. This result is also confirmed more directly by perturbation theory about the conformal theory.

In a $\mathcal{G}$-Toda theory or an affine $\mathcal{G}$-Toda theory, the infinitely many integrals of motion have spins m_i mod g where $m_i,\ i = 1, \ldots, \ell$ are the exponents of $\mathcal{G}$ and g is the dual Coxeter number of $\mathcal{G}$. These integrals can be constructed in a number of ways. One approach is to start from the Lax pair for the theory and then use the "connection matrices" to generate the conserved quantities and express them in terms of the bosonic fields ϕ [2]. While this approach readily demonstrates a relationship between the spins of the integrals of motion and the degrees of the Casimirs of $\mathcal{G}$, it is not very useful for obtaining explicit expressions for the required conserved quantities. Another way of proceeding is to regard the affine Toda theory as a very special perturbation of a conformal theory [1]. This perturbation has the property that the potential terms in the resulting affine Toda Hamiltonian possess a symmetry between the part corresponding to the unperturbed theory and the part corresponding to the perturbation [4] [5] [2]. All holomorphic fields of the conformal theory commute with the integrated Toda potential; thus all we need to do in order to ensure that any putative charge density is conserved is to require that it respect the symmetry just mentioned. We show in section 4 that the quantum integrals for the supersymmetric theories we consider can be found from appropriate

[1] We have to fix the Toda coupling constant to a particular, purely imaginary value in order to make the correspondence to a perturbed conformal field theory.

[2] The charge at infinity that is needed to make the theory a conformal one actually breaks this symmetry, but this is of no consequence for the structure of the local conserved charge densities.

symmetric combinations of Miura-transformed variables arising from the free bosons describing the underlying Toda theory.

As mentioned above, the $N = 2$ supersymmetric cosets based on G/H' that can be directly related to Toda theories are those for which G is level one and G/H' is hermitian and symmetric. This is also precisely the class of cosets for which there is a known Landau-Ginzburg formulation [6]. As a consequence of this and the results of [3] one might expect some close relationship between Landau-Ginzburg solitons and particles or solitons in Toda theories. In section 5 we discuss some preliminary results on the Landau-Ginzburg soliton structure of our particular perturbed coset models, and we discuss how these solitons can appear in the spectrum of the corresponding Toda therory.

Finally, Section 6 contains some further discussion of the issues raised in this paper.

2. $N = 2$ Supersymmetric Cosets, W-Algebras and Toda Theories.

In [7] it was shown that if G/H' is a Kählerian coset manifold with rank$G =$ rankH', then the usual GKO construction could be extended to obtain an $N = 2$ superconformal theory from the coset theory $\mathcal{M} = \frac{G \times SO(d)}{H'}$. In this coset model, d is the (real) dimension of G/H', and the factor $SO(d)$ represents the bosonized fermions. The group H' is diagonally embedded into G and $SO(d)$, the latter embedding being induced by the H'-action on the tangent space of G/H'. The index of the embedding of H' into $SO(d)$ is $g - h$ where g and h are the dual Coxeter numbers of G and H' respectively [3]. Thus if the current algebra of G has level k then the current algebra of H' has level $k+g-h$. (The group $SO(d)$ represents bosonized fermions and thus always has level one.)

In this paper we wish to consider the $N = 2$ coset models such that G has level one, and G/H' is a hermitian, symmetric space. As a convenient

[3] The groups G and H' will be semi-simple, except possibly for $U(1)$ factors. The dual Coxeter number of $U(1)$ is defined to be zero, and if G or H' has more than one simple or $U(1)$ factor then g, h and $g - h$ are to be viewed as vectors in the obvious manner.

abbreviation we shall refer to such $N=2$ models as *minimal, hermitian* models. We will also assume for simplicity of exposition that G is simple. The fact that G/H' is a symmetric space means that the embedding of H' into $SO(d)$ is conformal [8] and thus representations of the $SO_1(d)$ current algebra are finitely reducible in terms of H'_{g-h} representations. (Subscripts on the label of a group will always denote the level of the current algebra in question). Moreover, because G is level one and G and H' have the same rank, it follows that G_1 representations are finitely reducible as representations of H'_1. (For simply-laced algebras this is a trivial consequence of the vertex operator construction.) Thus, up to questions of finite reducibility of the representation theory, the minimal, hermitian model $\mathcal{M}$ is equivalent to

$$\frac{H'_1 \times H'_{g-h}}{H'_{g-h+1}}. \tag{2.1}$$

The fact that G/H' is Kählerian means that H' contains at least one $U(1)$ factor. This $U(1)$ factor defines the complex structure on G/H'. For hermitian, symmetric spaces with G simple, one finds that $H' = H \times U(1)$, where H is either simple or is the product of at most two simple factors [7][6]. Let ℓ denote the rank of of H and $\partial\phi$ denote the $\ell+1$ component vector of free bosonic currents that generate the Cartan subalgebra of G or, equivalently, of H'. The $U(1)$ factor of H' is defined by $(\rho_G - \rho_H)\cdot\partial\phi$, where, as usual, ρ_A denotes half the sum of the positive roots of a group A.

One can cancel $U(1)$ factors between the numerator and denominator of (2.1) (provided that one rescales the momenta of the remaining $U(1)$ appropriately) and hence the original coset is equivalent to the coset model:

$$\frac{H_1 \times H_{g-h}}{H_{g-h+1}} \times U(1), \tag{2.2}$$

where the $U(1)$ factor now represents the $U(1)$ current of the $N=2$ superconformal algebra. It is elementary to compute the central charge of this theory:

$$c = \frac{3d}{2(g+1)} = 1 + \sum_i \ell_i \Big[1 - \frac{h_i(h_i+1)}{g(g+1)}\Big], \tag{2.3}$$

where the sum over i is over the simple factors $H^{(i)}$ of H, and ℓ_i and h_i are the ranks and dual Coxeter numbers of these factors.

The importance of the foregoing decomposition of the minimal, hermitian models is that the coset models $\frac{H_1^{(i)}\times H_p^{(i)}}{H_{p+1}^{(i)}}$ are intimately related to Toda theories and to W-algebras. Since we will need to employ some of the detailed properties of this relationship we will now briefly review the relevant material. Consider the coset model $\frac{H_1\times H_p}{H_{p+1}}$ for H simple. The representations of such a coset theory are labelled by three highest weights, w_1, w_2 and w_3 of H. These weights label affine H representations of levels $1, p$ and $p+1$ respectively, and correspond to the numerator and denominator factors in $\frac{H_1\times H_p}{H_{p+1}}$. The fact that H_{p+1} is embedded in $H_1\times H_p$ means that one must have $w_1 = w_3 - w_2 + r$ where r is some co-root vector, and since w_1 is a label of a level one algebra it follows that w_1 is uniquely determined by the labels w_2 and w_3. One should also note that there is an action of the center of H that acts trivially upon the coset model, but can be used to change the label w_1 to zero. That is, because of "spectral flow" by the center of H, all representations of this particular coset model can be obtained by merely taking the representation of H_1 to be the one that has a singlet ground state [9] [6]. Making such a choice of w_1 means that w_2 and w_3 must differ by a (co-)root of H.

It is convenient at this juncture to assume that H is simply-laced, and we will comment upon the generalizations later. Consider a state of the coset model labelled by w_1, w_2 and $w_1 = w_2 - w_3 + r$. Then, up to an integer, the conformal dimension of this state is given by [7]:

$$\begin{aligned}\Delta &= \frac{1}{2}(w_2-w_3)^2 + \frac{1}{2(p+h)}\Big[w_2^2 + 2\rho_H\cdot w_2\Big] \\ &\qquad - \frac{1}{2(p+h+1)}\Big[w_3^2 + 2\rho_H\cdot w_3\Big] \\ &= \frac{1}{2}\Big[\Big(\tilde\beta w_3 - \frac{1}{\tilde\beta}w_2\Big)^2 + 2\Big(\tilde\beta - \frac{1}{\tilde\beta}\Big)\rho_H\cdot\Big(\tilde\beta w_3 - \frac{1}{\tilde\beta}w_2\Big)\Big],\end{aligned} \tag{2.4}$$

where

$$\tilde\beta^2 \equiv \frac{p+h}{p+h+1}. \tag{2.5}$$

Note that (2.4) is also *precisely* (*i.e.* not modulo integers) the conformal dimension of the operator

$$exp\Big(-i\Big(\tilde{\beta}w_3 - \frac{1}{\tilde{\beta}}w_2\Big)\cdot\phi(z)\Big) \tag{2.6}$$

in a free bosonic theory whose energy momentum tensor is

$$T(z) \;=\; -\frac{1}{2}\big(\partial\phi(z)\big)^2 + i\Big(\tilde{\beta} - \frac{1}{\tilde{\beta}}\Big)\rho_H\cdot\partial^2\phi. \tag{2.7}$$

This is not a coincidence. It is known, at least when H is simply-laced, that the coset model $H_1\times H_p/H_{p+1}$ can be obtained as a unitary subsector of the Hilbert space of ℓ free bosons [10]. The vertex operators (2.6) then represent primary fields in the physical Hilbert space of the coset model, and the energy momentum tensor of the theory is given by (2.7).[4]

Finally, one can also connect the foregoing with Toda theories with an imaginary coupling constant [11] [12]. Consider the theory whose action on the complex plane is:

$$S_0 \;=\; \frac{1}{2\pi}\int d^2z\,\Big[(\partial\phi)(\overline{\partial}\phi) \;+\; \frac{1}{\tilde{\beta}^2}\sum_{j=1}^{\ell} exp(i\tilde{\beta}\alpha_j\cdot\phi)\Big], \tag{2.8}$$

where ℓ is the rank of H and the α_j are the simple roots of H. After adding improvement terms (which correspond to the non-vanishing background charge) to the generally covariant form of this action, one can argue that this Toda theory corresponds to a conformal field theory with central charge $c=\ell\Big[1-\frac{h(h+1)}{(p+h)(p+h+1)}\Big]$. The energy momentum tensor is given by (2.7), where $\phi=\phi(z,\overline{z})$ is now a Toda field (and is thus *not* free). The physical vertex operators still have the form (2.6) (with $\phi(z)$ replaced by the Toda field $\phi(z,\overline{z})$), and the conformal dimension is still given by equation (2.4), in spite of the fact that ϕ is now a Toda field. More details about the relationship between the Toda theory and the free field theory can be found in [11][12]. The physical interpretation of a theory with non-real action is not clear, but recent work has shown that what we are studying is a unitary projection of the theory [13].

[4] The generalization of this result to the Lie algebra B_n is fairly straightforward. The energy momentum tensor is modified by adding to (2.7) the energy momentum tensor of a Majorana fermion. In the non-spinor representations the vertex operators are identical to their simply-laced counterparts, but in the spinor representations one must append the spin field of the Majorana fermion to the bosonic vertex operators [10].

With these observations we can represent the minimal, hermitian model based on the coset G/H' as a product of an H-Toda theory and a single, free, $U(1)$ conformal field theory (the latter representing the $U(1)$ current of the $N=2$ algebra as outlined above). We can construct the perturbations of these supersymmetric models as vertex operators of the Toda fields combined with vertex operators of the free $U(1)$ boson. From this viewpoint there are obvious integrable perturbations, namely those perturbations that extend the H-Toda theory to some larger Toda theory, or to some affine Toda theory. Indeed, the original $N=2$ coset model suggests two natural integrable perturbations:

$$V_- \;=\; exp(i\tilde{\beta}\gamma\cdot\phi);\quad V_+ \;=\; exp(-i\tilde{\beta}\psi\cdot\phi), \tag{2.9}$$

where γ is a simple root of G that extends the simple roots of H up to a simple root system for G. The vector ψ is the highest root of G. In equation (2.9), and henceforth, we take ϕ to denote a vector of $\ell+1$ bosons that generate the Cartan subalgebra of G. The bosons of the H theory are obtained by making the appropriate projections. Perturbing by V_- extends the H-Toda theory to a G-Toda theory, and the further perturbation by V_+ extends this to an affine G-Toda theory. Note that the intermediate G-Toda theory will not be conformal since the energy momentum tensor of this theory is that of the free $U(1)$ plus (2.7), and (2.7) has a charge at infinity given by ρ_H and not ρ_G. Moreover, the coupling constant $\tilde{\beta}$ is tuned to the conformal value for the H-coset and not for the G-coset. As an incidental point, we shall see below that there is a symmetry between V_+ and V_- and one can obtain a completely equivalent intermediate G-Toda theory by perturbing with V_+ first.

We now show that perturbing by V_- and V_+ preserves the $N=2$ supersymmetry. Because G/H is a hermitian, symmetric space, we have:

$$(\rho_G-\rho_H)\cdot\gamma \;=\; (\rho_G-\rho_H)\cdot\psi \;=\; \frac{1}{2}g.$$

In particular, $(\psi-\gamma)$ lies on the root lattice of H. Furthermore, one can show that $\{\alpha_j,-\psi\}$ is also a system of simple roots for G. Hence there is a Weyl rotation that maps $\{\alpha_j,-\psi\}$ onto $\{\alpha_j,\gamma\}$. It follows that γ is necessarily of the same length as ψ, that is, they must both be long roots (and thus have a canonically normalized length of $\sqrt{2}$) [5]. The canonically normalized left-moving $U(1)$ current, $J(z)$, of an $N=2$ superconformal theory has $J(z)J(w)\sim\frac{c}{3}\frac{1}{(z-w)^2}$, and in our models it is therefore represented by:

$$J(z)=\frac{2i}{\sqrt{g(g+1)}}(\rho_G-\rho_H)\cdot\partial\phi\;. \tag{2.10}$$

One should also remember that the Toda theories that make up the $N=2$ theory must all have $\tilde\beta=\sqrt{\frac{g}{g+1}}$. From this one finds that the operators $V_\pm$ have an $U(1)$ charges of $\pm\left(\frac{1}{(g+1)}-1\right)$. Moreover, the conformal weights of $V_\pm$ are now elementary to compute using (2.4) and one finds that V_+ and V_- both have left and right conformal weights of $\frac{1}{2}+\frac{1}{2(g+1)}$. The charges and dimensions are exactly consistent with V_+ and V_- representing $G^-_{-\frac{1}{2}}\widetilde G^-_{-\frac{1}{2}}x$ and $G^+_{-\frac{1}{2}}\widetilde G^+_{-\frac{1}{2}}\overline{x}$ where x and $\overline{x}$ are the most relevant chiral, primary fields of the $N=2$ coset model, and $G^\pm_{-\frac{1}{2}}$ and $\widetilde G^\pm_{-\frac{1}{2}}$ respectively represent the left-moving and right-moving super-charges.

To render the identification incontrovertable, we first observe that x and $\overline{x}$ are the unique fields in the $N=2$ coset model having conformal weight $\frac{1}{2(g+1)}$ and $U(1)$ charges $\pm\frac{1}{(g+1)}$. These operators can therefore be unambiguously identified with the Toda (or free field) vertex operators:

$$W_-=exp\Big(i\Big(\tilde\beta-\frac{1}{\tilde\beta}\Big)\;\gamma\cdot\phi\Big)\;,\quad W_+=exp\Big(-i\Big(\tilde\beta-\frac{1}{\tilde\beta}\Big)\;\psi.\phi\Big)\;. \tag{2.11}$$

It is also elementary to identify the supercurrents in terms of Toda vertex operators. In the hermitian, symmetric space models one has [7]

$$G^\pm(z)\;=\;\sum_{\alpha\in t^+}J^{\mp\alpha}(z)\Psi^{\pm\alpha}(z) \tag{2.12}$$

[5] Note that we are *not* assuming that G is simply-laced and so it is not obvious that γ has length $\sqrt{2}$.

where $t^+=\Delta^+(G)\backslash\Delta^+(H)$, the $J^{\pm\alpha}$ are the currents of G, and $\Psi^{\pm\alpha}$ are the fermions of $SO(d)$. Consider how this is represented in the H-decomposition. The group G decomposes into $K^+\oplus K^-\oplus H'$ under the H action, where K^+ and K^- are conjugate, irreducible H representations. The highest weights of these representations are, in fact, γ and $-\psi$ respectively. The corresponding vertex operators are thus those with $w_2=\gamma$ or $w_2=-\psi$. Moreover, the operators $G^\pm(z)$ have no components in the denominator of the coset, and hence one must have $w_3=0$ and thus $w_1=w_2$, as is evident from (2.12). Thus one has:

$$G^-(z)\;=\;exp\Big(-\frac{i}{\tilde\beta}\;\gamma\cdot\phi\Big)\;,\quad G^+(z)\;=\;exp\Big(\frac{i}{\tilde\beta}\;\psi\cdot\phi\Big)\;. \tag{2.13}$$

One can readily check that the $U(1)$-charges are ±1 while the conformal dimension is $3/2$ for both.

If one now considers the operator products of the vertex operators in (2.13) and (2.9) one finds: $G^\pm(z)\;V_\pm(w)=\frac{1}{(z-w)^2}W_\pm(w)$, and thus $W_\pm$ and $V_\pm$ are superpartners of each other. (By associativity we also expect that $G^\mp W_\pm\sim V_\pm$, but this will only become manifest after employing the screening operators so as to make the operator product local.) Combining left movers and right movers and passing to the Toda form of the theory, the foregoing leads directly to the identification of V_+ and V_- with $G^+_{-\frac{1}{2}}\widetilde G^+_{-\frac{1}{2}}\overline{x}$ and $G^-_{-\frac{1}{2}}\widetilde G^-_{-\frac{1}{2}}x$.

The fact that these operators can be written in this form means that perturbing by them preserves the supersymmetry. Consider the perturbed action:

$$\begin{aligned}S\;&=\;S_0\;+\;\int d^2z(\lambda V_-(z,\overline{z})+\lambda^*V_+(z,\overline{z}))\\&=\;S_0\;+\;\lambda\;\int d^2zd^2\theta^+\;\mathcal{X}\;+\;\lambda^*\int d^2zd^2\theta^-\;\overline{\mathcal{X}}\end{aligned} \tag{2.14}$$

where S_0 is given by (2.8), and $\mathcal{X}$ and $\overline{\mathcal{X}}$ are the appropriate superfields: for example,

$$\mathcal{X}\;=\;\big(x\;+\;\theta^+\;G^-_{-\frac{1}{2}}x\;+\;\tilde\theta^+\;\widetilde G^-_{-\frac{1}{2}}x\;+\;\theta^+\tilde\theta^+\;G^-_{-\frac{1}{2}}\widetilde G^-_{-\frac{1}{2}}x\big)\;.$$

Substituting (2.8) and (2.9) into (2.14) one obtains the the action of an affine G-Toda theory. The action is not quite in the standard form: one must rescale

λ and λ^*. This can be accomplished by shifting the bosons according to $\phi \to \phi + \mu_1(\rho_G - \rho_H) + \mu_2\rho_H$ for some appropriately chosen constants μ_1 and μ_2. Therefore, the most relevant supersymmetric perturbation of the $N = 2$ supersymmetric, hermitian symmetric space model based on G/H' corresponds to an affine G-Toda theory, with $\tilde{\beta}^2 = \left(\frac{g}{g+1}\right)$.

It should be noted that we are considering an intrinsically different sort of perturbation to those considered in [4]. In [4] the energy pertubations of the W-algebras led to affine H-Toda theories with $\tilde{\beta}^2 = \frac{h+k}{h+k+1}$. Here we start with an H-Toda theory with $\tilde{\beta}^2$ tuned to the appropriate value for the conformal field theory of the H-coset. Supersymmetry and CPT then require that we use *two* perturbing vertex operators, one of which extends the theory to a G-Toda theory at a non-conformal value of the coupling constant, while the second operator takes the theory to the corresponding *affine* G-Toda theory. Putting this another way, an affine G-Toda theory can serve to describe various kinds of perturbed conformal field theories. If the coupling is chosen to be $\tilde{\beta}^2 = \frac{g+k}{g+k+1}$, and the background charge is chosen to have the appropriate form, then the affine Toda theory describes a conformal W_{G_k} theory perturbed by Φ_{13} [4]. When $\tilde{\beta}^2 = \frac{g}{g+1}$, then the affine Toda theory describes a perturbed $\frac{G \times SO(d)}{H'}$, $N = 2$ superconformal, minimal, hermitian model. The different choices of H' correspond to different choices of the background charge that enter in the stress energy tensor (2.7).

The affine G-Toda theories have precisely n_s integrals of motion of spin s, where n_s is the number of exponents, m_i, of G such that $n_s \equiv m_i \bmod g$. The exponents of a Lie algebra can be defined in several ways: there are precisely $\ell + 1 = \text{rank}\ G$ of them and the numbers $m_i + 1$ are the degrees of the Casimirs of G. In particular $m_1 = 1$, $m_{\ell+1} = g - 1$, and $m_1 < m_i < m_{\ell+1}$, $i = 2, \ldots, \ell$. One also has $m_i = g - m_{\ell+2-i}$. For $SU(n)$, $\{m_i\} = \{1, 2, \ldots, n-1\}$, for $SO(2n)$, $\{m_i\} = \{1, 3, 5, \ldots, 2n-3, (n-1)\}$. For E_6, E_7 and E_8 the exponents are $\{1, 4, 5, 7, 8, 11\}$, $\{1, 5, 7, 9, 11, 13, 17\}$ and $\{1, 7, 11, 13, 17, 19, 23, 29\}$ respectively. These therefore represent, $\bmod g$, a complete list of the spins of the integrals of motion for the perturbed minimal, hermitian models.

We will discuss some applications of the foregoing observations in subsequent sections of this paper. We conclude here by noting that in our discussion of the connections between conformal theories and Toda theories, we restricted H to be simply-laced (but no such restrictions were made upon the group G). We did this primarily because this restriction was made in the relevant literature on Toda theories. It appears that this restriction has been made basically due to the technical details, rather than because of some fundamental obstruction. Thus it is highly probable that the results above can be generalized to conclude that the most relevant, chiral, primary perturbations of *all* the minimal, hermitian coset models will be directly related to affine Toda theories, and hence are integrable.

3. Integrable Perturbations of the $N = 2$ Minimal Series.

In [3] it was shown that the most relevant supersymmetric perturbation of the A_{k+1} modular invariant, minimal, N=2 supersymmetric model yields an integrable field theory. These minimal conformal models can be constructed as level one, $N = 2$ supersymmetric cosets with $G = SU(k+1)$ and $H' = SU(k) \times U(1)$. Thus, by employing the results of the foregoing section one can easily see that the integrability of the most relevant supersymmetric perturbation of the kth minimal model is a direct consequence of the integrability of affine $SU(k+1)$ Toda theory. It also follows that these perturbed minimal models possess precisely one integral of motion of every integer spin, except when the spin is a multiple of $k+1$. This was indeed what was found in [3], and in addition it was also shown there that the soliton scattering matrices were closely related to those of the affine $SU(k+1)$ Toda theories. It was to some extent this observation that prompted the work contained in this paper.

There are also other $N = 2$ coset models that correspond to the $N = 2$ minimal series. Most particularly, if one takes the level one, super-coset model with $G = SO(m+2)$ and $H' = SO(m) \times SO(2)$, one obtains a theory with central charge $c = \frac{3m}{m+1}$. This infinite sequence of theories corresponds to every second model in the usual $N=2$ minimal series, *i.e.* $k=2m$, except that the partition function is that of the type D modular invariant. The most relevant chiral, primary field of such a type D conformal field theory has conformal weight $h=\frac{1}{2(g+1)}=\frac{1}{2(m+1)}$. The corresponding type A theory (with the same central charge) is a Z_2 orbifold of the type D theory, and the most relevant chiral, primary field of this type A theory has conformal weight $\frac{1}{2(k+2)}=\frac{1}{4(m+1)}$. In the language of the corresponding Landau-Ginzburg theory, the type A theory has superpotential $W(x)=x^{2(m+1)}$ and has x as its most relevant chiral, primary field, while the type D theory has superpotential $W(x_1,x_2)=x_1^{m+1}+x_1x_2^2$ with x_1 as its most relevant chiral, primary field. The orbifoldization that takes the type A theory to the type D theory projects out all odd powers of x, leaving x^2 as the most relevant, chiral, primary field which can thus be identified with x_1. (The new chiral, primary field x_2 of the type D theory appears in the twisted sector of the orbifold).

Using the results of the previous section one can relate the type D minimal models with $m=2p$ to $SO(2p)$ Toda theories tensored with a single free boson. Furthermore the x_1 perturbation of this model will be related to an affine $SO(2p+2)$ Toda theory, and it will thus possess non-trivial integrals of motion of spin s whenever s modulo $2p$ is an element of $\{1,3,5,\ldots,(2p-1),p\}$ [6].

The question naturally arises as to whether these integrals of motion survive orbifoldization back to the type A model. That is, is the perturbed $N=2$ superconformal field theory with potential $W(x)=x^{4p+2}+\lambda x^2$ an integrable field theory? More generally, what can be said about $W(x)=x^{k+2}+\lambda x^2$? Direct computation, using the techniques of [14] and [3] shows that there is *indeed* a non-trivial integral of motion at spin 3 for the x^2 perturbation of the general A_{k+1} minimal model with $k\geq 2$. The holomorphic part of this conserved current is

$$G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big[L_{-2}J_{-1}\;-\;\frac{(k+2)}{12}(J_{-1})^3\Big]|0>, \qquad (3.1)$$

[6] In section 2 we restricted ourselves to the consideration of simply-laced groups, and thus we make a similar restriction here. It does, however, seem highly plausible that the result will generalize to all the type D minimal models.

for the theory with central charge $c=\frac{3k}{(k+2)}$. The J_n are the moments of the $U(1)$ current of the $N=2$ superconformal field theory normalized so that $J(z)J(w)\sim\frac{c}{3}\frac{1}{(z-w)^2}$. As in [3] we have written (3.1) in the form $G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\mathcal{O}|0>$ so as to explicitly exhibit the supermultiplet structure.

It should be noted that the foregoing integral of motion has been computed to first order in the perturbation x^2. As was pointed out in [3], a simple argument, based on a unitarity bound for the dimension and charge of a field in an $N=2$ S.C.F.T., reveals that there can be only one other possible higher order correction. Specifically, if $\mathcal{T}(z)$ is a holomorphic conserved current in the conformal theory, and $\mathcal{O}$ is some collection of operators of the theory, then the second order correction to $\partial_{\bar z}\langle\mathcal{T}(z)\;\mathcal{O}\rangle$ is of the form

$$\partial_{\bar z}\Big\langle\int d^2\theta_1^+d^2z_1\;\lambda\;\mathcal{X}^2(\theta_1^+,z_1)\int d^2\theta_2^-d^2z_2\;\lambda^*\;\overline{\mathcal{X}}^2(\theta_2^-,z_2)\;\;\mathcal{T}(z)\;\mathcal{O}\Big\rangle. \qquad (3.2)$$

The unitarity argument shows that the only possible correction to $\partial_{\bar z}\langle\mathcal{T}(z)\;\mathcal{O}\rangle$ occurs in the limit $z_1\to z_2$, where $\mathcal{X}^2$ and $\overline{\mathcal{X}}^2$ fuse into the field Φ_0^2. (See, for example [15] for notation: Φ_0^2 denotes the primary field with quantum numbers $l=2$, $m=0$.) Now suppose that $\mathcal{T}(z)$ is the holomorphic component of a current that is conserved at least to first order in perturbation theory. Recall that such a current and the perturbation to the $N=2$ S.C.F.T. are both top components of some superfield, and so by the Ward identities for supersymmetry, the quantum corrections to the current $\mathcal{T}$ should also be top components of a superfield. However, Φ_0^2 is manifestly not the top component of a superfield, and therefore such a quantum correction would explicitly violate supersymmetry. We thus expect that the second order correction to the conserved current vanishes when the divergence in (3.2) is regulated in an appropriate, supersymmetry preserving manner. Therefore, despite a potential resonance, first order perturbation theory suffices to compute the integrals of motion for the x^2 perturbation. As further evidence for this it should be noted that the corresponding perturbation of the type D modular invariant theory is certainly integrable because of its relationship to a Toda field theory. Moreover, the $l=2,m=0$ state (which is not excluded by unitarity from appearing as second order correction to the integral of motion) is not projected out under orbifoldization. Thus any problems with such resonances in the type A model would also be manifest in the type D model.

Returning to the consideration of the integrals of motion of the type D theories, one finds that at least one of these integrals does not survive orbifoldization. Consider the model with supercoset with $G = SO(4)$ and $H' = SO(2) \times SO(2)$ (*i.e.* we take $p = 1$ in the models discussed above). Since $SO(4) \equiv SU(2) \times SU(2)$ it follows that this model is two copies of the $c = 1$, $N = 2$ superconformal theory. Indeed it has long been known that $k = 4$, type D minimal model decomposes in this manner. The orbifoldization that takes this to the A_5 modular invariant, minimal model (with $W(x) = x^6$) is the Z_2 interchange of the two $c = 1$ models [16]. Thus the x^2 perturbation of the A_5 modular invariant model corresponds to the symmetric sum of the most relevant, chiral, primary fields of the two $c = 1$ models. The corresponding Toda theory tells us that there should be exactly two integrals of motion at every odd spin. At spin 1, one of the corresponding conserved currents is obviously the energy-momentum tensor, and the other current is simply the anti-symmetric combination of the energy-momentum tensors of the two $c = 1$ models. Obviously this second integral of motion will be projected out in orbifoldization back to the type A modular invariant model.

More generally, we have checked that for the x^2 perturbation of the $k = 4p$ type A minimal models the only integrals of motion up to and including spin 4 are those associated with the $N = 2$ superconformal algebra and the spin 3 integral described above. In particular, there is no spin 2 integral for $p = 2$, no spin 4 integral for $p = 4$ and only one spin 3 integral for $p = 3$. Thus it appears to be generally true that the spin p integral of motion of the $SO(2p + 2)$ affine Toda theory does not survive orbifoldization. The presence or absence of the spin p integral of motion is, presumably, intimately connected with the fact that the type D chiral algebra is extended beyond the type A chiral algebra (which is simply the $N = 2$ superconformal algebra) by an operator of conformal weight p. One should also note that the spin p integral of motion of the $SO(2p + 2)$ affine Toda theory is connected with the Casimir of degree $p + 1$ [2] and it is

also probably not a coincidence that this Casimir cannot be constructed directly from the Casimirs of the unitary, semi-simple subgroups of $SO(2p + 2)$.

As regards computing further integrals of motion, it is, as yet, too complicated to use the methods of [14] and [3] to obtain the integrals of spin greater than four. However the fact that one has at least one higher spin integral of motion gives us strong cause to believe that there is an infinite tower of such integrals of motion, and thus, the x^2 perturbation of the x^{k+2} models lead to integrable field theories.

As a parenthetic note, there seems to be an integrable perturbation of the minimal series that does not appear to correspond to a Toda theory. Consider the perturbed conformal theory with Landau-Ginzburg superpotential $x^{k+2} + \lambda x^k$. [7] There are no integrals of motion at spins 2 and 4, but for spin 3 one obtains:

$$G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} \left\{ 2(J_{-1})^3 \ + \ 12 L_{-2} J_{-1} \ + \ (c-6)\, G^-_{-\frac{3}{2}}\, G^+_{-\frac{3}{2}} \right\} |0> \ .$$

Again, first order perturbation theory suffices. Using the fact that $G^-_{-\frac{1}{2}} x^k$ is a bosonic exponential (up to a trivial parafermionic current), one can see that there is no multiple fusion of several of such operators that could possibly generate a higher order correction. (Higher order chiral corrections would necessarily have to be in a non-trivial representation of the parafermionic algebra.) On the other hand, fusion of chiral with antichiral fields can produce only supersymmetry violating contributions that must be excluded by the same reasoning as above. It seems that these non-renormalization type features are generic and should apply also more generally than to merely the $N = 2$ minimal models.

4. Quantum conserved currents from Toda theory

We now turn to the structure of the conserved currents in the non-critical field theories discussed above. We wish to show that for these $N = 2$ supersymmetric models the conserved currents can be obtained in terms of the Miura transform on the free bosons ϕ used in section 2.

[7] Note that this perturbation does not resolve the singularity of the Landau-Ginzburg potential completely, and thus this model has some massless excitations.

The basic idea behind the construction of the conserved currents is the following [5][4]. The classical affine Toda theory based on a Lie group G has a discrete symmetry that is essentially generated by the center of the group. In the quantum theory this symmetry relates the BRST charges of the conformally invariant Toda theory to the perturbing operators that take the conformal theory to the off-critical, affine Toda theory. All operators in the conformal theory commute with the BRST charges. If we now construct an operator that is invariant under the discrete symmetry, then this operator has to commute with the perturbing Hamiltonian as well as with the BRST charges. Therefore such an operator must give rise to an integral of motion. For example, consider $G = SU(k+1)$. The simple roots can be represented by vectors $e_i - e_{i+1}, i = 1, \ldots, k$, while the negative of the highest root is $-\psi = e_{k+1} - e_1$. These $k+1$ roots have a Z_{k+1} symmetry corresponding to the cyclic permutations of $e_1, \ldots, e_{k+1}$. It was observed in section 3 that the off-critical field theory obtained from the most relevant, $N = 2$ supersymmetric perturbation of the G/H' coset model could be represented as an H-Toda theory perturbed by the operators (2.9). These perturbing operators can obviously be cyclically permuted into vertex operators involving the simple roots of H, and the latter operators are simply the BRST currents in the free field language. It follows that any operator that lies in the physical part of the free field Hilbert space and that is Z_{k+1} symmetric must be an integral of motion for the off-critical theory. The Miura transformation provides a method of constructing such operators.

To illustrate the procedure we will consider the $N = 2$ supersymmetric, minimal series with the type A modular invariant. We will also only consider the perturbation of this conformal model by the real combination of the most relevant fields, that is, in the Landau-Ginzburg language, by a hermitian combination of the chiral, primary field x and its anti-chiral conjugate $\overline{x}$. Let us define

$$A(B)(z) \equiv \oint_z A(z')B(z)(z'-z)^{-1}dz' \tag{4.1}$$

where the contour runs around the point z. The first two conserved currents for these models have the following holomorphic components:

$$T(z) \quad \text{and} \quad 2J(T(z)) - G^+(G^-(z)) \ . \tag{4.2}$$

It should be noted that these expressions are only unique up to total derivative terms.

We now show that these expressions encode the structure of the corresponding integrals of motion in Toda field theories. It is interesting to note that the conserved currents (4.2), and their higher spin counterparts, are constructed from only four basic local fields: $J(z), T(z), G^+(z), G^-(z)$. The Toda field theory and its integrals of motion, on the other hand, are constructed from ℓ bosons, where ℓ is the rank of the group in question. The fact that we are able to express the integrals of motion in terms of fewer fields than one needs to describe the Toda theory is another manifestation of the fact that there is extra symmetry (in particular, supersymmetry) at the value of the coupling constant that we have chosen.

To get the type A minimal models one takes $G = SU(k+1)$ and $H' = SU(k) \times U(1)$ as in section 3. The coset $(SU_1(k) \times SU_1(k))/SU_2(k)$ is realized in terms of $k-1$ free bosons, which we will represent by k bosonic fields, ϕ_i, satisfying the constraint:

$$\sum_{i=1}^{k} \phi_i = 0 \tag{4.3}$$

The extra $U(1)$ factor of (2.2) will be represented by a free bosonic field ϕ_0. As we saw in section 2, the perturbed $N = 2$ theory can be regarded as an affine $SU(k+1)$-Toda theory at a specific value of the coupling. In order to simplify the action of the Z_{k+1} symmetry, it is convenient to introduce a new basis for the bosonic fields. Define $\hat{\phi}_i, i = 1, \ldots, k+1$ by

$$\hat{\phi}_i = \phi_i + \nu\phi_0 \ ; \qquad \hat{\phi}_{k+1} = -k\nu\phi_0 \tag{4.4}$$

with

$$\nu = \frac{1}{\sqrt{k(k+1)}} \tag{4.5}$$

These fields satisfy a constraint similar to (4.3):

$$\sum_{i=1}^{k+1} \hat{\phi}_i = 0 \tag{4.6}$$

The Z_{k+1} then acts by cyclic permutation of the $\hat{\phi}_i$. In [11][4] it was shown that the operators of the W-algebra of the coset model $(SU_1(k)\times SU_1(k))/SU_2(k)$ have a simple realization in terms of the bosonic fields. This realization is obtained directly from the Miura transformation of the Toda theory. The operators of the W-algebra are Z_k symmetric, and one finds that their bosonic realizations also display this Z_k symmetry up to total derivative terms [8]. Motivated by this we consider the Miura transformation extended to the k free bosons of the $N=2$ theory. That is, consider operators defined by:

$$\begin{aligned}\prod_{i=1}^{k+1}(i\alpha\partial-\partial\hat{\phi}_i) \;&\equiv\; (i\alpha\partial-\partial\hat{\phi}_{k+1})\ldots\ldots(i\alpha\partial-\partial\hat{\phi}_1)\\ &\equiv\; \sum_{j=0}^{k+1}\hat{U}_j^{(k+1)}(i\alpha\partial)^{k+1-j}\end{aligned} \tag{4.7}$$

One finds $\hat{U}_0^{(k+1)}=1$, $\hat{U}_1^{(k+1)}=-\sum_{i=1}^{k+1}\partial\hat{\phi}_i\equiv 0$. We also have

$$\begin{aligned}\hat{U}_2^{(k+1)} \;&=\; -\frac{1}{2}\sum_{j=1}^{k+1}(\partial\hat{\phi}_j)^2 \;-\; i\alpha\sum_{j=1}^{k+1}(k+1-j)\partial^2\hat{\phi}_j\\ \hat{U}_3^{(k+1)} \;&=\; -\frac{1}{3}\sum_{j=1}^{k+1}(\partial\hat{\phi}_j)^3 \;-\; i\alpha\sum_{j=1}^{k+1}(k+1-j)(\partial^2\hat{\phi}_j)(\partial\hat{\phi}_j)\\ &\quad+ i\alpha\sum_{1\le i\le j\le k+1}(\partial\hat{\phi}_i)(\partial^2\hat{\phi}_j) \;-\; \frac{1}{2}(i\alpha)^2\sum_{j=1}^{k+1}(k+1-j)(k-j)\partial^3\hat{\phi}_j\end{aligned} \tag{4.8}$$

The operators $\hat{U}_j^{(k+1)}$ are Z_{k+1} symmetric, up to total derivatives. In order to relate these operators to the W-algebra of the minimal models, we must take

$$\alpha \;\equiv\; \frac{1}{\sqrt{(k+1)(k+2)}}. \tag{4.9}$$

Using the relations (4.4) between the $\hat{\phi}$ and the ϕ, we find

$$\hat{U}_2^{(k+1)} \;=\; T \;-\; \frac{1}{2}\partial J \tag{4.10}$$

[8] The bosonic realization need not have the same manifest symmetries as the operator it represents—the bosonic realization will, in general, only display such symmetries up to trivial states in the BRST cohomology.

where

$$J \;\equiv\; i\sqrt{\frac{k}{k+2}}\;\partial\phi_0 \tag{4.11}$$

is the $U(1)$ current of the $N=2$ theory, and T is the energy-momentum tensor. That is,

$$T \;=\; T_{pf} \;-\; \frac{1}{2}(\partial\phi_0)^2 \tag{4.12}$$

where

$$T_{pf} \;=\; -\frac{1}{2}\sum_1^k\,(\partial\phi_i)^2 \;-\; i\alpha\sum_{j=1}^k\,(k-j)\;\partial^2\phi_j \tag{4.13}$$

is the energy momentum tensor of Z_k parafermions. Rewriting $\hat{U}_3^{(k+1)}$ in terms of the decompostion of the $N=2$ theory according to (2.2), one obtains:

$$\begin{aligned}\hat{U}_3^{(k+1)} \;&=\; U_3^{(k)} \;-\; \frac{1}{3}(k-1)\nu\;(\partial\phi_0)^3 \;-\; 2\nu\;U_2^{(k)}\;\partial\phi_0\\ &\quad-\; \frac{i}{2}\alpha(2k-1)(\partial\phi_0)(\partial^2\phi_0) \;+\; \frac{1}{3\nu}(\partial^3\phi_0)\;.\end{aligned} \tag{4.14}$$

The operator $U_3^{(k)}$ is known [11] to be a linear combination of the spin 3 current W and the derivative of T_{pf}. The precise combination can be determined from the OPE of U_3^k with itself and with $U_2^{(k)}\equiv T_{pf}$. One finds

$$U_3^{(k)} \;=\; -i\sqrt{\frac{(k-2)(3k+4)}{2k(k+1)}}\;W + \frac{i}{2}\alpha(n-2)\;\partial T_{pf}\,, \tag{4.15}$$

where W is normalized, as usual, by

$$W(z)\;W(w) \;\sim\; (z-w)^{-6}\;\Big(\frac{2(k-1)}{3(k+2)}\Big). \tag{4.16}$$

Using the representation

$$G^+(z) \;=\; \sqrt{\frac{2k}{k+2}}\;\psi(z)e^{i\mu\phi(z)},\quad G^-(z) \;=\; \sqrt{\frac{2k}{k+2}}\;\psi^\dagger(z)e^{-i\mu\phi(z)} \tag{4.17}$$

with

$$\mu \;=\; \sqrt{\frac{k+2}{k}} \tag{4.18}$$

we find

$$G^+ \,(G^-) \;=\; \sqrt{\frac{2(k-2)(3k+4)}{k(k+2)}}\; W \;+\; 2i\mu\; T_{pf}\;(\partial\phi_0) \;-\; \frac{i}{3}\mu\;(\partial\phi_0)^3 \\ -\;(\partial\phi_0)\;(\partial^2\phi_0)\;+\;\frac{i}{3\mu}\;(\partial^3\phi_0) \tag{4.19}$$

where we have used the parafermion OPE as given in [17].

Putting together (4.14) and (4.19) we find that, up to total derivatives:

$$\hat{U}_3^{(k+1)} \;=\; \frac{i}{2}\sqrt{\frac{k+2}{k+1}}\;\Big[2J\;(T) - G^+\;(G^-)\Big] \tag{4.20}$$

where the right-hand side is (up to normalization) exactly the second integral of motion in equation (4.2). We have thus recovered the spin 2 quantum integral of motion directly from the Miura transformation.

Because all of the operators $\hat{U}_j^{(k+1)}$ lie within the conformal field theory, they must commute with the BRST charges in the free bosonic formulation, and because the operators $\hat{U}_j^{(k+1)}$ are all Z_{k+1} symmetric (up to total derivatives) they must also commute with the perturbation. It follows that the operators $\hat{U}_j$ must be current densities for quantum integrals of motion for the perturbed conformal field theory. Moreover, some, and probably all of the higher spin (*i.e.* spin $\geq k+2$) integrals of motion can be obtained from polynomials in the $\hat{U}_j^{(k+1)}$ and their derivatives, provided that these polynomials are Z_{k+1} symmetric up to total derivatives.

For the $N = 2$ minimal series ($c < 3$) there is another method for obtaining the higher spin integrals of motion. In this instance one knows that the holomorphic components of all the integrals of motion can be written as polynomials in $J(z)$, $G^+(z)$, $G^-(z)$, $T(z)$ and their derivatives. The coefficients in these polynomials will themselves be polynomials of a known bounded degree in the central charge, c, of the theory. The precise form of a general integral of motion of a given spin, s, may be determined by obtaining the corresponding expression for a sufficient number of models in the minimal series, thereby completely determining the general form of the relevant polynomials in c. On the other hand, for the kth member of the minimal series, with $k > s$, the

holomorphic component of the spin s integral of motion is simply $\hat{U}_{s+1}^{(k+1)}$. The operators $\hat{U}$ can be written in terms of J and the W-algebra, and thus in terms of J, $G^\pm$ and T. Thus one can use one's explicit knowledge of the integrals of motion for $k > s$ to interpolate and obtain the integrals of motion for $k < s$.[9]

5. Some Observations about Landau-Ginzburg Solitons.

In [6] it was shown that the class of minimal, hermitian models has an $N = 2$ supersymmetric, Landau-Ginzburg description. It is thus interesting to investigate the spectra of the Landau-Ginzburg models that correspond to the perturbed minimal, hermitian theories and to compare them with the spectra of the corresponding affine Toda theories. More precisely, we want to compare minimal sets of excitations that form closed scattering theories. In the Landau-Ginzburg models an obvious candidate for such a minimal set are the fundamental solitons, and in particular those that saturate the Bogomolny bound of the supersymmetry algebra [3]. On the other hand, in the affine G-Toda theories the massive particle excitations form a natural closed scattering sector. There are $\ell \equiv$ rank G such particles, and their masses are: $\mathbf{m} = (m_1, ..., m_\ell)$, where $\mathbf{m}$ is the Frobenius-Perron eigenvector of the Cartan matrix, C, of G. That is [18]:

$$C \cdot \mathbf{m} \;=\; 4\sin^2\big(\frac{\pi}{2g}\big)\,\mathbf{m}\;.$$

It was shown in [3] that for the most relevant supersymmetry-preserving perturbation of the $N = 2$ minimal series, the Landau-Ginzburg solitons do indeed have masses that correspond to an affine $SU(k+1)$ Toda theory. Moreover, apart from supersymmetric pairings of states, the scattering matrix of the Landau-Ginzburg solitons was that of the massive particle excitations of the affine

[9] It is possible that the polynomial expression for the integral of motion found in the foregoing manner could become trivial via null vectors. Explicit computations for the first few integrals of motion shows that this only happens when the integral itself disappears. Thus no new expressions for the integrals of motion are needed, and we assume here that this is always the case.

Toda theory. It is therefore tempting to suggest that there should be a correspondence in general between the Landau-Ginzburg solitons and the massive excitations of the Toda theory. To provide further evidence for this, and also because of their intrinsic interest, we present here some preliminary results on the Landau-Ginzburg soliton structure for the most relevant perturbations of some more general minimal, hermitian coset models.

The solitons of an $N = 2$ Landau-Ginzburg theory interpolate between pairs of isolated degenerate vacua. Let x denote the fundamental Landau-Ginzburg fields, and $W(x)$ denote the superpotential. Then in the i-th vacuum the fields, x, have an expectation value x_i^0, where x_i^0 is a solution to $\nabla W(x_i^0) = 0$. Employing specific properties of $N = 2$ Landau-Ginzburg models, it can be shown [19] [3] that the mass of a fundamental soliton linking the i-th vacuum with the j-th vacuum is bounded by the distance between the images of x_i^0 and x_j^0 in the complex W-plane, *i.e.*:

$$m(i,j) \;\geq\; |W(x_i^0) - W(x_j^0)| \; . \tag{5.1}$$

It was shown in [3] that for models with one Landau-Ginzburg field, the soliton masses saturate this Bogomolny bound, and that the soliton trajectories correspond to straight lines in the W-plane. While the bound (5.1) remains valid for Landau-Ginzburg theories with several fields, it is not clear whether the correspondence of soliton trajectories with straight lines in the W-plane remains true. The lengths of these straight lines still represent lower bounds for the soliton masses, and this bound will be saturated by solitons that are annihilated by two of the four supercharges. We will investigate the Landau-Ginzburg potential to find the lengths of these lines and hence the minimal possible soliton masses, and we will compare these masses to those of the Toda theory.

First of all one needs to determine the superpotentials that describe the unperturbed models. For the minimal, hermitian model based on G/H', the algebra of chiral, primary fields is closely related to the cohomology ring $\mathcal{R}$ on G/H' [6]. Both rings can be obtained from a generating function W, and can be realized as the quotient $\mathcal{R} = \frac{C[x]}{\nabla W = 0}$. In fact, there are general arguments and

also explicit examples that support the conjecture that both rings are actually isomorphic to each other, and that the functions W coincide.

Adopting this hypothesis (and our results will strongly support it), the superpotentials can easily be computed by integrating the non-trivial vanishing relations $\nabla W = 0$ in the polynomial algebra of the cohomology. These vanishing relations can be obtained by noting that if rank G = rank H' and G/H' is Kähler, the cohomology of G/H' is generated by the Chern classes c_i of bundles that are defined by H' represenations, R. For any R, the c_i are defined by expanding the graded total Chern form

$$C_R(\Omega) \;=\; \det{}_R(1 + t\Omega) \;=\; \sum_{i=0} c_i t^i \; , \tag{5.2}$$

where Ω is the curvature two-form of the bundle. The point is that the bundles are trivial precisely when R is also a representation of G. That is, if $R_G = \oplus_l R_l$, we obtain the following relations for the c_i:

$$\tilde{C}_{R_G}(\Omega) \equiv \sum_{k=0} \tilde{c}_k t^k \;=\; \prod_l C_{R_l}(\Omega) \;=\; 1 \; . \tag{5.3}$$

The Chern classes c_i and $\tilde{c}_i$ are of course directly related to the Casimirs of H' and G respectively. From (5.3) we see that the $\tilde{c}_k$ are quasihomogeneous polynomials in the c_k, and that these polynomials represent vanishing relations. To obtain a complete set of such relations one may need to consider more than one choice of R_G. (For an example, see below.) The cohomology ring $\mathcal{R}$ is generated by all the c_i taken modulo the $\tilde{c}_k$. The Chern classes c_j correspond precisely to Landau-Ginzburg fields with $U(1)$ charge $j/(g+1)$. It is usually the case that by using the vanishing relations one can trivially express many of the c_k as polynomials in the c_j of lower degree. Thus it is convenient to eliminate these redundant variables and work with a reduced set of independent generators for $\mathcal{R}$. Corresponding to this there will be a reduced set of vanishing relations. One finds that for the minimal, hermitian models the (reduced set of) vanishing relations can be "integrated" to a potential [6]. That is, all the vanishing relations are equivalent to the equations $\nabla W = 0$ for some function W of the independent variables.

Consider for example $G/H' = SO(10)/U(5)$. From [6] one infers that the general form of the superpotential has to be $W = x^9 + y^3 + ax^3y^2$, for some particular value of the modulus a that we need to determine. Taking R_G to be the ten dimensional vector representation, with $\Omega = diag(x_1, -x_1, ..., x_5, -x_5)$, we have $\tilde{C}_{10}(\Omega) = \prod_{i=1}^{5}(1 - t^2x_i^2)$. Expressing (5.3) in terms of the Chern classes corresponding to the **5** of $U(5)$, $c_i = \sum_{k_1<...<k_i} x_{k_1}...x_{k_i}$, one obtains the following relations:

$$\begin{aligned} \tilde{c}_2 &= 2c_2 - c_1^2 \;=\; 0 \\ \tilde{c}_4 &= 2c_4 - 2c_1c_3 + c_2^2 \;=\; 0 \\ \tilde{c}_6 &= 2c_2c_4 - 2c_1c_5 - c_3^2 \;=\; 0 \\ \tilde{c}_8 &= c_4^2 - 2c_3c_5 \;=\; 0 \end{aligned}$$

The $SO(10)$ Casimir of degree 5 does not appear here, but from the **16** of $SO(10)$ we get the additional vanishing relation $\tilde{c}_5 = -12c_5 = 0$. Thus only c_1 and c_3 remain independent and they can be associated with the Landau-Ginzburg fields x and y above. These fields satisfy vanishing relations at grades 6 and 8, that give the equations we are looking for as $\partial_y W = \tilde{c}_6(x,y) = 0$ and $\partial_x W = \tilde{c}_8(x,y) - \frac{1}{2}x^2\tilde{c}_6(x,y) = 0$. Integration yields the superpotential, which can be brought by field redefinitions to the above normal form with modulus $a = -\sqrt{2}\left[\frac{81}{17+12\sqrt{2}}\right]^{1/3}$. Further examples are:

$$\begin{aligned} W_{\left[\frac{SU(5)}{SU(3)\times SU(2)\times U(1)}\right]} &= x^6 + y^3 + ax^2y^2 \;,\; a = -3\sqrt{5}(22+10\sqrt{5})^{-1/3},\; (\mu = 10) \\ W_{\left[\frac{SU(6)}{SU(4)\times SU(2)\times U(1)}\right]} &= x^7 + xy^3 + ax^3y^2 \;,\quad a = -7^{1/3}, \quad (\mu = 15) \\ W_{\left[\frac{SU(6)}{SU(3)\times SU(3)\times U(1)}\right]} &= x^7 + xz^2 + y^2z + ax^4z + bx^5y \;, \\ & \qquad a = 3/4\sqrt{7},\; b = 0, \quad (\mu = 20) \\ W_{\left[\frac{SU(7)}{SU(5)\times SU(2)\times U(1)}\right]} &= x^8 - y^4 + ay^2x^4 + bx^6y, \\ & \qquad a = 14\sqrt{\frac{2}{31}},\; b = 28\left(\frac{2}{31}\right)^{3/4} \;,\; (\mu = 21) \end{aligned} \tag{5.4}$$

In accordance with [6], the multiplicity $\mu = dim\mathcal{R} = Tr(-1)^F$ is always given by the dimension of a particular representation of G.

We consider perturbations of the superpotentials $W \to W + \lambda x$, where x is the lowest dimensional chiral, primary field with conformal weight $h = \frac{1}{2(g+1)}$. Under these perturbations, the μ-fold degenerate critical points of W resolve into μ non-degenerate critical points, so that we have distinct vacuum states x_i^0, $i = 1, ..., \mu$.

Studying the foregoing (and other) examples, the following general pattern emerges. The μ vacuum image points in the W-plane either lie on concentric circles or at the center of these circles. Some of these image points may coincide with each other. The W-plane picture is symmetric under Z_g rotations, and, in particular, each concentric circle has g points regularly spaced around it. The circles have specific radii whose ratios are fixed by the particular values of the moduli $a, b, ...$ above. The important observation from the point of view of solitons is that the radii are such that most of the distances $|W(x_i^0) - W(x_j^0)|$ give masses that correspond to the appropriate affine Toda theory. This is highly non-trivial, as there are many more possible links (*i.e.* solitons) between the μ vacua than the ℓ masses of the Toda particle spectrum.

As an example, consider the picture for $\frac{SU(6)}{SU(4)\times SU(2)\times U(1)}$ in more detail. The W-plane diagram is given by a regular star with 6 vertices, created by the intersection of two equal sized equilateral triangles. The $\mu = 15$ vacuum images, $W(x_i^0)$, are distributed as follows: one vauum maps to each of the 6 vertices of the star. One vacuum goes to each of the 6 intersections of the two equilateral triangles, and finally three vacua map to the center of the star. The affine $SU(6)$ Toda model has three different masses,

$$(m_a = m_{\bar{a}}) : (m_b = m_{\bar{b}}) : m_c \;=\; 1 : \sqrt{3} : 2 \;. \tag{5.5}$$

These ratios correspond precisely to the distance between two nearby vertices of the star (m_b), the distance between a vertex and a nearby intersection (m_a), and the distance between a vertex and a next-to-nearby intersection (m_c). There are also distances that do not fit with the affine Toda particle spectrum, for example, the distances between non-adjacent vertices. These distances are all larger than the ones cited above. Moreover, from a naive classical analysis it appears that there are no fundamental soliton trajectories linking these points, but only multi-soliton trajectories. Thus the $SU(6)$ affine Toda spectrum is exactly reproduced if the solitons saturate the Bogomolny bound. The same spectrum can also be obtained from the perturbed $\frac{SU(6)}{SU(3)\times SU(3)\times U(1)}$ theory, whose superpotential is given in (5.4) (and of course, also from the perturbed $\frac{SU(6)}{SU(5)\times U(1)}$ theory whose superpotential is $W = x^7 + \lambda x$). Here the W-plane picture consists of two concentric circles with radii 1 and 1/2, plus two vacuum images in the center. Six vacuum images build a regular hexagon on the outer circle, and there are 12 such images on the inner circle (the latter set of images come in pairs that are mapped to the same point); the hexagons have the same orientation. One can check that if the fundamental solitons saturate the Bogomolny bound then they do not have masses other than those determined by (5.5).

For the other examples, the situation is more complicated. Though most distances $|W(x_i^0) - W(x_j^0)|$ indeed do correspond to Toda masses, there exist in general also distances that do not fit and are smaller than the Toda masses. However, not every line between two dots need correspond to a soliton; in fact, in [3] a simple example of this phenomenon was displayed. Thus it is possible that these smaller lengths do not correspond to solitons and in fact the Toda masses give the entire spectrum of the Landau-Ginzburg model.

6. Conclusions

By exploiting the relationship between Toda field theories and conformal field theories, we have shown that a large class of $N = 2$ supersymmetric models admit integrable deformations. It is intriguing to note that the class of $N = 2$ supercoset models that can be related to Toda theories coincides precisely with the class of $N = 2$ supercoset models that are known to have a Landau-Ginzburg description [6]. This means that all of the integrable models that we have described here also have an equivalent Landau-Ginzburg form, and as the results in section 5 suggest, one can learn a considerable amount from this Landau-Ginzburg formulation.

The connection with Toda theories enables one to trivially read off the spins of the integrals of motion of the perturbed conformal model. Moreover, by using the Miura transform of the Toda field theory one can explicitly construct the integrals of motion in terms of the free bosonic formulation of the conformal model. Thus the realization of perturbed theories in terms of Toda theories affords valuable insight into the structure of the theories.

In this paper we have studied the application of Toda technology to perturbed $N = 2$ superconformal theories. These techniques have already been applied with great success to non-supersymmetric theories [4][12]. There are further applications that we are now investigating. One of these is to look at the rôle of generalized Toda theories [2] in perturbed conformal field theory. The simplest example of this is the Bullough-Dodd theory, which can be thought of as a Z_4 reduction of SO(8) affine Toda theory [18], and is intimately related to the spin perturbation of the bosonic minimal series. Another application is essentially a generalization, to non-supersymmetric theories, of the results presented here. That is, we find new integrable models that can be thought of as multiple perturbations of products of conformal models.

In most of this paper we have considered what can be learned about conformal field theory from Toda theories. There is also something to be learned by reversing this procedure. First, although the classical integrability of Toda theories is well understood, the quantum integrabilty, and even the quantization of Toda theories with imaginary coupling constants is, at best, poorly understood. The fact that the integrals of motion of the perturbed, minimal, $N = 2$ models that we derived here coincide with those of [3] provides some confirmation that there is a sensible, unitary conformal field theory within the Hilbert space of the quantized Toda theory, and that the corresponding affine Toda theories are quantum integrable. Another consequence of our work is that for certain values of the coupling constant, affine Toda theories have a quantum $N = 2$ supersymmetry. This $N = 2$ supersymmetry has no analogue in the classical theory, and appears in the quantum theory via the vertex operators (2.13). In the simplest example of this, namely in sine-Gordon theory, the supersymmetry appears at a value of the coupling constant that lies is the strong coupling region. We believe that this is generally true, that is, we believe that the supersymmetry point of an affine Toda theory will lie somewhere in the strong coupling domain. This domain is notoriously hard to analyze, but at this one isolated point the structure becomes simpler, and by utilizing the corresponding Landau-Ginzburg theory one should be able to learn much about the theory.

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