CERN-TH.7055/93
POLFIS-TH.09/93

Picard–Fuchs Equations, Special Geometry and Target Space Duality

A. Ceresole[1,2,⋆], R. D'Auria[1,2,∗], S. Ferrara[3,∗,†],
W. Lerche[3], J. Louis[4] and T. Regge[2,5,∗]

[1]Dipartimento di Fisica, Politecnico di Torino,
Corso Duca Degli Abruzzi 24, 10129 Torino, Italy
[2]INFN, Sezione di Torino, Italy
[3]CERN, 1211 Geneva 23, Switzerland
[4]Sektion Physik, Universität München, D-80333 München
[5]Dipartimento di Fisica Teorica, Università di Torino,
Via P. Giuria 1, 10125 Torino, Italy

Contribution to the second volume of
"Essays on Mirror Manifolds"

Abstract

We review the system of holomorphic differential identities implied by special Kählerian geometry of four dimensional $N = 2$ supergravity. For the special case of superstring compactifications on Calabi–Yau threefolds, these identities are equivalent to the Picard–Fuchs equations of algebraic geometry that are obeyed by the periods of the holomorphic three–form. The monodromy group of these equations is closely related to the target space duality symmetry group. Examples with one and two moduli are considered. The connection of special geometry with the moduli space of $N = 2$ superconformal field theories is also discussed.

December 1993

⋆ Supported in part by EEC Research program SC1-CT92-0789.
† Supported in part by DOE grants DE-AC0381-ER50050 and DOE-AT03-88ER40384,Task E.

1 Introduction and Summary

It is well known that four–dimensional heterotic string vacua which are $N=1$ space–time supersymmetric have necessarily an $N=2, c=9$ superconformal field theory (SCFT) in the left moving sector [1]. Furthermore, the couplings of the corresponding low energy effective Lagrangian are directly related to correlation functions in the $N=2$ SCFT. In general, such correlation functions are quite difficult to compute, and usually more insight into their structure can be gained by using the formulation of the (2,2)–string vacua in terms of the Landau–Ginzburg superpotential [2] (possibly in its twisted version corresponding to $N=2$ topological field theory [3]), which is in many cases directly related to the geometrical description in terms of Calabi–Yau 3–fold compactification of the heterotic string [4].

One is actually faced with the problem of computing low–energy quantities in terms of the properties of the moduli space of a given SCFT field theory. The moduli space is the space of all marginal deformations of the underlying N=2 SCFT or, in the geometrical picture of Calabi–Yau compactifications, it is the space of the parameters φ^α, ψ^a describing the deformations of the Kähler class and of the complex structure of the Calabi–Yau manifold [5]. In the low energy theory, the moduli parameters $\varphi^\alpha(x)$, $\psi^a(x)$ appear as neutral massless scalar fields with vanishing potential.

Recently, much progress has been made in the determination of the physical low–energy parameters without ever relying on the underlying SCFT, but rather by using techniques of algebraic geometry and topological field theories (TFTs) [6–19]. A crucial step in this new direction was done in ref. [6], where it was shown that the couplings could be obtained from the solution of a certain fourth order linear holomorphic differential equation. It was realized that this differential equation is a particular example of "Picard–Fuchs equations" obeyed by the periods of the holomorphic threeform Ω that exists on any Calabi–Yau threefold [7–9]. (Picard–Fuchs equations can be derived for general "Calabi–Yau" d–folds [8,7], but we consider mostly $d=3$ in the following.)

On the other hand, in the framework of TFTs one can derive from consistency considerations alone [10,11] differential equations that are equivalent

to the Picard–Fuchs equations. We should note that not only the Picard–Fuchs equations arise from these topological considerations, but there are further properties of the low energy effective Lagrangian encoded in the underlying topological field theory (TFT)[12,13,16].

A further step in uncovering the general structure behind the differential equation was undertaken in refs. [14,15]. It was realized that the Picard–Fuchs equations for a Calabi–Yau threefold are just another way of expressing a geometrical structure called "special geometry" [20–24]. It first arose in the study of coupling vector multiplets to $N=2$ supergravity in four dimensions [20]. In string theory, special geometry is related to the subclass of string vacua with $(2,2)$ worldsheet supersymmetry [25–27]. The additional right–moving world–sheet supersymmetry implies further constraints on the couplings of the effective Lagrangian. In particular, in the moduli sector of the theory these constraints can be expressed by the equations [20,22–24,27]

$$\begin{aligned} R^{\delta}_{\alpha\overline{\beta}\gamma} &= g_{\alpha\overline{\beta}}\delta^{\delta}_{\gamma} + g_{\gamma\overline{\beta}}\delta^{\delta}_{\alpha} - C_{\alpha\gamma\epsilon}g^{\epsilon\overline{\epsilon}}C_{\overline{\beta}\overline{\delta}\overline{\epsilon}}g^{\delta\overline{\delta}}\,, \\ C_{\alpha\beta\gamma} &= e^{K}W_{\alpha\beta\gamma}(z)\,. \end{aligned} \tag{1.1}$$

Here, $g_{\alpha\overline{\beta}}(z,\overline{z}) = \partial_\alpha\partial_{\overline{\beta}}K(z,\overline{z})$ is the Kähler metric on the moduli space and the (completely symmetric) Yukawa couplings $W_{\alpha\beta\gamma}$ are holomorphic functions of the moduli z^α ($\alpha = 1,\ldots,n$, where n is the dimension of the moduli space). A complex Kähler manifold whose metric obeys (1.1) is called a special Kähler manifold. (Further relevant formulas of special geometry are collected in Appendix A.)

For given $W_{\alpha\beta\gamma}$, eq. (1.1) can be viewed as a covariant and non-holomorphic differential equation for the Kähler potential. Its general solution can be expressed [24,22] in terms of $n+1$ holomorphic sections $X^A(z), A=0,1,\ldots,n$ which obey $\partial_{\overline{\alpha}}X^A = 0$:

$$K \;=\; -\ln i(X^A\overline{F}_A - \overline{X}^A F_A)\,, \tag{1.2}$$

where

$$W_{\alpha\beta\gamma} \;=\; \partial_\alpha X^A \partial_\beta X^B \partial_\gamma X^C F_{ABC}\,, \tag{1.3}$$

and $F_A(X) = \frac{\partial F(X)}{\partial X^A}$, *etc*, and $F(X)$ is a homogeneous function of X of degree two. We see that all information about K and $W_{\alpha\beta\gamma}$ are encoded in the holomorphic objects $X^A(z)$ and $F_A(z)$ and their complex conjugates.

In order to make contact with the differential equation of ref. [6] one observes [24,22,23] that eq. (1.1) is entirely equivalent to the following system of non–holomorphic first order equations

$$\begin{aligned} D_\alpha V &= U_\alpha \\ D_\alpha U_\beta &= -iC_{\alpha\beta\gamma} g^{\gamma\overline{\gamma}} \overline{U}_{\overline{\gamma}} \\ D_\alpha \overline{U}_{\overline{\beta}} &= g_{\alpha\overline{\beta}} \overline{V} \\ D_\alpha \overline{V} &= 0 \,, \end{aligned} \tag{1.4}$$

where $V(z) = \big(X^A(z), F_A(z)\big)$ and D_α is the Kähler and reparametrization covariant derivative. By successively inserting these equation into each other one can represent this system by

$$D_\alpha D_\beta (C^{-1\hat{\gamma}})^{\rho\sigma} D_{\hat{\gamma}} D_\sigma V = 0 \tag{1.5}$$

(assuming for the moment that the matrix $(C_\alpha)_{\beta\gamma}$ is invertible). Here, $\widehat{\gamma}$ is a priori not summed over (in contrast to σ). In ref. [14] it was shown that eq. (1.5) in one complex dimension (n=1) is actually holomorphic, although its building blocks are not. It is the covariant version of the (holomorphic) equation of ref. [6], and thus is the analogue of the Picard–Fuchs equations in special geometry. Its solution determines X^A and F_A and thus via eqs. (1.3),(1.2) also $W_{\alpha\beta\gamma}$ and K.

The existence of the covariant holomorphic differential equation (1.5) is intimately connected to the fact that the Christoffel as well as the Kähler–connection naturally split into the sum of two terms [14]. One of them is non–holomorphic and transforms as a tensor whereas the other term is holomorphic and transforms like a connection. Furthermore, the holomorphic pieces of these connections are flat and vanish in "special coordinates"

$$t^a(z) = \frac{X^a(z)}{X^0(z)} \qquad (a = 1, \ldots, n) \,. \tag{1.6}$$

A similar situation holds in topological Landau-Ginzburg models where the flat connection can be identified with the Gauss–Manin connection [28,13,8].

Ref. [15] generalizes the analysis of [14] to arbitrary dimensions n and sections 2–4 are based on this paper. We start in section 2.1 by showing that eq. (1.5) in one dimension is not the most general linear fourth order differential equation but rather is characterized by the vanishing of one of the "invariants", $w_3 = 0$. The other invariant w_4 measures the deviation from covariantly constant* Yukawa couplings, or in other words, the deviation from the large radius (classical) limit of the Calabi–Yau moduli space.

Every N–th order differential equation is equivalent to a first order matrix equation of the form $(\partial - \mathsf{A})\mathbf{V} = 0$, where the first row of $\mathbf{V}$ is the solution vector V in (1.5). In section 2.2 we show that $w_3 = 0$ translates into the statement that the gauge potential A of this matrix equation takes values in $sp(4)$. It is this fact which nicely generalizes to an n dimensional moduli space.

In chapter 3 we derive the Picard–Fuchs equations of special geometry for an arbitrary number of moduli. They are a direct consequence of (1.3), and are most easily displayed as n coupled first order holomorphic matrix equations

$$(\partial_\alpha - \mathsf{A}_\alpha)\mathbf{V} \;=\; 0 \,, \tag{1.7}$$

where A takes values in $sp(2n+2)$. This is the analog of the vanishing of w_3 in one dimension. A_α is a sum of a matrix $\mathbb{\Gamma}_\alpha$ which contains the flat connections plus the structure constants $\mathbb{C}_\alpha$ of a $2n+2$ dimensional chiral ring $\mathcal{R}^{(3)}$. The structure constants contain the Yukawa couplings $W_{\alpha\beta\gamma}$ and furthermore satisfy $[\mathbb{C}_\alpha, \mathbb{C}_\beta] = 0$, $\mathbb{C}^4 = 0$. Because the connection is symplectic, $\mathbf{V}$ can always be taken as an element of $Sp(2n+2)$. This means that the well–known symplectic structure of special geometry can ultimately be traced back to the identity (1.3).

In section 3.2 we display the relationship between equations (1.7) and (1.4). Eq. (1.4) can also be written as a first order matrix equation $(\partial_\alpha - \mathcal{A}_\alpha)\mathbf{U} = 0$, albeit with a non-holomorphic connection $\mathcal{A}$. Strominger observed [22] that the

* Covariantly constant with respect to the holomorphic connections.

system (1.1) is just the flatness condition for $\mathcal{A}$. We will show that eq. (1.7) corresponds to a gauge where $\overline{\mathcal{A}}=0, \mathcal{A}=\mathsf{A}, \overline{\partial}\mathsf{A}=0$.

In section 3.3 we consider particular cases where $\mathbb{C}$ is degenerate, which corresponds to decoupled chiral rings. Here the F–function is a direct sum whereas the metric of the moduli space is a complicated function and by no means the metric on a product space. This clearly shows that the fundamental object in special geometry is indeed the holomorphic function F and not the non-holomorphic metric on moduli space.

As we indicated above, the motivation for the present work was to analyse the holomorphic differential equations of special geometry. So far, the discussion was completely general and applied also to geometries that do not have an interpretation in terms of Calabi-Yau manifolds or TFT's. In chapter 4 we relate eqs. (1.4)–(1.7) to Calabi–Yau moduli spaces, where $V, U_\alpha, \overline{U_{\overline{\alpha}}}, \overline{V}$ correspond to basis elements of the third cohomolgy, $H^3 = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$ [22,23]. We then relate the formulas used in topological Landau-Ginzburg theory to the structures uncovered in special geometry in chapter 3. This is useful in order to explicitly compute the differential equations. In particular we verify that the computation of Yukawa couplings $W_{\alpha\beta\gamma}$ via Picard–Fuchs equations in special geometry is, as expected, identical to the computation of certain three–point correlators in topological Landau-Ginzburg theory.

We should stress that in the context of special geometry the deformations of the Kähler class and the deformations of the complex structure appear on an equal footing. This is because eq. (1.1) holds for both types of moduli [27,25], which is a manifestation of "mirror symmetry" [29]. In practice, it is often only possible to compute the Picard–Fuchs equations for one type of moduli. In order to find the Picard–Fuchs equation for the other class of moduli, one needs to make use of the mirror symmetry.

By using "topological anti–topological fusion" the geometrical structure implied by eq. (1.1) was extended in [13] to include relevant (massive) perturbations, in addition to the marginal (massless) moduli considered here. One of the main objectives of [13] was to construct non–holomorphic quantities (like the metric) from TFT. On the other hand, the emphasis of the present review

is on the structure of the holomorphic Picard–Fuchs equations in relation to special geometry.

In the remaining sections, based on refs. [30,31], we will be concerned with the non-trivial global properties of the moduli space $\mathcal{M}$, which are a consequence of the group of discrete isometries Γ of $\mathcal{M}$, generally referred to as the "target space duality group", or "modular group" [32]. The duality group describes quantum symmetries of the string effective action and is the discrete remnant of the non–compact symmetries of the well-known supergravity Lagrangians (no–scale supergravities[33]). The knowledge of the duality group is very important since the physical quantities appearing in the effective Lagrangian must transform in a definite way under this group: for example, the gauge coupling $g_a^{-2}(t,\bar{t})$ is a real function which must be modular invariant in $t, \bar{t}$. We will see that the target space duality group Γ is closely related to the monodromy group Γ_M of the Picard–Fuchs equations that determine the period matrix for Calabi–Yau manifolds.

A celebrated example is the compactification of the heterotic string on a 6–torus, modded out by some discrete symmetry that leaves a residual $N=1$ space-time supersymmetry. The Kähler class untwisted modulus (corresponding to the volume size), $t = 2(R^2 + i\sqrt{|g|})$, parametrizes the homogeneous space [34]:

$$\frac{SU(1,1)}{U(1)} \ . \tag{1.8}$$

The Kähler potential of the scalar fields t, ϕ_{27} is

$$K = -3 \log \left[i(t-\bar{t}) - \phi_{27}\overline{\phi_{27}} \right] , \tag{1.9}$$

and the target space superpotential W is characterized by a constant Yukawa coupling, $W = C\phi_{27}^3$. The theory is invariant under transformations of $PSL(2,\mathbb{R})$

$$t' = \frac{at+b}{ct+d} \quad , \quad \phi'_{27} = -\frac{1}{ct+d}\phi_{27} \quad , \quad ad-bc=1 \ . \tag{1.10}$$

However, when other (twisted) sectors are introduced and quantum corrections are taken into account, the duality group is reduced to $PSL(2,\mathbb{Z})$, with generators

$$t \to t+1 \qquad\qquad t \to -\frac{1}{t}\ . \tag{1.11}$$

More generally, for Calabi-Yau manifolds the superpotential W is no longer constant, but in general it depends on the moduli : $W_{\alpha\beta\gamma} = W_{\alpha\beta\gamma}(\phi,\psi)$, because of the instanton corrections of the σ–model. However, for one modulus it turns out that $t \to t+1$ is still an exact symmetry, if t is the special coordinate of special geometry, or equivalently, the flat coordinate of the associated topological field theory. That is, the Yukawa coupling [6]

$$W(t) = \sum_{n=0}^{\infty} d_n\ e^{2\pi int} \tag{1.12}$$

is periodic in t. Actually, the translational symmetry is supposed to be exact for any number of moduli, since the invariance under $t^a \to t^a + 1$, $a = 1, ..., n$ has a stringy origin in the existence of the antisymmetric axion field B_{ij}.

In the large radius limit, $t \to i\infty$, eq. (1.12) gives $W(t) = d_0 = const.$, which implies that $\mathcal{F}(t) \equiv |X^0|^{-2}F(X^A)$ is cubic in t. Actually the most general prepotential such that $W(t)$ and the Kähler potential (1.2) are invariant under $t \to t+1$ is [26]

$$F(X^A) = (X^0)^2[\frac{C}{3!}t^3 + \lambda(t) + P_2(t)] \tag{1.13}$$

where $P_2(t)$ is a polynomial of degree two with purely real coefficients and $\lambda(t)$ a periodic function of t. If instead invariance under arbitrary shifts is required, $t \to t + c$, $c \in \mathbf{R}$, then λ must be a constant. Note that a non vanishing λ is generated in σ-model perturbation theory at the four loop level [6].

Under the inversion $t \to -\frac{1}{t}$ the Kähler potential transforms as follows

$$K \to K + f(t) + f(\bar{t}) \tag{1.14}$$

so that W, which has a non trivial Kähler weight undergoes a non trivial transformation. In the following, we will show that the generalization of the inversion for a generic duality group is given by

$$t^a \to f^a(t^a, \partial_a\mathcal{F}, \mathcal{F}; A, B, C, D) \qquad a = 1, ..., n \tag{1.15}$$

where n is the number of moduli, $\mathcal{F}(t^a) = (X^0)^{-2}F(X^A)$ and A, B, C, D are $(n+1)\times(n+1)$ matrices parametrizing a generic $Sp(2n+2;\mathbf{Z})$ transformation M:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \in \quad Sp(2n+2;\mathbf{Z}) \tag{1.16}$$

As an example, we will then briefly recall the determination of the duality group for the quintic [6], which is generated by the following two transformations acting on the special coordinate:

$$\begin{aligned} &t \to t+1 \\ &t \to \frac{t}{t\mathcal{F}_t' - 2\mathcal{F} + 1}\ . \end{aligned} \tag{1.17}$$

Here, $\mathcal{F}(t)$ is the holomorphic prepotential of special geometry. The modular coordinate γ on which the two transformations described earlier act as $SL(2,\mathbb{R})$ transformations (for any $\mathcal{F}$) is described in chapter 5 and the relation of γ with the special coordinate t as well as with the Landau–Ginzburg coupling of the defining polynomial in CP_{d+1} will also be given.

The determination of the duality symmetry group Γ for Calabi–Yau compactifications, or more generally, for $(2,2)$ $c = 9$ SCFT, is in general a mathematically very complex problem. Only recently one has been able to solve a couple of examples in which the moduli space is two-dimensional [31,17,19]. However, we will see that just from the specific structure of the Picard–Fuchs equations, one can extract important information for any number of moduli:

i) The (holomorphic) Yukawa couplings can be read out directly from the leading coefficients in the order Picard–Fuchs equations.

ii) Using the matrix form of the Picard–Fuchs equations in the special geometry gauge, we argue that, for any number of moduli, the subgroup of the duality group that corresponds to translations in the special variables

$$t^a \to t^a + n^a \ , \qquad n^a \in \mathbb{Z}^n \ , \tag{1.18}$$

can be constructed in terms of the intersection numbers of the Calabi–Yau manifold.

Finally, in section 6 we will illustrate an efficient method for computing the duality group, exemplified by considering a model with two moduli. We will use algebro–geometric techniques which were developed and applied to the study of the monodromy group of Feynman integrals some time ago[35,36] . We will show that Γ is simply given by a 3–dimensional representation of a central extension of B_5, the braid group on five strands.

2 Differential equations for one variable

2.1 Linear differential equations and W–generators

In ref. [6] it was shown that the periods of a one–dimensional moduli space of a particular Calabi–Yau–threefold (a quintic in CP_4) are determined by a fourth order linear differential equation. The corresponding differential equation in special geometry —the one–dimensional version of eq. (1.5)— was derived in ref. [14]. In this section we add some observations concerning this one–dimensional case. This will prove advantageous for the study of the general situation.

Thus, let us first briefly review some facts about linear fourth order differential equations [37]. Their general form reads

$$\sum_{n=0}^{4} a_n(z)\, \partial_z^n V \ = \ 0 \ , \tag{2.1}$$

where the a_n obey well–defined transformation laws in order to render eq. (2.1) covariant under coordinate changes $z \to \tilde{z}(z)$, $\partial \to \xi^{-1}\partial$, $\xi \equiv \partial\tilde{z}/\partial z$. Not all

of the a_n are relevant. First, one can scale out a_4, and furthermore drop the coefficient proportional to a_3 by means of the redefinition $V \to V e^{-1/4\int \frac{a_3(u)}{a_4(u)}du}$. This puts the differential equation into the form

$$\mathcal{D}V \equiv (\partial^4 + c_2\partial^2 + c_1\partial + c_0)V = 0 \ , \tag{2.2}$$

where the new coefficients c_n are combinations of the a_n and their derivatives. In this basis V transforms as a $-3/2$ differential, but the transformation properties of the c_n are not very illuminating. However, one can find combinations of the c_n's and their derivatives which transform like tensors:

$$\begin{aligned} w_2 &= c_2 \\ w_3 &= c_1 - c_2' \\ w_4 &= c_0 - \tfrac{1}{2}c_1' + \tfrac{1}{5}c_2'' - \tfrac{9}{100}c_2^2 \ . \end{aligned} \tag{2.3}$$

A straightforward computation shows

$$\begin{aligned} \tilde{w}_2 &= \xi^{-2}[w_2 - 5\{\tilde{z};z\}] \\ \tilde{w}_3 &= \xi^{-3}w_3 \\ \tilde{w}_4 &= \xi^{-4}w_4 \ , \end{aligned} \tag{2.4}$$

where $\{\tilde{z};z\} = (\frac{\partial^2\xi}{\xi} - \frac{3}{2}(\frac{\partial\xi}{\xi})^2)$ is the Schwarzian derivative. Actually w_2, w_3, w_4 form a classical W_4–algebra (see, for instance, [38]), a fact that we will not make use of in this paper. Using (2.2) and (2.3) one finds

$$\mathcal{D}\,V = \left[\partial^4 + w_2\partial^2 + (w_3 + w_2')\partial + \tfrac{3}{10}w_2'' + \tfrac{9}{100}{w_2}^2 + \tfrac{1}{2}w_3' + w_4\right]V \ . \tag{2.5}$$

The advantage of rewriting a differential equation in terms of W-generators is that this is a convenient way to display the particular properties of the equation in a reparametrization-covariant way. From eq. (2.4) we learn that there is always a coordinate system in which $w_2 = 0$ holds. On the other hand, w_3 and w_4 do characterize the fourth order differential operator $\mathcal{D}$ in any coordinate frame.

Let us return to special geometry: in ref. [14] it was shown that there is a holomorphic fourth order differential equation that expresses the constraint of special geometry and thus is equivalent to eq. (1.1). This equation is the one-dimensional version of (1.5) and reads

$$DD\,W^{-1}\,DDV \;=\; 0\,, \tag{2.6}$$

where D is the Kähler– and reparametrization covariant derivative defined in (A.11), and W is the one-dimensional Yukawa coupling. In special coordinates (1.6), this equation becomes very simple,

$$\partial^2\,W^{-1}\,\partial^2\,V \;=\; 0\,. \tag{2.7}$$

Equation (2.6) can be written in the form (2.1) and one finds that the coefficients are not arbitrary but are related as follows [14]: $a_3 = 2\partial a_4$, $a_4 = W^{-1}, a_1 = \partial a_2 - \frac{1}{2}\partial^2 a_3$. The coefficients a_2 and a_0 are complicated functions of W and the connections. The above relations translate into the invariant statement*

$$w_3 \;\equiv\; 0\,. \tag{2.8}$$

Furthermore, the other W-generators are given (in special coordinates) by

$$\begin{aligned} w_2 \;&=\; \tfrac{1}{2W^2}\big(4WW'' - 5W'^2\big) \\ w_4 \;&=\; \tfrac{1}{100W^4}\big(175W'^4 - 280WW'^2W'' + 49W^2W''^2 + 70W^2W'W''' \\ &\qquad - 10W^3W''''\big)\,. \end{aligned} \tag{2.9}$$

Thus, all special geometries in one dimension lead to a fourth order linear differential equation that is characterized by $w_3 = 0$. This is in close relation to the fact that the solution vector V does not consist of four completely independent elements, but rather has a restricted structure. More precisely, by construction four linear independent solutions are given by the components of the vector (cf., (A.22))

$$V = \big(X^A(z), F_A(z)\big)\;, \qquad A = 0,1\,, \tag{2.10}$$

with

$$F_A(z) \;=\; \frac{\partial}{\partial X^A(z)}F(z), \tag{2.11}$$

* This was noted in [8] for the special example of the quintic hypersurface in CP_4.

where F is a homogeneous function of X of degree 2. The reverse statement is however not true: $w_3 = 0$ does not imply that the solution V can always be written in the form (2.10). This is proven in Appendix B.

Note that the property (2.11) does not uniquely fix V. It is known [20,23,24,26] that precisely for symplectic rotations of V,

$$\big(\widetilde{X}^A, \widetilde{F}_A(\widetilde{X}^A)\big) \;=\; \big(X^A, F_A(X^A)\big)\cdot M\;, \qquad M \in Sp(4)\;, \tag{2.12}$$

one has $\widetilde{F}_A = (\partial\widetilde{F}/\partial\widetilde{X}^A)$ where $\widetilde{F}$ is again a homogeneous function of degree 2. Thus, the elements of V are defined only up to this kind of transformations. Of course, generic linear combinations of the four solutions are still solutions of (2.6), but for these the special structure of the solutions (that reflects $w_3 = 0$) is not manifest.

Symplectic transformations belonging to $Sp(2n+2,\mathrm{I\!R})$ have a particular meaning in special geometry. They represent changes of special coordinate bases and are exactly those transformations which leave K form–invariant and consequently do not change any physical quantity. (This can be easily seen from eq. (A.24) which displays manifestly the symplectic structure of K). We will show in the following sections how this symplectic structure of special geometry is encoded in the differential equations.

One can similarly discuss the properties of $\mathcal{D}$ when in addition:

$$w_4 \;=\; 0\,. \tag{2.13}$$

From (2.9) it is clear that this applies in particular if $W = const$. However, $w_4(W) = 0$ is a non-trivial differential equation that possesses also other solutions than $W = const$. One might thus ask about the significance of general solutions of $w_4(W) = 0$ with non-constant superpotential.

If $w_4 = 0$, eq. (2.5) simplifies to

$$\mathcal{D}V = \big(\partial^4 + w_2\partial^2 + w_2'\partial + \tfrac{3}{10}w_2'' + \tfrac{9}{100}w_2^2\big)\,V\,, \tag{2.14}$$

and the solutions are given by $\{\theta_1^3, \theta_1^2\theta_2, \theta_1\theta_2^2, \theta_2^3\}$ [37] . Here, $\theta_{1,2}$ are the independent solutions of the second order equation,

$$(\partial^2 + \tfrac{1}{10}w_2)\theta_{1,2} = 0 \ . \tag{2.15}$$

One easily determines a symplectic basis to be

$$\begin{aligned} X^0 &= \theta_1^3 \,, \qquad F_0 = -\tfrac{1}{6}\frac{(X^1)^3}{(X^0)^2} + 2c_{00}X^0 + 2c_{01}X^1 \ , \\ X^1 &= \theta_1^2\theta_2 \,, \qquad F_1 = \tfrac{1}{2}\frac{(X^1)^2}{X^0} + 2c_{01}X^0 + 2c_{11}X^1 \ , \end{aligned} \tag{2.16}$$

where c_{AB} are arbitrary constants. Using the homogeneity property $X^A F_A = 2F$ or integrating F_A we find

$$F = \tfrac{1}{6}\frac{(X^1(z))^3}{X^0(z)} + c_{AB}X^A X^B \ . \tag{2.17}$$

From this F we can compute (using (1.3)) the Yukawa coupling and find that it is covariantly constant: $\widehat{D}W = 0$. For $c_{AB} = 0$ (2.17) is the F–function* corresponding to the homogeneous moduli space $SU(1,1)/U(1)$ (which satisfies the stronger constraint $DW = 0$). Moreover, it follows from the inhomogeneous transformation behavior (2.4) of w_2 that one can always find a "schwarzian" coordinate where w_2 vanishes, by solving a Schwarzian differential equation $\{t; z\} = \frac{1}{5}w_2(t)$. Then one has $\theta_1 = 1, \theta_2 = t$ and thus can take ($c_{AB} = 0$)

$$V \ = \ (1, t, \tfrac{1}{2}t^2, \tfrac{1}{6}t^3) \ , \qquad F \ = \ \tfrac{1}{6}t^3 \ , \qquad W \ = \ \partial^3 F = 1 \, . \tag{2.18}$$

It is clear that $t = X^1/X^0$ is precisely the special coordinate of eq. (1.6) (note that the coincidence of special coordinates with Schwarzian coordinates holds only if $w_4 = 0$). There is an analogous group action that preserves the relationship among the solutions of (2.14). This group is just the invariance group

$\star$ The quadratic terms in F can have the interpretation as perturbative σ–model corrections in Calabi–Yau compactifications [6].

of the Schwarzian derivative, which is $SL(2,\mathbb{R})$: $\theta' = \frac{a\theta+b}{c\theta+d}$, $ad - bc = 1$. The action on the solutions of (2.14) is easily found through the mapping $V = \theta^3$:

$$M \ = \ \begin{pmatrix} a^3 & a^2c & ac^2/2 & c^3/6 \\ 3a^2b & 2abc + a^2d & bc^2/2 + acd & c^2d/2 \\ 6ab^2 & 2b^2c + 4abd & 2bcd + ad^2 & cd^2 \\ 6b^3 & 6b^2d & 3bd^2 & d^3 \end{pmatrix} \ , \tag{2.19}$$

which is part of $Sp(4,\mathbb{R})$ [39]. Thus, the specific structure of the solutions is unique up to such $SL(2)$ transformations.

Summarizing, the above means that if $w_4 = 0$, the situation for generic w_2 is reparametrization equivalent to $w_2 = 0$, in which case the solutions are given by (2.18). This corresponds to a cubic F-function and to constant Yukawa coupling W. In general coordinates where w_2 does not vanish, W is not constant (but covariantly constant with respect to the holomorphic connections).

Thus, for covariantly constant Yukawa couplings the differential equation is essentially reduced to the differential equation of a torus. This is similar to the situation for the K_3 surface where the only non–trivial W-generator is w_2 [8]. The possibility of having non–trivial Yukawa couplings, or $w_4 \neq 0$, is the new ingredient in special geometry. It reflects the possibility of having instanton corrections to W. Specifically, it is easy to see from (2.7) that in special coordinates the solutions have the general structure

$$V \ = \ (\,1\,, t\,, \tfrac{1}{2}t^2 + \mathcal{O}(t^3)\,, \tfrac{1}{6}t^3 + \mathcal{O}(t^4)\,) \ , \tag{2.20}$$

where the higher order "instanton" terms arise from a non-trivial w_4. Thus, the invariant w_4 measures the deviation from $W = const$, which is the large–radius limit of the Calabi–Yau moduli space. One can actually check that the contribution of a given rational curve of degree k to the Yukawa couplings corresponds to a "w_4–surface", i.e., to a covariantly constant w_4 generator. That is, from (2.9) one finds that in special coordinates: $w_4(W = e^{kt}) = (const)k^4$ (see also Appendix B).

We now turn to another way of understanding the significance of $w_3 = 0$. This will also allow us to introduce some concepts which nicely generalize to multi–dimensional moduli spaces (section 3).

2.2 First order equations

Any fourth order linear differential equation (2.1) is equivalent to a first order matrix equation [40]

$$\Big[\,\mathbb{1}\partial \;-\; \mathsf{A}\Big]\cdot\mathbf{V} \;=\; 0, \tag{2.21}$$

(for a particular choice of the matrix A) where $\mathbf{V}$ is a 4×4 matrix whose first row is V. A matrix of the form

$$\mathsf{A} \;=\; \begin{pmatrix} * & 1 & 0 & 0\\ * & * & 1 & 0\\ * & * & * & 1\\ * & * & * & * \end{pmatrix}\;, \tag{2.22}$$

corresponds to a fourth order operator with $a_4 = 1$ whereas $tr\mathsf{A} = 0$ leads to $a_3 = 0$. However, $\mathcal{D}$ is left invariant by local gauge transformations acting as $\mathbf{V}\to S^{-1}\cdot\mathbf{V}$ and $\mathsf{A}\to S^{-1}\mathsf{A}S - S^{-1}\partial S$, where S has the form

$$S \;=\; \begin{pmatrix} 1 & 0 & 0 & 0\\ * & 1 & 0 & 0\\ * & * & 1 & 0\\ * & * & * & 1 \end{pmatrix} \in N\subset SL(4)\;. \tag{2.23}$$

This is just the usual matrix of lower triangular transformations generated by a nilpotent subalgebra of $sl(4)$. The top row of $\mathbf{V}$ corresponds to the highest weight and thus also is N–invariant (the other rows of $\mathbf{V}$ are gauge dependent). That is, the solutions of (2.5) are completely invariant under the local transformations (2.23).

Note also that the more general gauge transformations belonging to a Borel subgroup B of $SL(4)$, where

$$S = \begin{pmatrix} * & 0 & 0 & 0\\ * & * & 0 & 0\\ * & * & * & 0\\ * & * & * & * \end{pmatrix} \;\in\; B\;, \tag{2.24}$$

do not leave $\mathcal{D}$ invariant but induce $a_3\neq 0$ and $a_4\neq 1$. However, this just corresponds to a rescaling of the solution $V\to f(z)V$ (and corresponds to an irrelevant Kähler transformation in this context).

Using the gauge freedom one can put the connection into the form:

$$\mathsf{A} \;=\; \mathsf{A}_w \;\equiv\; \begin{pmatrix} 0 & 1 & 0 & 0\\ -\frac{3}{10}w_2 & 0 & 1 & 0\\ -\frac{1}{2}w_3 & -\frac{4}{10}w_2 & 0 & 1\\ -w_4 & -\frac{1}{2}w_3 & -\frac{3}{10}w_2 & 0 \end{pmatrix} \in sl(4,R)\;. \tag{2.25}$$

To understand this form, recall the well-known relationship[⋆] between W-algebras and a special, "principally embedded" $SL(2)$ subgroup $\mathcal{K}$ [41] of $G = SL(N)$ (in fact, G can be any simple Lie group). The generators of $\mathcal{K}$ are

$$J_- = \sum_{\substack{simple\\ roots\ \alpha}} b_\alpha E_\alpha\;,\quad J_+ = \sum_{\substack{simple\\ roots\ \alpha}} c_\alpha(b_\alpha)E_{-\alpha}\;,\quad J_0 = \rho_G\cdot H\;, \tag{2.26}$$

where b_α are arbitrary non-zero constants, c_α depend on the b_α in a certain way and ρ_G is the Weyl vector. An intriguing property [41] of $\mathcal{K}$ is that the adjoint of any group G decomposes under $\mathcal{K}$ in a very specific manner:

$$adj(G)\to\bigoplus r_j\;, \tag{2.27}$$

where r_j are representations of $SL(2)$ labelled by spin j, and the values of j that appear in the r.h.s. are equal to the exponents of G. The exponents are just the degrees of the independent Casimirs of G minus one (for $SL(N)$, they are equal to $1,2,\ldots,N-1$).

Recalling that the Casimirs are one-to-one to the W generators associated with G, one easily sees that the decomposition (2.27) corresponds to writing the connection (2.25) in terms of W-generators; more precisely, for an N–th order equation related to $G = SL(N)$, the connection (2.25) can be written as [42,38]:

$$\mathsf{A}_w \;=\; J_- - \sum_{m=1}^{N-1} w_{m+1}(J_+)^m\;, \tag{2.28}$$

⋆ We thank R.Stora for discussions on this point.

where $J_\pm$ are the $SL(2)$ step generators (2.26) (up to irrelevant normalization of the w_n).

In our case[†] with $N=4$, the decomposition (2.27) of the adjoint of $SL(4)$ is given by $j=1,2,3$, which corresponds to w_2, w_3 and w_4. We noticed above that $w_3 \equiv 0$ for special geometry and this means that A_w belongs to a Lie algebra that decomposes as $j=1,2$ under $\mathcal{K}$. It follows that this Lie algebra is $sp(4)$. Indeed, remembering that the algebra $sp(n)$ is spanned by matrices A that satisfy $A\,Q + Q\,A^T = 0$, we can immediately see from (2.25) that

$$\mathsf{A}_w \;\in\; sp(4) \qquad \longleftrightarrow \qquad w_3 \;\equiv\; 0\ . \tag{2.29}$$

Above, the symplectic metric Q is taken as in (A.25).

We chose the gauge in (2.25) such that the symplectic structure is manifest. General gauge transformations conjugate the embedding of $sp(4)$ in $sl(4)$, and in general gauges the fact that $\mathsf{A}_w \in sp(4)$ is not obvious. The invariant way to express this fact is to state that $w_3 = 0$ in the gauge invariant scalar equation.

Similarly, if in addition $w_4 = 0$ (which corresponds to a covariantly constant Yukawa coupling), A_w further reduces to an $SL(2)$ connection. This $SL(2)$ is identical to the principal $SL(2)$ subgroup, $\mathcal{K}$, since according to (2.28) the entries labelled by w_2 and 1 in (2.25) are directly given by the $\mathcal{K}$ generators J_+ and J_-. It consists precisely of the transformations (2.19) that preserve the non–trivial relationship between the solutions.

3 Differential equations for arbitrary many moduli

3.1 Holomorphic Picard-Fuchs equations and special geometry

In this section we generalize the previous analysis to many variables. The basic identities of special geometry are given by the system (1.4).[*] We already

† The choice (2.25) for A corresponds to an embedding (2.26) with $b_1 = b_2 = b_3 = 1$, and $c_1 = c_3 = 3/10, c_2 = 4/10$.

∗ The relevant formulas of special geometry are collected in appendix A.

mentioned in the introduction that, assuming that $(C_\alpha)_{\beta\gamma}$ is invertible, these identities are equivalent to

$$D_\alpha D_\beta (C^{-1\hat\gamma})^{\rho\sigma} D_{\hat\gamma} D_\sigma V \;=\; 0\ , \tag{3.1}$$

where $\hat\gamma$ is a priori not summed over.

Since the solution vector $V \equiv (X_A(z), F^A(X))$ is holomorphic, we expect that the non–holomorphic pieces in (3.1) that come from the connections in D cancel, so that (3.1) is effectively a purely holomorphic identity. We will prove below that this is indeed the case by showing that V also satisfies manifestly holomorphic identities that are equivalent to (3.1). These equations contain only the holomorphic connections $\widehat\Gamma$ and $\partial\widehat K$ (defined in Appendix A).

Let us choose special coordinates $t^a = X^a/X^0$ and the Kähler gauge $X^0 = 1$, and consider the following set of equations:

$$\begin{aligned}
\partial_a V \;&=\; V_a \\
\partial_a V_b \;&=\; W_{abc} V^c \\
\partial_a V^b \;&=\; \delta_a^b V^0 \\
\partial_a V^0 \;&=\; 0\ ,
\end{aligned} \tag{3.2}$$

where (V, V_a, V^a, V^0) are all holomorphic and W_{abc} are the Yukawa couplings in special coordinates. The last two equations of (3.2) give

$$V^0 \equiv (1,0)\ , \qquad\qquad V^a \equiv (t^a, \quad 1, \quad 0)\ , \tag{3.3}$$

while the first two are solved by setting

$$\begin{aligned}
V &\equiv (1, \quad t^a, \quad \partial_a\mathcal{F}, \quad t^a\partial_a\mathcal{F} - 2\mathcal{F})\ , \\
V_a &\equiv (0, \quad \delta_a^b, \quad \partial_a\partial_b\mathcal{F}, \quad t^b\partial_a\partial_b\mathcal{F} - \partial_a\mathcal{F})\ .
\end{aligned} \tag{3.4}$$

The holomorphic function $\mathcal{F}$ is defined in eq. (A.33) and satisfies (in special coordinates)

$$\partial_a\partial_b\partial_c\mathcal{F} = W_{abc}\ . \tag{3.5}$$

This identity is the only non–trivial input in solving the differential equations. The system (3.2) can also be written in matrix form,

$$
\begin{aligned}
(\,\mathbb{1}\partial_a - \mathbb{C}_a\,)\,\mathbf{V} \;&=\; 0\,,\\
\mathbb{C}_a \;&=\; \begin{pmatrix} 0 & \delta_a^c & 0 & 0\\ 0 & 0 & W_{abc} & 0\\ 0 & 0 & 0 & \delta_a^b\\ 0 & 0 & 0 & 0 \end{pmatrix}\,,
\end{aligned}
\tag{3.6}
$$

and from the above we see that this is solved by the columns of the following $(2n+2)\times(2n+2)$–dimensional matrix:

$$
\mathbf{V} = \begin{pmatrix} V\\ V_b\\ V^b\\ V^0 \end{pmatrix} \;=\; \begin{pmatrix} 1 & t^a & \partial_a\mathcal{F} & t^a\partial_a\mathcal{F} - 2\mathcal{F}\\ 0 & \delta_b^a & \partial_a\partial_b\mathcal{F} & t^a\partial_a\partial_b\mathcal{F} - \partial_b\mathcal{F}\\ 0 & 0 & \delta_a^b & t^b\\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{3.7}
$$

From eqs. (3.2), (A.17) we can infer the transformation properties of $\mathbf{V}$ under coordinate and Kähler transformation and thus it is straightforward to write down the covariant and holomorphic version of eqs. (3.2):

$$
\begin{aligned}
\widehat{D}_\alpha V \;&=\; V_\alpha\\
\widehat{D}_\alpha V_\beta \;&=\; W_{\alpha\beta\gamma}V^\gamma\\
\widehat{D}_\alpha V^\beta \;&=\; \delta_\alpha^\beta V^0\\
\widehat{D}_\alpha V^0 \;&=\; 0\,.
\end{aligned}
\tag{3.8}
$$

where $\widehat{D}$ is defined in eq. (A.29) and contains the holomorphic connections given in eq. (A.28). This system can also be written as

$$
(\mathbb{1}\partial_\alpha - \mathsf{A}_\alpha)\,\mathbf{V} \;\equiv\; 0\,, \qquad \mathbf{V} \;=\; \begin{pmatrix} V\\ V_\beta\\ V^\beta\\ V^0 \end{pmatrix}\,, \tag{3.9}
$$

which contains the holomorphic "connection"

$$
\mathsf{A}_\alpha \;=\; \begin{pmatrix} -\partial_\alpha\widehat{K} & \delta_\alpha^\gamma & 0 & 0\\ 0 & (\widehat{\Gamma}_\alpha - \partial_\alpha\widehat{K}\mathbb{1})_\beta^\gamma & (W_\alpha)_{\gamma\beta} & 0\\ 0 & 0 & (\partial_\alpha\widehat{K}\mathbb{1} - \widehat{\Gamma}_\alpha)_\gamma^\beta & \delta_\alpha^\beta\\ 0 & 0 & 0 & \partial_\alpha\widehat{K} \end{pmatrix}\,. \tag{3.10}
$$

The general solution of (3.9) is just the covariant version of eq. (3.7) and thus corresponds to the columns of the matrix

$$
\mathbf{V} = \begin{pmatrix} X^0 & X^a & X^0\, e_a^\alpha\partial_\alpha\mathcal{F} & X^a\; e_a^\alpha\partial_\alpha\mathcal{F} - 2\mathcal{F}X^0\\ 0 & X^0\; e_\beta^a & X^0 e_a^\alpha\; \widehat{D}_\alpha\; \partial_\beta\mathcal{F} & X^a\; e_a^\alpha\; \widehat{D}_\alpha\; \partial_\beta\mathcal{F} - X^0\; \partial_\beta\mathcal{F}\\ 0 & 0 & (X^0)^{-1}\; e_a^\alpha & (X^0)^{-2}\; X^a\; e_a^\alpha\\ 0 & 0 & 0 & (X^0)^{-1} \end{pmatrix}\,. \tag{3.11}
$$

Here $e_\alpha^a = \partial_\alpha t^a(z)$ which satisfies $\widehat{D}_\beta e_\alpha^a = 0$. Furthermore, in arbitrary coordinates $\mathcal{F}$ is Kähler invariant and obeys

$$
\widehat{D}_\alpha\widehat{D}_\beta\widehat{D}_\gamma\mathcal{F} = (X^0)^{-2}W_{\alpha\beta\gamma}\qquad . \tag{3.12}
$$

The system (3.8) implies the following manifestly holomorphic equation for V:

$$
\widehat{D}_\alpha\widehat{D}_\beta(W^{-1})^{\widehat{\gamma}\rho\sigma}\widehat{D}_{\widehat{\gamma}}\widehat{D}_\sigma V = 0. \tag{3.13}
$$

Using eq. (A.33) one checks that the first row of (3.11) indeed coincides with $V \equiv (X^A, F_A)$. We conclude, therefore, that eq. (3.13) is the same as eq. (3.1), except that it is written in a manifestly holomorphic way.

As for one variable, the correspondence between eq. (3.13) and the linear system (3.9) is not unique. Indeed, (3.13) is invariant under gauge transformations (up to Kähler transformations) acting on $\mathbf{V}$ and A via

$$
S \;=\; \begin{pmatrix} *_{1\times1} & \mathbf{0} & \mathbf{0} & 0\\ * & *_{n\times n} & \mathbf{0} & \mathbf{0}\\ * & * & *_{n\times n} & \mathbf{0}\\ * & * & * & *_{1\times1} \end{pmatrix} \in B, \tag{3.14}
$$

which belong to a Borel subgroup B of $SL(2n+2,\mathbb{C})$.

It is easy to check that for one variable, the connection A in (3.10) can be gauge transformed to the form (2.25) that displays the W-generators. More precisely, under a symplectic transformation

$$
S \;=\; \mathrm{diag}\big(W^{-1/2}, W^{-1/2}, W^{1/2}, W^{1/2}\big) \tag{3.15}
$$

the connection A takes the form

$$\mathsf{A} = \begin{pmatrix} -\partial\widetilde{K} & 1 & 0 & 0 \\ 0 & \widehat{\Gamma} - \partial\widetilde{K} & 1 & 0 \\ 0 & 0 & -\widehat{\Gamma} + \partial\widetilde{K} & 1 \\ 0 & 0 & 0 & \partial\widetilde{K} \end{pmatrix} \tag{3.16}$$

where $\widetilde{K} = \widehat{K} + \frac{1}{2}\ln W = -\ln(X^0 W^{-1/2})$. To bring further $\widetilde{K}$ to the gauge (2.25) one obviously needs an additional $Sp(4)$ transformation that belongs to the nilpotent subgroup N. This transition from (3.16) to (2.25) is nothing but a Miura–transformation [40].

We have seen in section 2 that the Picard-Fuchs equation for one variable can invariantly be characterized by the vanishing of classical W-generators. The vanishing of w_3 was related to $\mathsf{A}_w \in sp(4)$. For many variables, we do not know how to characterize the differential equation (3.13) in terms of covariant quantities like w_n. But in analogy to the one–variable equation, we expect that the statement that corresponds to $w_3 = 0$ is just that $\mathsf{A}_\alpha \in sp(2n+2)$ in (3.10). Indeed, the gauge in which we wrote (3.10) is manifestly symplectic: one easily verifies that $Q\mathsf{A} = (Q\mathsf{A})^T$, where Q is the symplectic metric given in (A.25).

More generally, we expect that a multi-variable equation can invariantly be characterized by the subalgebra $g \subset sp(2n+2)$ in which the set of connections actually takes values, for given $W_{\alpha\beta\gamma}$ (just like for $n=1$ where the additional vanishing of w_4 implies that $\mathsf{A}_w \in sl(2)$). For large n, there exists obviously a large number of distinct possible subgroups. (Note that it is in general not easy to determine g, as the embedding in $sp(2n+2)$ is gauge dependent and thus not always obvious. One is missing a gauge–invariant criterion for many variables, in analogy to the vanishing of certain W-generators for one variable).

The solution vectors can accordingly be viewed as representations of $Sp(2n+2)$ (or of some subgroup G). The set of solution vectors when written as a matrix $\mathbf{V}$ can always be chosen in a way such that this matrix becomes a group element, by multiplying $\mathbf{V}$ with an appropriate constant matrix from the right. One can easily check that our choice of solution matrix (3.11) is indeed symplectic with respect to the metric (A.25). In this way, one can

regard $\mathbf{V}$ as a vielbein $V_{\hat\alpha}^{\hat A}$ with a well-defined symplectic action on both indices ($\hat A, \hat\alpha = 1, \dots, 2n+2$).⋆ Under coordinate and Kähler transformations $z \to \widetilde{z}(z)$, $K \to K + f(z) + \overline{f}(\overline{z})$, the matrix $\mathbf{V}$ transforms as follows:

$$V_{\hat\beta}{}^{\hat A}(\widetilde{z}) = S^{-1}_{\hat\beta}{}^{\hat\alpha}(z) V_{\hat\alpha}{}^{\hat B}(z) M_{\hat B}{}^{\hat A} \tag{3.17}$$

where S is the symplectic block diagonal matrix

$$S = \begin{pmatrix} e^{-f} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & e^{-f}\xi^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & e^f\xi & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & e^f \end{pmatrix} \in B\ , \tag{3.18}$$

with $\xi \equiv \xi^\alpha_\beta = \partial\widetilde{z}^\alpha/\partial z^\beta$. Furthermore, M is a constant matrix that can always be taken as an element of $Sp(2n+2)$. One easily infers form (3.10) that these transformations are nothing but gauge transformations of the holomorphic connections $\partial\widehat{K}$ and $\widehat{\Gamma}$:

$$\begin{aligned} \partial_\alpha\widehat{K} &\longrightarrow \partial_\alpha\widehat{K} + \partial_\alpha f \\ \widehat{\Gamma}_\alpha &\longrightarrow \xi^{-1}\widehat{\Gamma}_\alpha\xi + \partial_\alpha \ln\,\xi\ . \end{aligned} \tag{3.19}$$

This point of view allows us to also understand how global $Sp(2n+2)$ transformations acting on the index $\hat A$ induce local frame rotations acting on the index $\hat\alpha$: the local rotations are induced by the requirement that A_α stays in the gauge (3.10). More explicitly, symplectic transformations: $\widetilde{\mathbf{V}} = \mathbf{V}\cdot M$, which act in particular on the solution vector as

$$\begin{aligned} (\widetilde{X}^A, \widetilde{F}_A(\widetilde{X}^A)) &= (X^A, F_A(X^A))\cdot M \\ M &= \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in Sp(2n+2)\ , \end{aligned} \tag{3.20}$$

induce the following reparametrizations of special coordinates:

$$\widetilde{t}^a = \frac{A^a_B X^B + B^{aB}F_B}{A^0_B X^B + B^{0B}F_B}(t) \tag{3.21}$$

⋆ The index $\hat A$ corresponds to a symplectic basis of the Hodge bundle $\mathcal{H}$ and the index $\hat\alpha$ to the flat bundle $\mathcal{E}$ defined in ref. [22].

These reparametrizations induce local, compensating gauge transformations (3.19) with $f = \mathrm{Tr}(\ln\xi)$ and $\xi = \partial\widetilde{t^a}/\partial t^b$.

Note that the transformations (3.18) belong to the part of the Borel gauge group (3.14) that is not fixed by the gauge choice (3.10); that is, they lie in (the complexification of) the maximal compact subgroup $U(n)\times U(1)$ of $Sp(2n+2)$. This implies that the group element $\mathbf{V}$ can be thought of as an element of G/H, where $G\subset Sp(2n+2)$ and $H\subset U(n)\times U(1)$. More specifically, one can decompose [22,13]

$$\mathsf{A}_\alpha \;=\; \mathbb{\Gamma}_\alpha + \mathbb{C}_\alpha \;, \tag{3.22}$$

where the diagonal part, $\mathbb{\Gamma}_\alpha$, consists of the connections $\widehat{\Gamma}$ and $\partial\widehat{K}$ (which are flattened by special coordinates $t^a = X^a\,, X^0 = 1$). Furthermore, $\mathbb{C}_\alpha$ is the covariant version of (3.6) and generates an abelian, n-dimensional subalgebra of $sp(2n+2)$ that is nilpotent of order three: $\mathbb{C}_\alpha\mathbb{C}_\beta\mathbb{C}_\gamma\mathbb{C}_\delta = 0$. Thus, G is determined by the subalgebra of $sp(2n+2)$ in which $\mathbb{C}_\alpha$ takes values, and H is determined by the subgroup of $U(n)\times U(1)$ that is gauged by $\mathbb{\Gamma}_\alpha$ [13].

More precisely, $\mathbf{V}$ is an element of G^c/B (which is, essentially, isomorphic to G/H), where G^c is the complexification of G and B the Borel subgroup (3.14), which contains the complexification of H. From this viewpoint one can easily make contact to supersymmetric σ-models on moduli spaces. According to [43], Kähler potentials for homogeneous Kähler manifolds G/H can be written in terms of holomorphic CCWZ type coset representatives $L\in G^c/B$ as arbitrary functions of $K_0\equiv vLQL^\dagger v^\dagger$. Here, L transforms under global G transformations as: $L(z)\,g = S(z)L(z)$, where $g\in G$ and $S\in B$. Furthermore, Q is the metric of G and v denotes an isotropy vector, which is left invariant under S (up to a $U(1)$ factor, which corresponds to Kähler transformations). Note that K_0 is manifestly invariant under global G and under local S transformations (except for the Kähler transformations). Taking for Q the symplectic metric (A.25), $v = (1,0,0\ldots,0)$ and $L=\mathbf{V}$, the logarithm of $(-i)K_0$ gives precisely the Kähler potential (1.2) of special geometry: $v\mathbf{V}Q\mathbf{V}^\dagger v^\dagger = -(X^A\overline{F}_A - \overline{X}^A F_A)$.

In the generic case, $G/H = Sp(2n+2)/U(n)\times U(1)$, but the moduli space in which $\mathbf{V}(z)$ actually takes values is a complicated subvariety of this space. However, there are special cases where G and H are effectively smaller subgroups; one example are the theories with cubic F–function where the moduli spaces are directly given by G/H. For instance for $n=1$, the generic moduli space is some complicated one-dimensional submanifold of $\frac{Sp(4,\mathbb{R})}{U(1)^2}$ whose complex dimension is four. But for constant coupling W (and for $c_{AB}=0$ in (2.17)), the moduli space in which $\mathbf{V}$ takes values is the one–dimensional submanifold $G/H = \frac{SL(2,\mathbb{R})}{U(1)}$. The special geometry of cubic F–functions are further discussed below in Appendix C.

3.2 Non–holomorphic Picard–Fuchs equations

In this section we establish the relationship between the first order systems (1.4) and (3.8). Let us first note that the gauge group (3.14) can also be extended to non–holomorphic gauge transformations $\mathcal{S}=\mathcal{S}(z,\overline{z})$ that leave V invariant. The point is that eqs. (1.4) and (3.8) are precisely related by such a non–holomorphic gauge transformation. That is, the non–holomorphicity of the supergravity equations (1.1) and (1.4) is a gauge artifact, corrsponding to the fact that all quantities in special geometry are determined entirely in terms of holomorphic quantities.

More specifically, one can rewrite the non–holomorphic system (1.4) in first order form

$$\mathcal{D}_\alpha\,\mathbf{U} \;\equiv\; (\mathbb{1}\partial_\alpha - \mathcal{A}_\alpha)\mathbf{U} \;=\; 0\;, \tag{3.23}$$

where $\mathbf{U} = (V, U_\alpha, \overline{U}_{\overline{\alpha}}, \overline{V})^T$ and

$$\mathcal{A}_\alpha \;=\; \begin{pmatrix} -\partial_\alpha K & \delta^\beta_\alpha & 0 & 0 \\ 0 & -\delta^\beta_\gamma\partial_\alpha K + \Gamma^\beta_{\gamma\alpha} & -iC_{\alpha\beta\gamma}g^{\gamma\overline{\gamma}} & 0 \\ 0 & 0 & 0 & g_{\alpha\overline{\beta}} \\ 0 & 0 & 0 & 0 \end{pmatrix}\;. \tag{3.24}$$

In addition $\mathbf{U}$ also satisfies

$$\mathcal{D}_{\overline{\alpha}}\,\mathbf{U} \;\equiv\; (\mathbb{1}\partial_{\overline{\alpha}} - \mathcal{A}_{\overline{\alpha}})\mathbf{U} \;=\; 0\;, \tag{3.25}$$

where

$$\mathcal{A}_{\overline{\alpha}} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ g_{\overline{\alpha}\beta} & 0 & 0 & 0 \\ 0 & iC_{\overline{\alpha}\overline{\beta}\overline{\gamma}}g^{\overline{\gamma}\gamma} & -\delta^{\overline{\beta}}_{\overline{\gamma}}\partial_{\overline{\alpha}}K + \Gamma^{\overline{\beta}}_{\overline{\gamma}\overline{\alpha}} & 0 \\ 0 & 0 & \delta^{\overline{\beta}}_{\overline{\alpha}} & -\partial_{\overline{\alpha}}K \end{pmatrix}\;. \tag{3.26}$$

It is easy to verify that as a consequence of (1.1) the connections $\mathcal{A}_\alpha$ and $\mathcal{A}_{\overline{\alpha}}$ have vanishing curvature [22]*:

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] = [\mathcal{D}_{\overline{\alpha}}, \mathcal{D}_{\overline{\beta}}] = [\mathcal{D}_\alpha, \mathcal{D}_{\overline{\beta}}] = 0 \,. \tag{3.27}$$

It follows that via non–holomorphic transformations $\mathcal{S}(z,\overline{z})$ that leave V invariant ($\mathcal{S}V = V$), one can gauge away $\mathcal{A}_{\overline{\alpha}}$ and make $\mathcal{A}_\alpha$ purely holomorphic. As a consequence of eq. (3.27) one can go to a gauge where

$$\mathcal{A}_{\overline{\alpha}} = \mathcal{S}\partial_{\overline{\alpha}}\mathcal{S}^{-1} \,. \tag{3.28}$$

This implies

$$\partial_{\overline{\alpha}}(\mathcal{S}\mathbf{U}) = 0 \qquad \text{and} \qquad \partial_{\overline{\alpha}}\left[\mathcal{S}\mathcal{A}_\alpha\mathcal{S}^{-1} - \mathcal{S}\partial_\alpha\mathcal{S}^{-1}\right] = 0 \,, \tag{3.29}$$

so that the non–holomorphic system (3.23) becomes the holomorphic system (3.9) with

$$\mathsf{A} = \mathcal{S}\mathcal{A}_\alpha\mathcal{S}^{-1} - \mathcal{S}\partial_\alpha\mathcal{S}^{-1} \,, \qquad \mathbf{V} = \mathcal{S}\,\mathbf{U} \,, \tag{3.30}$$

which displays a residual gauge symmetry of holomorphic transformations. Of course, one could also have chosen gauge transformations $\overline{\mathcal{S}}$ that leave the lowest row of $\mathbf{V}$, ie. $\overline{V}$, invariant; in that instance one would have produced a purely anti–holomorphic connection $A_{\overline{\alpha}}$. The point is that there is no invariant subspace with respect to both $\mathcal{S}$ and $\overline{\mathcal{S}}$, so that the connection cannot be completely flattened.

3.3 Singular Picard-Fuchs systems

In the previous considerations we have assumed that the matrices $(W_\alpha)_{\beta\gamma} \equiv W_{\alpha\beta\gamma}$ are invertible for all α. It is interesting to find the implications of degenerate fourth order partial differential equations (3.13).

* Vice versa [22], one can start from a covariantly constant basis $(V, U_\alpha, U_{\overline{\alpha}}, \overline{V})$ of a flat $Sp(2n+2,\mathbb{R})$ vector bundle $\mathcal{E}$ with connection $\mathcal{A}$ and derive the fundamental identities (3.23) and (3.25) of special geometry.

i) The simplest situation is when $W_{\alpha\beta\gamma} \equiv 0$. Then, from (3.8) or alternatively from (1.4) one can see that the Picard-Fuchs identities become of second order

$$\widehat{D}_\alpha \widehat{D}_\beta V = 0 \,. \tag{3.31}$$

In special coordinates we get

$$\partial_a \partial_b V = 0 \tag{3.32}$$

with solutions $(1\,, t^a)$. This corresponds to $\mathcal{F} = (t^a)^2 - \frac{1}{2}$, and implies that the symplectic connection (3.10) becomes block diagonal in two $(n+1)^2$ blocks. The matrices $\mathbb{C}_a$ are nilpotent of order two ($\mathbb{C}_a\mathbb{C}_b = 0$), and the solution matrix is given by

$$\mathbf{V} = e^{t^a C_a} \,. \tag{3.33}$$

The moduli space is (locally) $G/H = U(1,n)/U(1) \times U(n)$ with the embedding $2n+2 = (n+1) \oplus \overline{(n+1)}$ of $U(1,n)$ in $Sp(2n+2)$.

ii) We now consider the situation in which $W_{\alpha\beta\gamma}$ does not vanish but is degenerate. This is best discussed in special coordinates where $\Gamma_\alpha = 0$ and $\mathsf{A}_\alpha = \mathbb{C}_\alpha$. Let us first consider $W_{ijk} = 0$ for some subset of indices i, j, k, and also $W_{iab} = 0, W_{ijb} = 0$. Then, assuming that the remaining couplings W_{abc} give rise to an invertible matrix $(W^a)^{bc}$, we have two sets of decoupled equations

$$\begin{aligned} \partial_i \partial_j V &= 0 \\ \partial_a \partial_b (W^{-1})^{cde} \partial_c \partial_d V &= 0 \end{aligned} \tag{3.34}$$

with solutions given by the prepotential

$$\mathcal{F}(t^i, t^a) = c + (t^i)^2 + \mathcal{F}(t^a) \,. \tag{3.35}$$

To write these equations in arbitrary coordinates we note

$$\widehat{D}_\alpha V_\beta = W_{\alpha\beta\gamma} V^\gamma \equiv \partial_\alpha t^A \partial_\beta t^B \partial_\gamma t^C W_{ABC} V^\gamma \tag{3.36}$$

(where A, B, C here stands for either a, b, c or i, j, k). Equivalently, multiplying by the inverse vielbeins $e^\alpha_A \equiv (e^A_\alpha)^{-1} = (\partial_\alpha t^A)^{-1}$ one gets:

$$e^\alpha_A e^\beta_B \widehat{D}_\alpha V_\beta = W_{ABC} V^\gamma e^C_\gamma \tag{3.37}$$

Supposing $W_{ABC} = W_{abc}, W_{ijk} = W_{aij} = W_{abj} = 0$, we get

$$\begin{aligned} e_i^\alpha e_j^\beta \widehat{D}_\alpha \widehat{D}_\beta V &= 0 \ , \\ \widehat{D}_\rho \widehat{D}_\mu (W^{-1})^{abc} e_a^\alpha e_b^\beta \widehat{D}_\alpha \widehat{D}_\beta V &= 0 \ . \end{aligned} \tag{3.38}$$

Using $\widehat{D}_\alpha e_\beta^a = 0$, the last equation can also be written as

$$e_d^\rho e_e^\mu e_a^\alpha e_b^\beta \widehat{D}_\rho \widehat{D}_\mu (W^{-1})^{abc} \widehat{D}_\alpha \widehat{D}_\beta V = 0 \, . \tag{3.39}$$

In (3.37) and (3.38) all moduli coordinates appear, but the structure of the equations is such that they become indeed decoupled by making a coordinate transformation. The coordinate independent statement on the Yukawa couplings is

$$e_i^\alpha e_j^\beta e_k^\gamma W_{\alpha\beta\gamma} = 0 \tag{3.40}$$

for a subset $(i, j, k \subset A, B, C)$; $A, B, C = 1, \cdots, n$.

iii) Two more special cases are worth of mention. One corresponds to two non-vanishing Yukawa couplings for different sets of indices $W_{ijk} \neq 0, W_{abc} \neq 0$ with $W_{iab} = W_{ija} = 0$. In this case one gets two sets of fourth order equations of the type (3.39).

iv) The other case is when $W_{ijk} = 0, W_{ija} = 0$ but $W_{iab} \neq 0$. Here the matrix W_i is invertible in the subblock (a, b) and the matrix $W_{a,BC}$ is fully invertible. The prepotential, in special coordinates, is of the form

$$\mathcal{F}(t^i, t^a) = C_{ij} t^i t^j + h_i(t^a) t^i + h(t^a) \qquad C_{ij} = \text{const.} \tag{3.41}$$

In this case we get three sets of decoupled differential equations

$$\begin{aligned} e_i^\alpha e_j^\beta \widehat{D}_\alpha \widehat{D}_\beta V &= 0 \\ e_a^\mu e_A^\rho \widehat{D}_\alpha \widehat{D}_\beta (W^{-1})^{aAB} \widehat{D}_\mu \widehat{D}_\rho V &= 0 \\ e_a^\rho e_i^\mu \widehat{D}_\alpha \widehat{D}_\beta (W^{-1})^{iab} \widehat{D}_\rho \widehat{D}_\mu V &= 0. \end{aligned} \tag{3.42}$$

The purpose of these exercises was to find the differential equations for decoupled chiral rings. In special coordinates, this reflects in a simple additive structure of F. On the other hand, the corresponding Kähler metrics do by no means have the structure of direct product manifolds, and rather are quite complicated. This shows that special geometry is most easily characterized by F and not by the geometry of the underlying manifold.

4 Relation to Calabi-Yau manifolds and topological field theory

The discussion of sections 2 and 3 has been completely general and without any reference to Calabi-Yau moduli spaces or to more general $c = 9$, $(2, 2)$ superconformal field theories. In this section, we relate our discussion to the special case of Calabi-Yau manifolds and to topological Landau-Ginzburg theories. We understand that part of the material of this section is well-known (see, for example, [44,22,45,46,13,28,8]), but we think it is important to give the precise relationship to special geometry. This relationship is useful for practical computations.

We like first to review some properties of the period matrix [44,45]. For those special geometries for which there exists an underlying Calabi-Yau space $\mathcal{M}$, the sections $V, U_\alpha, \overline{U}_{\overline{\beta}}$ and $\overline{V}$ in the non-holomorphic system (1.4) can be viewed as basis elements of the third real cohomology of $\mathcal{M}$, that is, of $H_{\overline{\partial}}^{(3,0)}(\mathcal{M}, \mathbb{R})$, $H_{\overline{\partial}}^{(2,1)}(\mathcal{M}, \mathbb{R})$, $H_{\overline{\partial}}^{(1,2)}(\mathcal{M}, \mathbb{R})$ and $H_{\overline{\partial}}^{(0,3)}(\mathcal{M}, \mathbb{R})$, respectively. Furthermore, the solutions (X^A, F_A) of the Picard-Fuchs equation (3.13) are just the periods of the holomorphic three–form, Ω [44,45,23]:

$$X^A \ = \int_{\gamma_A} \Omega \ , \qquad F_B \ = \int_{\gamma_B} \Omega \tag{4.1}$$

$(A, B = 0, \ldots, n$ where $n \equiv h^{2,1})$. Here, γ_A, γ_B are basis cycles of $H_3(\mathcal{M}, \mathbb{R})$. More generally, the complete solution matrix $V_{\hat\alpha}{}^{\hat A}$ of the first order system (3.9) can be interpreted as the period matrix of $\mathcal{M}$,

$$\Pi_{\hat\alpha}{}^{\hat A} \ = \begin{pmatrix} \int_{\gamma_{\hat A}} \Omega \\ \int_{\gamma_{\hat A}} \chi_\alpha \\ \int_{\gamma_{\hat A}} \overline{\chi}_{\overline{\beta}} \\ \int_{\gamma_{\hat A}} \overline{\Omega} \end{pmatrix} \ , \qquad \hat\alpha, \hat A = 1, \ldots, (2n+2) \ . \tag{4.2}$$

It is well-known [44] that the period matrix is defined only up to local gauge transformations,

$$\Pi \ \sim \ S\,\Pi \ , \qquad S = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix} \ \in B \ , \tag{4.3}$$

and this is precisely the gauge symmetry (3.14) of the first order system. Thus, we can represent the period matrix also in the holomorphic gauge (3.11), where the non-holomorphic sections $V, U_\alpha, \overline{U_{\overline{\alpha}}}, \overline{V}$ are replaced by holomorphic basis elements $(V, V_\alpha, V^\alpha, V^0)$ of H^3.

In addition, the period matrix is equivalent under conjugation by an integral matrix, Λ: $\Pi \sim \Pi\Lambda$. These transformations $\Lambda \in Sp(2n+2, \mathbb{Z})$, which correspond to changes of integral homology bases, preserve the symplectic bilinear intersection form Q of $H_3(\mathcal{M}, \mathbb{Z})$, that is: $\Lambda\, Q\, \Lambda^T = Q$ (the subset of these transformations that leave F invariant up to redefinitions constitute the "duality group", as we will see in the sequel). This intersection form is at the origin of the symplectic structure of the period matrix. More precisely, denoting the $(n+1) \times (2n+2)$-dimensional submatrix ${\Pi_\ell}^{\hat{A}}$, $\ell = 0, \ldots, n$ by $\widehat{\Pi}$, then one has in general* [44]

$$\widehat{\Pi}\, Q\, \widehat{\Pi}^T \;=\; 0\;. \tag{4.4}$$

This equation is analogous to Riemann's bilinear identity for period matrices of 2d surfaces, and is satisfied for $\Pi \in Sp(2n+2)$. This is indeed a general property of the solution matrix $\mathbf{V}$ (3.11) in special geometry†.

On the other hand, in special geometry there is no intrinsic notion of homology cycles. Rather, the symplectic structure arises from the appearance of Q in the Kähler potential,

$$K = -\ln\langle\, \Omega \,|\, \overline{\Omega}\, \rangle \;=\; -\ln[V\,(-iQ)\,V^\dagger]\;, \tag{4.5}$$

where $\langle\, x \,|\, y\, \rangle \equiv \int_{\mathcal{M}} x \wedge y$ and V is equal to the first row of the symplectic matrix $\mathbf{V}$. As we have seen in the previous sections, the existence of such a solution matrix is guaranteed by the fact that the connection in the first order system

* For d-dimensional complex manifolds, the invariance group of Q is $Sp(b^d)$ if d is even, and is equal to $SO(b^d_+, b^d_-)$ is d is odd [44]. As a consequence of this and according to our discussion in section 2, it follows that in the differential equations for one variable one necessarily has $w_n \equiv 0$ for odd n [13]. This generalizes $w_3 = 0$ for Calabi-Yau spaces.

† Of course, not every solution matrix $\mathbf{V}$ of special geometry needs to correspond to a period matrix of some Calabi-Yau space; this is a variant of the Schottky problem. Thus, special geometry is more general than compactification on Calabi–Yau manifolds.

(3.9) is symplectic. This matrix equation, in turn, is a consequence of $W_{\alpha\beta\gamma} = \partial_\alpha X^A \partial_\beta X^B \partial_\gamma X^C F_{ABC}$. It is this identity that is ultimately responsible for the symplectic structure of special geometry.

Of particular physical interest are the Yukawa couplings, $C_{\alpha\beta\gamma} = e^K W_{\alpha\beta\gamma}$. According to [45,22], for a given Kähler potential (4.5) the holomorphic sections $W_{\alpha\beta\gamma}$ can be written as:

$$W_{\alpha\beta\gamma} \;=\; \langle \Omega | \partial_\alpha \partial_\beta \partial_\gamma \Omega \rangle\;. \tag{4.6}$$

It is crucial to note that equations (4.5) and (4.6) determine the gauge of $W_{\alpha\beta\gamma}$ in terms of the gauge of K, so that there is no ambiguity in the physical couplings $C_{\alpha\beta\gamma}$. One can easily check that the derivatives in (4.6) can be replaced by covariant derivatives for free, reflecting the fact that $W_{\alpha\beta\gamma}$ is Kähler and reparametrization covariant. From the first order systems (3.23) or (3.9) it is clear that differentiation by ∂_α on H^3 is equivalent to multiplication by the matrix A_α. Thus, the holomorphic couplings can be written as

$$W_{\alpha\beta\gamma} \;=\; \left(\mathsf{A}_\alpha \mathsf{A}_\beta \mathsf{A}_\gamma\right)_V^{\;\overline{V}} \langle V | \overline{V} \rangle\;. \tag{4.7}$$

Considering the form of A_α in either the non-holomorphic gauge (3.24) or in the holomorphic gauge (3.10), it is easy to see that eq. (4.7) is indeed identically satisfied.

Let us now discuss how $W_{\alpha\beta\gamma}$ and K can be computed explicitly. One method is to evaluate the period integrals (4.1), using $X^A F_A = 2F$ to obtain F. This is how the Yukawa couplings for the quintic have first been computed in [6]; this technique was subsequently generalized [9,17,18,47,19] and now allows to compute the Yukawa couplings for a large variety of Calabi-Yau manifolds. Another method is to solve [7,8] the Picard-Fuchs equations, and this is the method we will focus on below. These equations, though, just represent identities ultimately expressing the fact that $W_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma F$, and depend explicitly on the unknowns $W_{\alpha\beta\gamma}$. Thus, one needs additional information in order to pin down the explicit form of these equations. This additional input makes use of the fact that the period matrix can be represented in a very specific way.

To be more precise, consider first the perturbed, quasi-homogeneous potential

$$\mathcal{W}(y_i,\mu_\alpha) \;=\; \mathcal{W}_0(y_i) - \sum \mu_\alpha\, p_\alpha(y_i)\ , \qquad\qquad \alpha = 1,\ldots,h^{2,1}\ , \tag{4.8}$$

where $\mathcal{W}=0$ describes the Calabi-Yau manifold in question (for simplicity, we restrict our discussion to hypersurfaces in weighted projective 4-space). Above, $p_\alpha(y_i)$ denote the marginal operators (which are polynomials in the homogeneous coordinates y_i) in some given basis, and the dimensionless moduli μ_α are certain functions of the flat coordinates t^a. As is well-known, $\mathcal{W}$ can be viewed as the superpotential of a Landau-Ginzburg theory that describes the underlying $N=2$ superconformal field theory [48], but this interpretation is not necessary in the present context.

The non–trivial point is that the period matrix can be represented, in a particular gauge, as follows [44]:

$$\Pi_{\hat\alpha}{}^{\hat A} \;=\; \int_{\gamma_{\hat A}} \frac{\phi_{\hat\alpha}(y_i)}{\mathcal{W}^{\ell(\hat\alpha)}(y_i,\mu_\alpha)}\omega\ . \tag{4.9}$$

Here, the homology cycle $\gamma_{\hat A}$ is a basis element of $H_3(\mathcal{M},\mathbb{R})$, ω an appropriate volume form and $\ell(\hat\alpha)$ depends on the degree of the homogeneous polynomial $\phi_{\hat\alpha}(y_i)$. In general, these $\phi_{\hat\alpha}(y_i)$ can be any basis of the local ring $\mathcal{R}$ of $\mathcal{W}$, but we restrict here only to those polynomials that represent the third cohomology of the Calabi-Yau space. They generate a subring which we denote by $\mathcal{R}^{(3)}$. More specifically, we choose $\mathcal{R}^{(3)} = \{\phi_{\hat\alpha}\} = \{1, p_\alpha, p^\beta, \rho\}$, where p_α are the marginal operators in (4.8), ρ is the unique top element of $\mathcal{R}$, and p^β can be defined such that $p_\alpha p^\beta = \delta^\beta_\alpha \rho$. Clearly, $\phi_{\hat\alpha} = 1, p_\alpha, p^\beta, \rho$ are associated with differential forms belonging to $H^{(3,0)}, H^{(2,1)}, H^{(1,2)}, H^{(0,3)}$, respectively.

For example, for the quintic discussed in [6] with $\mathcal{W} = \sum_{i=1}^5 (y_i)^5 - \mu X$ (where $X \equiv y_1y_2y_3y_4y_5$), the subring $\mathcal{R}^{(3)}$ consists of elements $\phi_{\hat\alpha} = X^{\hat\alpha}$, $\hat\alpha = 0,\ldots,3$, which are associated with $H^{(3-\hat\alpha,\hat\alpha)}$, respectively.

The period matrix (4.9) identically satisfies the following holomorphic, first order "Gauß-Manin" system:

$$\Big[\, \mathbb{1}\frac{\partial}{\partial\mu_\alpha} \;-\; \mathsf{A}_\alpha(\mu)\Big]\,\Pi \;=\; 0\ , \quad \text{where} \quad \Pi \;=\; \begin{pmatrix} \int \frac{1}{\mathcal{W}}\omega \\ \int \frac{p_\beta}{\mathcal{W}^2}\omega \\ \int \frac{p^\gamma}{\mathcal{W}^3}\omega \\ \int \frac{\rho}{\mathcal{W}^4}\omega \end{pmatrix}\ , \tag{4.10}$$

and $\mathsf{A}_\alpha = \mathbb{\Gamma}_\alpha + \mathbb{C}_\alpha$, with

$$\mathbb{\Gamma}_\alpha = \begin{pmatrix} * & 0 & & \\ * & * & 0 & \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}\ , \qquad \mathbb{C}_\alpha = \begin{pmatrix} & 1 & & \\ & & (W^{(p)}_\alpha)_{\beta\gamma} & \\ & & & 1 \\ & & & \end{pmatrix}\ . \tag{4.11}$$

This system can be seen as a gauge and coordinate transform of the holomorphic special geometry system (3.9) (and also of the non-holomorphic system (3.24)). The matrices $\mathbb{C}_\alpha$ are the structure constants of the subring $\mathcal{R}^{(3)}$, and the couplings $(W^{(p)}_\alpha)_{\beta\gamma}$ are determined by simple polynomial multiplication modulo the vanishing relations, ie., by $p_\alpha p_\beta = W^{(p)}_{\alpha\beta\gamma} p^\gamma$ mod $\nabla\mathcal{W}$. The crucial point is that also the components of $\mathbb{\Gamma}_\alpha$ can be easily computed* directly from $\mathcal{W}$ (this is explained in detail in [7,8]).

One way to solve the system (4.11) is to go to flat coordinates $t^a \equiv X^a/X^0$ where the Gauß-Manin connection vanishes. As was shown in detail in [8], imposing this condition gives a differential equation that determines explicitly the dependence of the Landau-Ginzburg couplings μ_α on the t^a. (The precise form of this complicated, non-linear differential equation is not important here and can be inferred from [8].) In such flat coordinates and in an appropriate gauge, the first order system takes the form (3.6). In going to (3.6), we implicitly compute eq. (4.7)

$$W_{abc}(t) \;=\; W^{(p)}_{\alpha\beta\gamma}\,\frac{\partial\mu_\alpha}{\partial t^a}\frac{\partial\mu_\beta}{\partial t^b}\frac{\partial\mu_\gamma}{\partial t^c}\,\langle\, V\,|\,V^0\,\rangle\ , \tag{4.12}$$

* They arise from the $\nabla\mathcal{W}$ piece, by partial integration. Note that the above expansion of $p_\alpha p_\beta$ is in general by no means unique, reflecting the gauge freedom in (4.10).

where $V = \int \frac{1}{\mathbf{W}}\omega$ and $V^0 = \int \frac{\rho}{\mathbf{W}^4}\omega$. One can view (4.12) as a change from a topological basis (with indices α, β, γ) to a flat basis (with indices a, b, c), and $\langle V \,|\, V^0 \rangle$ as a change of Kähler gauge. It can be inferred from [45] that

$$\langle V \,|\, V^0 \rangle \;=\; \int_{\Gamma_5} \frac{\rho}{\partial_1 \mathcal{W} \ldots \partial_5 \mathcal{W}} d^5x \;\equiv\; \langle\langle \rho \rangle\rangle \;, \tag{4.13}$$

where Γ_5 is the direct product of five one-dimensional contours that wind around the five curves $\partial_i \mathcal{W} = 0$, and $\langle\langle \;\; \rangle\rangle$ denotes the Grothendieck residue [10,49]. It has the property: $\langle\langle H \rangle\rangle = 1$ (up to a constant), where H is the Hessian of the superpotential, $H \equiv \det[\partial_{y_i}\partial_{y_j}\mathcal{W}(y_k)]$. In general, ρ and H differ by some holomorphic function and vanishing relations, $\rho = f_H H \mod \nabla\mathcal{W}$, so that

$$\langle V \,|\, \overline{V} \rangle \;=\; \langle\langle \rho \rangle\rangle \;=\; f_H \;. \tag{4.14}$$

For example, for the quintic: $W^{(p)}_{111} = 1$ and $f_H \sim \frac{1}{1-\mu^5}$. It follows $W_{111} \sim \frac{(\mu')^3}{1-\mu^5}$, which is the result of [6] (in a particular gauge). Note that $W \sim \frac{1}{1-\psi^5}$ can be inferred directly from the specific form (2.6) of the Picard–Fuchs equation for one variable: up to a change of basis, W is given directly in terms of the coefficient of the 3rd derivative,

$$a_3 = -W^{-1}\frac{dW^{-1}}{dz} \;. \tag{4.15}$$

In topological Landau-Ginzburg theory [10,49] one considers three–point correlators:

$$\langle\langle \Phi_a\Phi_b\Phi_c \rangle\rangle \;=\; W^{(top)}_{abc} \langle\langle H \rangle\rangle \;, \tag{4.16}$$

where $\langle\langle \;\; \rangle\rangle$ has exactly the same meaning as in (4.13) and where the flat fields are defined [10] by: $\Phi_a(y_i, t^b) = -\frac{\partial}{\partial t^a}\mathcal{W}(y_i, \mu(t^b))$. Referring back to the form of the superpotential (4.8), one quickly sees that one indeed computes here absolutely the same thing as in (4.12), that is†: $W^{(top)}_{abc} \equiv W_{abc}$, and the Kähler potential in the corresponding gauge is: $K = -\log\langle V | \overline{V} \rangle$.

† A similar result was obtained in [50], but the precise relation of $W^{(top)}_{abc}$ to special geometry remained unclear.

This is of course as expected, since also $W^{(top)}_{abc}$ can be represented as triple derivative of some function F [10]. One might think that this fact already proves the equality of $W^{(top)}_{abc}$ and W_{abc} of special geometry, defined in (4.6). However in [27] it was shown that there generally exist at least two different coordinate systems where $W_{\alpha\beta\gamma} = \partial_\alpha\partial_\beta\partial_\gamma F$ with two different and inequivalent Kähler potentials that solve the defining equations (1.4) of special geometry. Given this potential ambiguity, we feel, therefore, more comfortable to display explicitly the relationship between the couplings $W^{(top)}_{abc}$ computed in topological field theory on the one hand, and W_{abc} and K of special geometry on the other.

Note that rescaling $\mathcal{W} \to e^{f(t)}\mathcal{W}$ gives an equivalent superpotential. From (4.9) we see that this just amounts to a Kähler transformation, $\Omega \to e^{-f}\Omega$. Therefore the correct way to specify flat fields in the presence of moduli is

$$\begin{aligned} \Phi_\alpha(y_i, t^b) \;=\; -\widehat{\nabla}_\alpha\, \mathcal{W}(y_i, t^b) \;\equiv\; -\big[\,\partial_\alpha - \partial_\alpha \widehat{K}\,\big]\,\mathcal{W}(y_i, t^b) \;, \\ \text{where} \;\; \partial_\alpha \widehat{K} \equiv -\partial_\alpha \log X^0 \;, \end{aligned} \tag{4.17}$$

such that $\Phi_\alpha \to e^f \Phi_\alpha$. This transformation behavior is actually required for consistency of (4.16) as $W_{abc} \to e^{-2f} W_{abc}$ and $H \to e^{5f}$.

5 Target space duality and monodromy properties of Picard-Fuchs equations

In this chapter we briefly review the monodromy properties of the Picard–Fuchs system when the number of moduli is equal to one. We consider two examples, the torus (1–dimensional Calabi-Yau manifold) and the quintic Calabi-Yau manifold (worked out by Candelas et al. [6]). This pedagogical review also allows us to make some subsequent remarks on the discrete translational symmetry of the special coordinates t^a, for any number of moduli. Furthermore, recalling that the map $t^a = t^a(\psi)$ describes the mirror map between the Calabi–Yau manifold and its mirror, we are able, making use of the special geometry form of the Picard–Fuchs system, to derive the general algebraic structure of the monodromy generators around $t^a = i\infty$; this corresponds to the large radius limit of the Calabi–Yau manifold. This puts into evidence that the monodromy around such points is essentially determined by the intersection numbers of the Calabi–Yau manifold.

Let us recall that the Picard–Fuchs system as determined by special geometry can be written as a Gauss–Manin system given by equations (3.9)–(3.10). The data appearing in (3.10) are the Yukawa couplings $W_{\alpha\beta\gamma}$ and the holomorphic connections $\partial_\alpha \widehat{K}$ and $\widehat{\Gamma}^\alpha_{\beta\gamma}$. Since we do not know these data a priori, we have to resort to the algorithms described in [7,8], (originally due to Dwark and Griffiths), which allow us to detemine the Picard–Fuchs system once the defining polynomial equation $W=0$ of the Calabi–Yau manifold is given. Notice, however, that once such equations have been obtained, in principle the Yukawa couplings $W_{\alpha\beta\gamma}$ can be read out from the coefficients of the fourth derivatives, by comparing the actual equations with the special geometry equations (3.13). In the following we concentrate on the global structure of the moduli space, which is encoded in the monodromy properties of the Picard–Fuchs system.

Following [8], let us denote by Γ the target space duality group (quantum modular group) and by Γ_W the invariance group of the superpotential $W(y^i, \psi^a)$. Γ_W consists of these diffeomorphisms of the moduli ψ^a which leave $W=0$ invariant except for a (quasi)–homogeneous change of the coordinates of CP_{d+1} :

$$W(y^i, \psi^\alpha) = 0 \qquad \underset{\Gamma_W}{\longrightarrow} \qquad W(\widetilde{y}^i(y); \widetilde{\psi}^a(\psi)) = 0 \ , \tag{5.1}$$

where $\widetilde{y}^i = U^i_j y^j$ and i,j run over all chiral fields with same $U(1)$ charge. Alternatively, one may define Γ_W as the group of diffeomorphisms of ψ^a that leave the Picard–Fuchs equations invariant up to rescaling.

Moreover, let us denote by Γ_M the monodromy group of the Picard–Fuchs system. To define it in the simplest way, we restrict our attention to the case of one single modulus, where the Picard–Fuchs equations are ordinary differential equations (the general definition of the monodromy group for n moduli is given in the sequel).

If we denote by $(f_1(z), ..., f_n(z))$ a basis of solutions of the differential equation at a point z, then by analytically continuing $(f_1, ..., f_n)$ along a closed

loop around a singularity z_1 of the equation we arrive at a new solution at z_1. This must be expressible as a linear combination of the basis $(f_1, ..., f_n)$:

$$(f_1, ..., f_n) \ \rightarrow \ (\widehat{f}_1, ..., \widehat{f}_n) = (f_1, ..., f_n) \ A_{z_1} \ , \tag{5.2}$$

where the $n \times n$ non-singular matrix A_{z_1} characterizes the monodromy around z_1. If the equation has r singular points we obtain r monodromy matrices $A_{z_1}, ..., A_{z_r}$, and if we compose closed loops around z_i and z_j in the usual way it is clearly seen that to the loop $\gamma_i \circ \gamma_j \equiv \gamma_{ij}$ encircling z_i and z_j corresponds the monodromy matrix $A_{z_j} \cdot A_{z_i}$. More generally $A_{z_1}, ..., A_{z_r}$ generate a group, the monodromy group Γ_M of the differential equation. (The inverse $A^{-1}_{z_i}$ is the matrix obtained by running around z_i in the opposite direction, and $\mathbf{1}$ corresponds to a circuit contractible to a point.)

For one modulus, it seems that the full target space duality group Γ can in general be obtained by extending the monodromy group Γ_M with the LG superpotential invariance group, Γ_W [8]. In the remainder of this section, we will demonstrate this reconstruction of Γ for two examples with one modulus. In the subsequent section, we will give a computation of Γ for a model with two moduli, which serves as a counter example: here the duality group will not be the semi-direct product of Γ_M with Γ_W, but rather a central extension of Γ_M. For further examples with two moduli, see [17,19].

The examples with one modulus are:

1) the 1–dimensional torus described by a cubic polynomial in CP_2:

$$W = \frac{1}{3}(y_1^3 + y_2^3 + y_3^3) - \psi\, y_1 y_2 y_3 \ = \ 0 \tag{5.3}$$

2) the 3–dimensional Calabi–Yau manifold described by a quintic polynomial in CP_4:

$$W = \frac{1}{5}(y_1^5 + y_2^5 + y_3^5 + y_4^5 + y_5^5) - \psi\, y_1 y_2 y_3 y_4 y_5 \ = \ 0 \ . \tag{5.4}$$

Above, $y^i, i = 1, ..., d+2$ are homogeneous coordinates in CP_{d+1} (d being the complex dimension), and ψ is a complex structure modulus. Note that while for the torus the space of complex structure deformations is one–dimensional, it is 101–dimensional for the quintic. The distinguished 1–dimensional subspace defined by (5.4) preserves the permutation symmetry among the coordinates, and consequently $\{1, y_1...y_5, (y_1...y_5)^2, (y_1...y_5)^3\}$ forms a closed subring of the ring $\mathcal{R}^{(3)}$ that describes the three-froms. This symmetry is the underlying reason why the Picard–Fuchs equation will be only of fourth and not of higher (generically 204th) order.

Let us start with the torus. Using the simple algorithm described in [7,8], one obtains from (5.3) the following Picard–Fuchs system:

$$\frac{d}{d\psi}\begin{pmatrix}\omega_0\\ \omega_1\end{pmatrix} = \begin{pmatrix}0 & 1\\ \frac{\psi}{1-\psi^3} & \frac{3\psi^2}{1-\psi^3}\end{pmatrix}\begin{pmatrix}\omega_0\\ \omega_1\end{pmatrix} \tag{5.5}$$

This can be traded for a single 2nd-order differential equation for ω_0,

$$\left(\frac{d^2}{d\psi^2} - \frac{3\psi^2}{1-\psi^3}\frac{d}{d\psi} - \frac{\psi}{1-\psi^3}\right)\omega_0 = 0 , \tag{5.6}$$

which exhibits four regular singular points at $\psi^3 = 1, \psi = \infty$.

The monodromy group of this equation can be studied as follows. First of all we note that it is sufficient to compute the monodromy matrix T_0 around $\psi = 1$. Indeed the effect of a closed loop around $\psi = \alpha$ and $\psi = \alpha^2$ ($\alpha = e^{2\pi i/3}$) can be computed from the monodromy matrix T_0 around $\psi = 1$ by conjugation with $\mathcal{A}$, where $\mathcal{A}$ represents the operation $\psi \to \alpha\psi$:

$$\begin{aligned} T_1 &= \mathcal{A}^1 T_0 \mathcal{A}^{-1}\\ T_2 &= \mathcal{A}^2 T_0 \mathcal{A}^{-2}\end{aligned} \tag{5.7}$$

Furthermore a closed loop which encloses all the singular points, including ∞, is contractible and therefore

$$T_\infty T_2 T_1 T_0 = 1 \quad \to \quad T_\infty = (T_2 T_1 T_0)^{-1} \tag{5.8}$$

To compute T_0 it is convenient to perform the substitution $z = \psi^3$ in the differential equation (5.6). We obtain

$$\left\{9z(1-z)\frac{d^2}{dz^2} + (6-15z)\frac{d}{dz} - 1\right\}\omega = 0 \tag{5.9}$$

This is a standard hypergeometric equation with parameters $a = b = 1/3, c = 2/3$ (in the usual notation), and thus a set of independent solutions around $z \equiv \psi^3 = 0$ is given by

$$\begin{cases} U_1 = \frac{\Gamma^2(1/3)}{\Gamma(2/3)} F(1/3, 1/3, 2/3; \psi^3)\\ U_2 = \frac{\Gamma^2(2/3)}{\Gamma(4/3)} \psi F(2/3, 2/3, 4/3; \psi^3)\end{cases} , \tag{5.10}$$

where $F(a, b, c; z)$ denotes the hypergeometric function. The two solutions can be continued around $\psi^3 = 1$ by known formulae [51]: one finds

$$\begin{aligned} U_1 &= -\log(1-z)F(1/3, 1/3, 1; 1-\psi^3) + B_1(1-\psi^3)\\ U_2 &= -\log(1-z)F(1/3, 1/3, 1; 1-\psi^3) + B_2(1-\psi^3)\end{aligned} \tag{5.11}$$

where B_1 and B_2 are regular series around $\psi^3 = 1$. (The logarithmic factor in (5.11) can be traced back to the equality of the roots around $z \equiv \psi^3 = 1$). A closed loop around $\psi = 1$ gives

$$\begin{pmatrix}U_1\\ U_2\end{pmatrix} \to \begin{pmatrix}U_1'\\ U_2'\end{pmatrix} = \begin{pmatrix}U_1\\ U_2\end{pmatrix} - 2\pi i\ F(\frac{1}{3}, \frac{1}{3}, 1; 1-\psi^3)\begin{pmatrix}1\\ 1\end{pmatrix} \tag{5.12}$$

The Kummer relations [51] among hypergeometric functions allow us to express $F(\frac{1}{3}, \frac{1}{3}, 1; 1-\psi^3)$ in terms of the original basis (U_1, U_2) around $\psi = 0$,

$$F\left(\frac{1}{3}, \frac{1}{3}, 1; 1-z\right) = \frac{\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})} F\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; z\right) + \frac{\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})} F\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; z\right) . \tag{5.13}$$

Therefore, using the relation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ one obtains

$$\begin{pmatrix}U_1'\\ U_2'\end{pmatrix} = \begin{pmatrix}1 + i\ tg\frac{2\pi}{3} & i\ tg\frac{2\pi}{3}\\ i\ tg\frac{2\pi}{3} & i - i\ tg\frac{2\pi}{3}\end{pmatrix} \begin{pmatrix}U_1\\ U_2\end{pmatrix} , \tag{5.14}$$

which means that the monodromy matrix around $\psi = 1$ is

$$T_0 = \begin{pmatrix}1 - i\sqrt{3} & i\sqrt{3}\\ -i\sqrt{3} & 1 + i\sqrt{3}\end{pmatrix} . \tag{5.15}$$

In order to find T_1, T_2 we need to represent $\mathcal{A} : \psi \to \alpha\psi$ on U_1, U_2. From (5.5), (5.6) and (5.10) we see that under $\psi \to \alpha\psi$ the differential operator is invariant while

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{5.16}$$

Since we are interested in a projective representation of the monodromy group, we may rescale our basis in such a way that $\det \mathcal{A} = 1$ (note that T_0 already satisfies $\det T_0 = 1$). Hence we have

$$\mathcal{A} = \begin{pmatrix} \alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2} \end{pmatrix} \tag{5.17}$$

and from (5.7)

$$T_1 = \begin{pmatrix} 1 - i\sqrt{3} & \alpha^{-1} i\sqrt{3} \\ -\alpha i\sqrt{3} & 1 + i\sqrt{3} \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 - i\sqrt{3} & \alpha^{-2} i\sqrt{3} \\ -\alpha^2 i\sqrt{3} & 1 + i\sqrt{3} \end{pmatrix} \tag{5.18}$$

Let us recall that the modular group is given by the group of transformations acting on ψ that leave the theory invariant. The monodromy group Γ_M of the Picard–Fuchs system must therefore be a subgroup of the modular group. The modular group of the torus is of course known to be $\Gamma = SL(2;\mathbb{Z})$, and therefore it must be possible to perform a change of basis on the periods U_i such that the entries of the generators T_0, T_1, T_2 are integer numbers. Actually it is known since the last century that Γ_M is isomorphic to $\Gamma(3)$, which is the group of matrices equivalent to the identity modulo 3. The basis $(\mathcal{F}_1, \mathcal{F}_2)$ where $\Gamma_M \simeq \Gamma(3)$ is obtained by the following linear transformation [52]

$$\begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} = \frac{1}{3(1+\alpha^{-1/2})} \begin{pmatrix} 3\alpha^{1/2} & -3 \\ 1+\alpha^{1/2} & \alpha^2 - 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{5.19}$$

The transformed monodromy generators $\widehat{T}_i$ then take the following form:

$$\widehat{T}_0 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad ; \widehat{T}_2 = \begin{pmatrix} -5 & 12 \\ -3 & 7 \end{pmatrix} \quad ; \widehat{T}_1 = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$$

$$\widehat{T}_\infty \equiv (\widehat{T}_2 \widehat{T}_1 \widehat{T}_0)^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \tag{5.20}$$

We now consider the invariance group Γ_W. The transformation $\mathcal{A} : \psi \to \alpha\psi$ is obviously an invariance of $W = 0$ (and of the differential operator (5.16)) since it can be undone by the coordinate transformation $y^i \to \alpha^{-1/3} y^i$. Less evident is the invariance under the transformation

$$\mathcal{B} : \psi' = -\frac{\psi + 2}{1 - \psi} , \tag{5.21}$$

which can be undone by the change of coordinates [53]

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} . \tag{5.22}$$

The representation of the transformation A on the transformed periods, $(\mathcal{F}_1, \mathcal{F}_2)$ is given by

$$\widehat{A} = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} . \tag{5.23}$$

We note that the Γ_W generators A, B satisfy the relations $A^3 = B^2 = (AB)^3 = +1$, and these are precisely the defining relations of the tetrahedral group, $\Delta \equiv SL(2, \mathbb{Z}_3)$. So indeed we have $\Gamma/\Gamma_M \equiv \Gamma_W = SL(2,\mathbb{Z})/\Gamma(3) \equiv \Delta$, as advertised.

We now consider the second example, which is given by the quintic (5.4). The Picard–Fuchs equation for the periods is given by [6]:

$$\frac{d^4V}{d\psi^4} - \frac{10\psi^4}{1-\psi^5}\frac{d^3V}{d\psi^3} - \frac{25\psi^3}{1-\psi^5}\frac{d^2V}{d\psi^2} - \frac{15\psi^2}{1-\psi^5}\frac{dV}{d\psi} - \frac{\psi}{1-\psi^5} V = 0 . \tag{5.24}$$

The four independent solutions of this equation represent the four periods of the (3,0)–form Ω. The periods are defined by

$$\int_{\gamma_A} \Omega = X^A(\psi) \; ; \; \int_{\gamma^A} \Omega = F_A(\psi) , \tag{5.25}$$

where

$$V = (X^0, X^1, F_0, F_1) \equiv (X^A, F_A) , \tag{5.26}$$

and where (γ_A, γ^B) is some basis of homology 3–cycles that satisfies

$$\gamma^A \cap \gamma_B = -\gamma_B \cap \gamma^A = \delta^A_B \qquad ; \qquad \gamma^A \cap \gamma^B = \gamma_A \cap \gamma_B = 0 \ . \tag{5.27}$$

The basis is defined only up to $Sp(4;\mathbb{Z})$ transformations, which leave the intersection numbers (5.27) invariant.

Let us first consider the duality group Γ_W of the defining polynomial equation, given in (5.4). It is obvious that $\mathcal{A} : \psi \to \alpha\psi$, where $\alpha \equiv e^{2\pi i/5}$, is a symmetry of $W = 0$ since it can be undone by a rescaling of the CP_4 homogeneous coordinates: $(y_1, y_2, y_3, y_4, y_5) \to (\alpha^{-1}y_1, y_2, y_3, y_4, y_5)$ (alternatively, one easily sees that (5.24) invariant under $\psi \to \alpha\psi$). Obviously $\mathcal{A}^5 = 1$ and this excludes a priori the possibility that the duality group is a subgroup of $SL(2;\mathbb{Z})$, because that group does not possess elements of order 5. There are no other ψ–transformations which can be undone by linear transformations of the y_i's, hence $\Gamma_W = \mathbb{Z}_5$. According to our previous discussion, in order to reconstruct the full duality group we must now also compute the monodromy group Γ_M of (5.24).

The monodromy group Γ_M will be represented by 4×4 matrices acting on the periods V in (5.26). The same is true for Γ_W since $\mathcal{A} : \psi \to \alpha\psi$ leaves invariant the differential operator of eq. (5.24) and therefore induces at most a linear combinations of the periods. By fixing the gauge (3.10), (or, the special geometry gauge), we may represent Γ_W by $Sp(4;\mathbb{Z})$–matrices as well.

Let us now compute Γ_M. We sketch briefly the procedure, for further details see ref. [6]. The differential equation (5.24) is a Fuchsian equation with regular singular points at $\psi = \alpha^k$, $(k = 0, 1, ..., 4)$, $\alpha = e^{2\pi i/5}$, and $\psi = \infty$. Like for the torus, it is sufficient to study the monodromy matrix T_0 around $\psi = 1$, since around $\psi = \alpha^k$ the corresponding monodromy matrices $T_k, k = 1, 2, 3, 4$ are given by

$$T_0 \to T_k = A^k T_0 A^{-k} \tag{5.28}$$

where A represents $\psi \to \alpha\psi$ (we represent here the period vector as a row, according to eq. (2.10)). The monodromy around $\psi = \infty$ depends on the other group elements around $\psi = \infty$ through the relations $T_0 T_1 T_2 T_3 T_4 T_\infty = 1$.

Like for the torus, it is convenient to transform eq. (5.24) into a generalized hypergeometric equation through the substitution $z = \psi^{-5}$. We obtain

$$\begin{aligned}\Big\{\frac{d^4}{dz^4} - \frac{2(4z-3)}{z(1-z)}\frac{d^3}{dz^3} - \frac{72z-35}{5z^2(1-z)}\frac{d^2}{dz^2} - \frac{24z-5}{5z^3(1-z)}\frac{d}{dz} -\\ - \frac{24}{625z^3(1-z)}\Big\}V(z) = 0\end{aligned} \tag{5.29}$$

which has singular Fuchsian points at $z = 0, 1, \infty$ with associated Riemann P–symbol

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & 1/5 & 0 & \\ 0 & 2/5 & 1 & ;\psi^{-5} \\ 0 & 3/5 & 2 & \\ 0 & 4/5 & 1 & \end{matrix}\right\} \tag{5.30}$$

We notice that in the variable $z = \psi^{-5}$ we have introduced a new singular point around $\psi = 0$, so that the monodromy around $z = \infty$ corresponds exactly to the representation of $\psi \to \alpha\psi$ on the periods. In other words the duality generator A becomes part of the monodromy generators of the new equation (5.29) (this can be done for the torus too.) A solution of (5.29) around $z = 0$ ($\psi = \infty$) is given by

$$\omega_0(\psi) = \ {}_4F_3\Big(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; 1, 1, 1; \psi^{-5}\Big) \tag{5.31}$$

In order to represent A in a simple way one may construct a basis of solutions around $\psi = 0$ as follows [6]. One first continues $\omega_0(\psi)$ around $\psi = 0$ by using a Barnes–type integral representation and obtains:

$$\omega_0(\psi) = -\frac{1}{5 \cdot 16\pi^4}\sum_{n=0}^{\infty}\frac{\Gamma^5(\frac{n}{5})}{\Gamma(n)}(\alpha^n - 1)^4(5\psi)^n \qquad (|\psi < 1) \tag{5.32}$$

Then one recalls that $\psi \to \alpha\psi$ leaves the differential operator (5.24) invariant so that

$$\omega_j(\psi) \doteq \omega_0(\alpha^j\psi) \qquad\qquad j = 0, 1, 2, 3, 4 \tag{5.33}$$

are also solutions of (5.29). The five functions ω_j are subject to the linear relations $\sum_{n=0}^{4}\omega_j = 0$, as it follows from their explicit expression by the power series (5.32) and the analogous ones derived from (5.33).

If we take $\omega_0, \omega_1, \omega_2, \omega_4$ as a basis of solutions around $\psi = 0$ it follows immediately that $\psi \to \alpha\psi$ is represented on $(\omega_2, \omega_1, \omega_0, \omega_4)^t$ as follows

$$A = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{5.34}$$

Next we examine the monodromy T_0 around $\psi = 1$. For this purpose we observe that since $z = 1$ has the double root $\rho = 1$ for the indicial equation, the continuation of the series $\omega_j(\psi), |\psi| < 1$, to the neighbourhood $|\psi - 1| < 1$ will contain logarithms. Indeed one can write

$$\omega_j(\psi) = \frac{1}{2\pi i} c_j \widetilde{\omega}(\psi) \log(\psi - 1) + reg. \tag{5.35}$$

where $\widetilde{\omega}$ is a linear combination of regular solutions around $\psi = 1$. It turns out that

$$\widetilde{\omega}(\psi) = -\frac{1}{c_1}(\omega_1(\psi) - \omega_0(\psi)) = \frac{1}{c_1}(\psi - 1) + O(\psi - 1)^2) \tag{5.36}$$

It follows

$$\begin{aligned} \frac{d\omega_j}{d\psi} &= \frac{1}{2\pi i} c_j \Big(\frac{d\widetilde{\omega}}{d\psi} \log(\psi - 1) + \widetilde{\omega}(\psi) \frac{1}{\psi - 1} + \ldots\ldots \Big) \\ &=_{\psi \to 1} \frac{1}{2\pi i} \frac{c_j}{c_1} \log(\psi - 1) + \cdots \end{aligned} \tag{5.37}$$

We see that in order to compute the monodromy coefficients c_j, one has to compute the asymptotic behaviour of $\frac{d\omega_j}{d\psi}$ and look at the coefficient of $\log(\psi - 1)$. Using the series expansion for ω_j derived from (5.32) and (5.33) one finds (see [6] for details)

$$c_j = (1, 1, -4, 6, -4) \tag{5.38}$$

From eqs. (5.35)–(5.38), one easily finds that the monodromy matrix around $\psi = 1$ acting on the basis $(\omega_2, \omega_1, \omega_0, \omega_4)^t$ is given by

$$T_0 = \begin{pmatrix} 1 & 4 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 4 & -4 & 1 \end{pmatrix} \tag{5.39}$$

The matrices A and T_0 given by eqs. (5.34)and (5.39) are integer–valued, but not symplectic, since the ω_j–basis is not a symplectic basis. According to our previous discussion there must exist a matrix m such that

$$\widehat{T}_0 = m T_0 m^{-1} \quad ; \quad \widehat{A} = m A m^{-1} \tag{5.40}$$

are not only integer–valued but also symplectic.

A solution, which is unique up to $Sp(4; \mathbb{Z})$ transformations, has been found in ref.[6]. Our choice of basis corresponds to the matrices

$$\widehat{A} = \begin{pmatrix} 1 & -1 & -5 & 3 \\ 0 & 1 & 8 & 5 \\ 1 & -1 & -4 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad , \quad \widehat{T}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{5.41}$$

which act on the right of the row vector (5.26). The other monodromy generators $\widehat{T}_k$, $\widehat{T}_\infty$ around $\psi = \alpha^k$ and $\psi = \infty$ are finally computed from eqs. (5.28) and $T_\infty = (T_0 T_1 T_2 T_3 T_4)^{-1}$.

Summarizing, the duality group Γ of the moduli space of the Calabi–Yau 3–fold (5.4) can be given an integer–valued and symplectic representation on some basis of the periods. It is given by the subgroup of $Sp(4; \mathbb{Z})$ that is generated by $\widehat{A}$ and $\widehat{T}_0$, where $\widehat{A}$ is a representation of $\Gamma_W = \mathbb{Z}_5 \subset Sp(4; \mathbb{Z})$ and where $\widehat{T}_0$ generates the monodromy group of the Picard–Fuchs equation (5.24).

It was already mentioned that the duality group Γ cannot be a subgroup of $SL(2, \mathbb{Z})$ since $SL(2, \mathbb{Z})$ does not contain elements of order 5. One may however represent Γ as a subgroup of $SL(2, \mathbb{R})$ since $\mathbb{Z}_5 \subset SL(2, \mathbb{R})$. To find a representation, one needs to find a complex variable $\gamma(\psi)$ such that Γ can be represented as a subset of $PSL(2, \mathbb{R})$ transformations acting on the upper γ-plane. A determination of $\gamma(\psi)$ and the associated 2×2 representation of $\Gamma \in SL(2, \mathbb{R})$ has been given in ref. [6]. We give here a different, but closely related derivation, which is entirely based on the monodromy structure of the Picard–Fuchs equation (5.24).

Let us observe that if we denote by $V(\psi(\gamma))$ the four–dimensional row vector of the periods as a function of γ, then we must have:

$$V\Big(\psi\Big(\frac{a_i\gamma+b_i}{c_i\gamma+d_i}\Big)\Big)=V(\psi(\gamma))\Gamma_i \tag{5.42}$$

where Γ_i is any of $\widehat{A},\widehat{T}_k$ and $S_i\equiv\frac{a_i\gamma+b_i}{c_i\gamma+d_i}$, the corresponding 2–dimensional action on γ. If V is required to be a uniform function of γ, then $\psi=\psi(\gamma)$ must be uniform and such that the entire ψ^5–plane is mapped into a fundamental region of the γ–plane for the group $\Gamma\equiv\{S_i\}$. That amounts to say that γ is a modular variable and ψ is an automorphic function of γ with respect to Γ. There is a general procedure to construct the uniformizing variable γ directly from the Picard–Fuchs equation for V. It consists in associating to the differential equation, eq. (5.29), a second order differential equation with the same singular points ($z=0,1,\infty$ in our case) and with exponents determined as follows.

If all the integrals of the main equation are regular around the given singularity (no two roots of the indicial equation differ by integers) and if all the roots are commensurable quantities multiple of $1/k$ (k integer), then the difference of the roots of the indicial equations of the associated 2nd–order equation is taken equal to $1/k$. In all the other cases the difference of roots is taken equal to zero. The uniformizing variable γ is then given by the ratio of two solutions of the associated 2nd–order equation. Let us see how this works for us. To adhere to the same notations as in [6], we perform the substitution $z\to\frac{1}{z}$ in the equation (5.29). The P–Riemann symbol (5.30) transformes into

$$P\begin{pmatrix} 0 & \infty & 1 & \\ 1/5 & 0 & 0 & \\ 2/5 & 0 & 1 & ;\psi^5 \\ 3/5 & 0 & 2 & \\ 4/5 & 0 & 1 & \end{pmatrix}\quad . \tag{5.43}$$

From (5.43) we see that at $z^{-1}\equiv\psi^5=0$, all the roots are multiple of $\frac{1}{k}\equiv\frac{1}{5}$ and do not differ by integers. At $z^{-1}=\infty$ and $z^{-1}=1$ instead we have at least two coincident roots. Therefore calling λ,μ,ν the differences of the roots of the indicial equation for the associated 2nd–order equation we have

$$\lambda=1/5\qquad;\qquad\mu=\nu=0 \tag{5.44}$$

Given the exponents we can immediately write down the associated 2nd–order equation, which, having regular singular points at $z=0,1,\infty$ is a hypergeometric equation of parameters

$$a=\frac{1}{2}(1-\lambda-\mu+\nu)=\frac{2}{5};\ b=\frac{1}{2}(1-\lambda-\mu-\nu)=\frac{2}{5};\ c=1-\lambda=\frac{2}{5}\ , \tag{5.45}$$

that is:

$$z(1-z)\mathcal{F}''+\Big(\frac{4}{5}-\frac{7}{5}z\Big)\mathcal{F}'-\frac{4}{25}\mathcal{F}=0 \tag{5.46}$$

The uniformizing variable is then given by

$$\gamma=\frac{\mathcal{F}_1}{\mathcal{F}_2} \tag{5.47}$$

where

$$\mathcal{F}_1\equiv\frac{\Gamma^2(\frac{2}{5})}{\Gamma(\frac{4}{5})}F(2/5,2/5,4/5;\psi^5) \tag{5.48}$$

$$\mathcal{F}_2\equiv\frac{\Gamma^2(\frac{3}{5})}{\Gamma(\frac{6}{5})}F(3/5,3/5,6/5;\psi^5) \tag{5.49}$$

are two linearly independent solutions of (5.46). From the theory of the automorphic functions we know that γ maps the ψ^5–plane onto a couple of adjacent triangles inside the circle $|\gamma|^2=1$ with internal angles $(0,0,\pi/5)$; they constitute a fundamental region for the projective action of the modular group Γ, and the inverse function $\psi=\psi(\gamma)$ is automorphic with respect to Γ.

It is now easy to derive the explicit representation of A and T_0 as a subgroup of $SL(2,\mathbb{R})$ on $\mathcal{F}_1,\mathcal{F}_2$: one has simply to study the monodromy group of the differential equation (5.32) exactly as for the torus, eq. (5.9). One obtains:

$$A=\begin{pmatrix} e^{-i\pi/5} & 0\\ 0 & e^{i\pi/5}\end{pmatrix}\quad;\quad T_0=\begin{pmatrix} 1-itg\frac{2\pi}{5} & itg\frac{2\pi}{5}\\ -itg\frac{2\pi}{5} & 1+itg\frac{2\pi}{5}\end{pmatrix} \tag{5.50}$$

and

$$T_k=A^kT_0A^{-k}\quad(k=1,2,3,4);\quad T_\infty=(T_0T_1T_2T_3T_4)^{-1}\equiv(AT_0)^{-5} \tag{5.51}$$

Note that the matrices quoted in [6] are related to those given in (5.36) by a change of basis

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} i & -i\alpha^2 \\ 1 & -\alpha^2 \end{pmatrix} \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} , \tag{5.52}$$

which maps the interior of the circle $|\gamma|^2 = 1$ into the upper half–plane $Im\gamma > 0$.

So far we have considered the representation of the modular group Γ acting on the periods in terms of 4×4 $Sp(4;\mathbb{Z})$ matrices, or in terms of $SL(2;\mathbb{R})$ matrices that act projectively on the unformizing variable γ. There is another important variable, which is the special variable t, in terms of which we may give a further representation of Γ.

Let us reconsider equation ((3.21)), which defines the flat coordinates in terms of the periods as follows:

$$\widetilde{t}^a = \frac{\widetilde{X}^a}{\widetilde{X}^0} = \frac{A^a_B X^B + B^{aB} F_B}{A^0_B X^B + B^{0B} F_B}(t) \ . \tag{5.53}$$

Here and in the following, we change our conventions for the symplectic metric Q which, instead of being given by eq. (A.25) of Appendix 1 is now given by

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \ . \tag{5.54}$$

As a consequence, the 2×2 sub-matrices of M in eq. (3.20) now satisfy

$$\begin{aligned} A^t B &= B^t A \\ C^t D &= D^t C \\ A^t D - B^t C &= \mathbf{1} \end{aligned} \qquad . \tag{5.55}$$

Recalling that $F_A = \frac{\partial F}{\partial X^A}$ where $F(X)$ is a homogeneous function of degree two, we have

$$\begin{aligned} F(X^A) &= (X^0)^2 \mathcal{F}(t^a) \\ F_0(X^A) &\equiv \frac{\partial F}{\partial X^0} = X^0[2\mathcal{F}(t^a) - t^a \partial_a \mathcal{F}(t^a)] \\ F_a(X^A) &\equiv \frac{\partial F}{\partial X^a} = X^0 \partial_a \mathcal{F}(t^a) \ , \end{aligned} \tag{5.56}$$

and substituting this in (5.53) we find

$$\widetilde{t}^a = \frac{A^a_b t^b + A^a_0 + B^{ab}\mathcal{F}_b + B^{a0}(2\mathcal{F} - t^b \mathcal{F}_b)}{A^0_b t^b + A^0_0 + B^{ab}\mathcal{F}_b + B^{00}(2\mathcal{F} - t^b \mathcal{F}_b)} \ , \tag{5.57}$$

where $\mathcal{F}_a \equiv \partial_a \mathcal{F}$. We can now restrict the $Sp(2n+2, \mathbb{R})$ matrix M to belong to $\Gamma \subset Sp(2n+2, \mathbb{Z})$, so that we obtain the representation of Γ on the flat coordinates t^a. In particular, the subgroup of $Sp(2n+2, \mathbb{R})$ that consists of matrices of the form

$$\begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \qquad \begin{aligned} D &= (A^t)^{-1} \\ C &= AC^t (A^t)^{-1} \end{aligned} \tag{5.58}$$

acts on the $t^{a'}$s as linear fractional transformations. For the quintic, the generators A and T_0 act as follows on the flat coordinate:

$$A : \widetilde{t} = \frac{X^1 - (X^0 + F^0)}{X^0 + F^0} = \frac{t - 1 - (2\mathcal{F} - t\mathcal{F}')}{2\mathcal{F} - t\mathcal{F}'} \tag{5.59}$$

$$T_0 : \widetilde{t} = \frac{X^1}{X^0 - F_0} = \frac{t}{1 - 2\mathcal{F} + t\mathcal{F}'} \tag{5.60}$$

Furthermore, we have from (5.41)

$$(T_0 A)^{-1} = \begin{pmatrix} 1 & 1 & 5 & -8 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} , \tag{5.61}$$

so that

$$\widetilde{t} \equiv \frac{\widetilde{X}^1}{\widetilde{X}^0} = \frac{X^1}{X^0} + 1 \equiv t + 1 \ . \tag{5.62}$$

Note that while $(T_0 A)^{-1}$ corresponds to a circuit around $z = 0$, the monodromy around $\psi = \infty$ is represented by $(T_0 A)^{-5} : \widetilde{t} \to t + 5$.

5.1 An abelian subgroup of the duality group

Integer shifts such as in (5.62) play a distinguished rôle in string compactifications. Indeed, the following term in the σ–model:

$$\int_{W.S.} d^2\sigma \partial_\alpha Y^i \partial_\beta \overline{Y}^{\bar{j}} B_{i\bar{j}} \epsilon^{\alpha\beta} = \int_T B_{i\bar{j}} dY^i \wedge d\overline{Y}^{\bar{j}} \tag{5.63}$$

(where T is the image of the world-sheet in the Calabi–Yau 3–fold) is topological, and the B_{ij} are analogous to the θ–parameters of Q.C.D. If the complexified Kähler (1,1)–form is parametrized as

$$g_{i\bar{j}} + iB_{i\bar{j}} = -i\sum_{a=1}^{n} t^a L^a_{ij} \ , \tag{5.64}$$

where $L^a_{i\bar{j}}$ is a basis of the (1,1)–cohomology, then shifts

$$\mathcal{T}: \ t^a \to t^a + n^a \quad ; \quad n^a \in \mathbb{Z}^n \tag{5.65}$$

induce topologically non-trivial mappings from the world sheet to the Calabi–Yau manifold (world-sheet instantons). Such integral shifts are supposed to be an invariance of the quantum action, and thus one would expect $\mathcal{T}$ to be always contained in Γ.

Recalling the homogeneity relation $2F(X) = X^A F_A$, one finds for generic $Sp(2n+2)$–transformations (5.43) [54]:

$$\begin{aligned} 2\widetilde{F}(\widetilde{X}) &= (X^A, F_A)\begin{pmatrix} AC^t & AD^t \\ BC^t & BD^t \end{pmatrix}\begin{pmatrix} X^A \\ F_A \end{pmatrix} \\ &= 2F(X) + 2F_A(BC^t)^A_B X^B + X^A(AC^t)_{AB}X^B + F_A(BD^t)^{AB}F_B \end{aligned} \tag{5.66}$$

where we have used the conditions of symplecticity of the transposed matrix M^t in order to reconstruct $F(X)$ on the r.h.s. of (5.66).

If $M \subset \Gamma$, then $\widetilde{F} = F$, since a modular transformation is a discrete isometry. In particular for a translation (5.43) gives

$$F(X^B A^A_B) = F(X^A) + X^A(AC^t)_{AB}X^B \tag{5.67}$$

since $B=0$ and $A = \begin{pmatrix} 1 & n^a \\ 0 & \delta^a_b \end{pmatrix}$. In terms of the flat variables (5.67) becomes

$$\mathcal{F}(t^a + n^a) = \mathcal{F}(t^a) + (AC^t)_{ab} t^a t^b + 2(AC^t)_{0b} t^b + (AC^t)_{00} \tag{5.68}$$

Thus F (or $\mathcal{F}$) is periodic in the X^A (or t^a) up to quadratic additions.In particular the Yukawa coupling $W_{abc} = \frac{\partial^3 \mathcal{F}}{\partial t^a \partial t^b \partial t^c}$ is periodic, $W_{abc}(t^a + n^a) = W_{abc}(t^a)$, and thus can be expanded in a multiple Fourier series in $q_a \equiv e^{2\pi i t^a}$:

$$W_{abc}(q_a) = \sum_{\vec{m} \in \mathbb{Z}^n} d_{abc}(\vec{m}) \Pi^n_{a=1} q_a^{m_a} \ . \tag{5.69}$$

Note that $q_a = 0$ means $t^a \to i\infty$, which is the large radius limit; this corresponds to the "classical" intersection numbers $d_{abc}(0)$. The terms with $\vec{m} \neq 0$ give the instanton corrections to the classical result. For the quintic, $d_{111}(m)$ has been related with the number of rational curves of degree m on the quintic [6].

An interesting observation is that the n abelian elements of the quantum duality group are entirely determined by the intersection numbers $d_{abc}(0)$. This can be explicitly shown for one modulus, using the monodromy properties of the Picard–Fuchs equation around $q_a = 0$. It has also been verified for a model with two moduli by Candelas et al. [17].

For one modulus, the monodromy matrices can actually be computed directly from the Gauß–Manin connection of the linear system (3.9). Using the change of variable $\psi \equiv t \to \frac{1}{2\pi i}\log\ q$, the linear system becomes:

$$\Big[q\frac{\partial}{\partial q} + \frac{1}{2\pi i}A(q)\Big]V(q) = 0 \tag{5.70}$$

The monodromy generator $\mathbf{T}$ around $q=0$ is then given by (see Morrison, ref. [9]):

$$\mathbf{T} = \exp[A(q=0)] \ , \tag{5.71}$$

and has the property: $(\mathbf{T} - \mathbf{1})^4 = 0$, which corresponds to the maximal nilpotency dictated by the order of the differential equation.

If we now recall the structure of the Gauß–Manin connection in special coordinates (see (3.22),(3.6) with $\mathsf{A}=0$):

$$\frac{d}{dt}V = \mathbf{C}(t)\,V \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & W_{ttt} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} V\ , \tag{5.72}$$

then we see that for any W:

$$(\exp[\mathbf{C}] - \mathbf{1})^4 = 0\ , \tag{5.73}$$

where, as discussed in section 3, the matrix $\mathbf{C}$ satisfies $\mathbf{C^4} = 0$ (cf., 2.44). Therefore the symmetry $(T_0 A)^{-1} : t \to t+1$, is identified, up to a symplectic transformation, with $\exp[\mathbf{C}(t=i\infty)]$, with $W(i\infty) \equiv d_{111}$.

More generally, for n moduli the matrixes $\mathbf{C}_\alpha$ satisfy the relations $\mathbf{C}_\alpha\mathbf{C}_\beta\mathbf{C}_\gamma\mathbf{C}_\delta = 0$ and $[\mathbf{C}_\alpha\ ,\ \mathbf{C}_\beta] = 0$, and therefore the n–monodromy generators $\mathbf{T}_i = \exp[\mathbf{C}_i(i\infty)] = \mathbf{1} + \mathbf{L}_i$, obey:

$$\begin{aligned} [\mathbf{L}_i, \mathbf{L}_j] &= 0 \\ \mathbf{L}_i\mathbf{L}_j\mathbf{L}_k &= d_{ijk}E \\ \mathbf{L}_i\mathbf{L}_j\mathbf{L}_k\mathbf{L}_l &= 0 \end{aligned} \tag{5.74}$$

Here, E is the matrix with an "1" in the upper right corner.

6 Duality group of an example with two moduli

6.1 Introduction

In this chapter we present an efficient method for determining the duality group, by considering the following 2–moduli deformation of the quintic:

$$\mathcal{W} = \mathcal{W}_0 - a\,\mathrm{y}_4^3\mathrm{y}_5^2 - b\,\mathrm{y}_4^2\mathrm{y}_5^3\ , \tag{6.1}$$

This deformation is a special subspace of the general 101–dimensional deformation space of $\mathcal{W}_0$, which gives rise to zero Yukawa couplings for the associated effective Lagrangian. Because of that, it constitutes only a toy model as far as the low energy Lagrangian is concerned. However, it provides an example with two moduli for which the duality group can be easily determined, and thus allows to display the power of some general techniques of algebraic geometry which were previously developed and applied to the study of the monodromy groups of Feynman integrals [35],[36]. In ref. [17], [19] the target space duality group for other examples of two–moduli deformations have been worked out using techniques different from the ones presented here.

Our result for the duality group is surprisingly simple: Γ is given by an $U(1,2)$ valued (projective) representation of B_5, the braid group on five strands.

In terms of the defining polynomial $\mathcal{W}$, the fundamental period can be defined by

$$\omega_0(a,b) = \oint_\gamma \frac{\omega}{\mathcal{W}(\mathrm{y};a,b)}\ , \tag{6.2}$$

where ω is the volume element

$$\omega = \sum (-1)^i \mathrm{y}_i\, d\mathrm{y}_1 \wedge \cdots \wedge \widehat{d\mathrm{y}_i} \wedge \cdots \wedge d\mathrm{y}_5 \tag{6.3}$$

(the hat means that the corresponding differential must be omitted), and γ is an element of the basis for the homology cycles of $H_{(4)}(CP_4 - \mathcal{W};\mathbb{Z})$. There are as many independent integrals ω_0^I as there are elements of the basis, $\gamma^I \subset H_{(4)}(CP_4 - \mathcal{W};\mathbb{Z})$.

Quite generally, if L^{N-1} is the singularity locus of an algebraic variety $\mathcal{W}$ parametrized by N moduli, the monodromy group Γ_M acting on the periods of $\mathcal{W}$ is given by a representation of the fundamental group π_1 of the embedding space CP_N.

The computation of the homotopy group π_1 is based on the use of the following two theorems [55,56].

Theorem 1 (Picard-Severi). *Let L^{N-1} be the $N-1$ complex dimensional singular locus of a given algebraic variety. If the (complex) projective line $CP_1 \subset CP_N$ is generic with respect to L^{N-1} (i.e., it avoids all singular points of L^{N-1}), then we have the isomorphism*

$$\pi_1\left(CP_1 - (CP_1 \cap L^{N-1}); B\right)/G \approx \pi_1\left(CP_N - L^{N-1}; B\right)\ , \tag{6.4}$$

where B is the base point and G is an invariant subgroup of

$$\pi_1 \left(CP_1 - \left(CP_1 \cap L^{N-1} \right); B \right) \ . \tag{6.5}$$

In other words, $\pi_1 \left(CP_N - L^{N-1}; B \right)$ is obtained from (6.5) by imposing the relations satisfied by the monodromy generators. A method for deriving such identities has been provided by Van Kampen [57], and we shall use it in our particular case to obtain the isomorphism of eq. (6.4). As the singular locus of the algebraic variety $\mathcal{W}$ is given by an equation of the form $L(a,b) = 0$, we are interested in $N = 2$.

The second theorem allows to understand that, as far as the identification of the fundamental group π_1 is concerned, the situation with more than two moduli can be essentially reduced to $N = 2$, so that the general computation of π_1 in will not be much more difficult than the one under study. However, for several variables it is in general much harder to find the singular locus and the behaviour of the algebraic variety in its neighbourhood, and therefore the determination of the monodromy group can be more involved.

Theorem 2 (Zariski [56]). *If the complex projective plane CP_2 is generic with respect to L^{N-1} and if $B \in \left(CP_2 - CP_2 \cap L^{N-1} \right)$, then the map*

$$\pi_1 \left(CP_2 - CP_2 \cap L^{N-1}; B \right) \to \pi_1 \left(CP_N - L^{(N-1)}; B \right) \tag{6.6}$$

is an isomorphism.

We see that, in virtue this theorem, the study of the homotopy group of $CP_N - L^{N-1}$ is reduced to the study of the homotopy of the complement of the 1–dimensional curve $L^1 = CP_2 \cap L^{N-1}$ on a generic two-dimensional section. Since the singular locus of the variety (6.1) is already one-dimensional ($N = 2$), Theorem 1 is sufficient for our present purposes.

Before proceeding to the actual determination of the duality group, we recall what the local geometry associated to the moduli space $\mathcal{M}$ of $\mathcal{W}$ is. The two-moduli family of Calabi–Yau deformations given in (6.1) was first discussed in ref. [27], where the underlying conformal field theory was constrcuted as a tensor product of five copies of $N = 2$ minimal models with $k = 3$ and $c = 9/5$.

It was observed that there are restrictions due to $U(1)$ charge conservation, which imply $W_{\alpha\beta\gamma} = 0$ for the Yukawa couplings.

Because of that, the constraint of special geometry [22,24] (see appendix A)

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = g_{\alpha\overline{\beta}} g_{\gamma\overline{\delta}} + g_{\alpha\overline{\delta}} g_{\gamma\overline{\beta}} - e^{2K} W_{\alpha\gamma\rho} W_{\overline{\beta}\overline{\delta}\overline{\sigma}} g^{\rho\overline{\sigma}} \tag{6.7}$$

reduces to

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = g_{\alpha\overline{\beta}} g_{\gamma\overline{\delta}} + g_{\alpha\overline{\delta}} g_{\gamma\overline{\beta}} \qquad , \qquad \alpha, \overline{\beta}, \gamma, \overline{\delta} = 1, 2 \ . \tag{6.8}$$

Thus, the local geometry of $\mathcal{M}$ is given by a 2–dimensional Kähler manifold with constant curvature, which according to the classification of ref. [58] corresponds to the coset space $\frac{U(1,2)}{U(1)\otimes U(2)}$. On the other hand, the global structure of the moduli space is given by modding out the isometries given by the duality group Γ.

Notice that our example falls in the class of singular Picard–Fuchs systems that we have treated in section 3.3. Indeed, for two moduli, one would expect a $2N + 2 = 6$–dimensional representation of the modular group. In fact, the dimension of the H^3 cohomology group is given by $2h_{21} + 2$, where $h_{21} = dim\, H^{(2,1)}$, the number of complex structure moduli of the Calabi–Yau manifold. It turns out, however, that our representation is only 3–dimensional, because this specific model is singular due to the vanishing of the Yukawa couplings $W_{\alpha\beta\gamma}$. Thus, the Picard-Fuchs equations for the periods of $\mathcal{W}$ are of second order rather than fourth-order. The 6–dimensional representation, which in the symplectic basis takes values in $Sp(6, \mathbb{Z})$, splits into two 3–dimensional representations of $U(1,2)$, according to the embedding $6 = 3 + \overline{3}$ of $U(1,2)$ in $Sp(6)$. We shall later verify explicitly that the 3–dimensional representation of B_5 indeed takes values in $U(1,2)$.

6.2 The fundamental group of $\mathcal{W}(\mathrm{y}; a, b)$

In this section we compute the fundamental group $\pi_1(CP_2 - L; B)$ of the algebraic variety $\mathcal{W} = 0$. The first step consists of finding the singular locus L of eq. (6.1), which is given by solving simultaneosly the equations

$$\frac{\partial \mathcal{W}}{\partial \mathrm{y}_i} = 0 \qquad i = 1, \cdots, 5\,. \tag{6.9}$$

A straightforward computation yields the 1–dimensional complex curve

$$L(a,b) = 108(a^5 + b^5) - 80a^3b^3 - 165a^2b^2 + 30ab - 1 = 0\ , \tag{6.10}$$

which represents the locus of the complex points of the original variety $\mathcal{W} = 0$ where two or more of the roots coincide.

For the derivation of the Van Kampen relations among the homotopy generators around the various branches of $L(a,b)$, it is important to know where $L(a,b)$ itself has multiple points. These multiple points are found by solving the equations $L(a,b) = \frac{\partial L}{\partial a} = \frac{\partial L}{\partial b} = 0$, which give the location of the multiple roots

$$\begin{aligned} (a,b) &= (\rho^k\ , \rho^{-k}) \\ (a,b) &= -\frac{1}{4}(\rho^k\ , \rho^{-k}) \qquad k = 0, \cdots, 4 \end{aligned}\ , \tag{6.11}$$

where $\rho = e^{2\pi i/5}$. The first set of roots in (6.11) corresponds to nodes with two distinct complex conjugate tangents (which are isolated points for the real section of (6.10) represented by real values of a and b). The second set instead represents (second order) cusps, since the Hessian $\frac{\partial^2 L}{\partial a \partial b}$ is degenerate at these points.

We may obtain a more elegant and geometrically intuitive representation of the curve $L(a,b) = 0$ by choosing new coordinates (p,q) such that the real section corresponding to real values of p and q exhibits the previous singular points in the real (p,q) plane. It is sufficient to set

$$\begin{cases} a &= \mathrm{p} + \mathrm{i}\,\mathrm{q} \\ b &= \mathrm{p} - \mathrm{i}\,\mathrm{q} \end{cases} \tag{6.12}$$

and we find

$$\begin{aligned} L(p,q) =& -1 + 30(p^2+q^2) - 165(p^4+q^4) - 80(p^6+q^6) + 216p^5 \\ & - 330p^2q^2 - 2160p^3q^2 - 240(p^4q^2 + p^2q^4) + 1080pq^4\,. \end{aligned} \tag{6.13}$$

In the real plane of (p,q) the multiple points (6.11) take now the real values

$$\begin{aligned} (p,q) &= (\cos\frac{2\pi k}{5}, \sin\frac{2\pi k}{5}) \\ (p,q) &= -\frac{1}{4}(\cos\frac{2\pi k}{5}, \sin\frac{2\pi k}{5}) \end{aligned} \tag{6.14}$$

respectively. Actually, the curve $L(p,q) = 0$ can be put in a parametric form by setting

$$\begin{cases} p = \frac{1}{5}(3\cos 2t + 2\cos 3t) \\ q = \frac{1}{5}(3\sin 2t - 2\sin 3t) \qquad 0 \leq t \leq 2\pi \end{cases} \tag{6.15}$$

and can be recognized as a pentacuspidal hypocycloid (the curve described by a point of a circle of radius $R = 1/5$ rolling inside a circle of $R = 1$), whose graph is represented in Fig. 1.

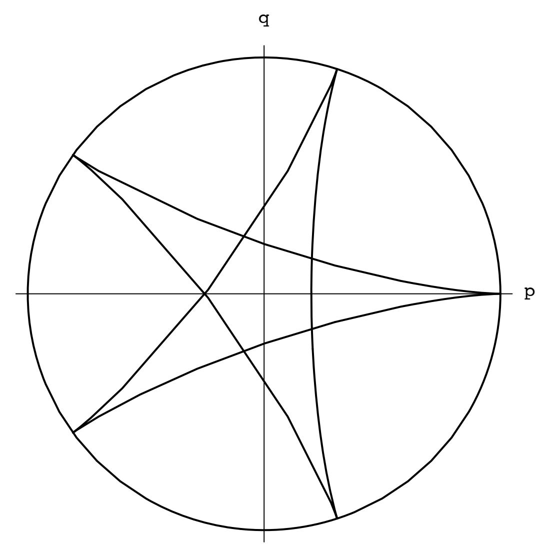

Fig. 1 The pentacuspidal hypocycloid in the unit circle.

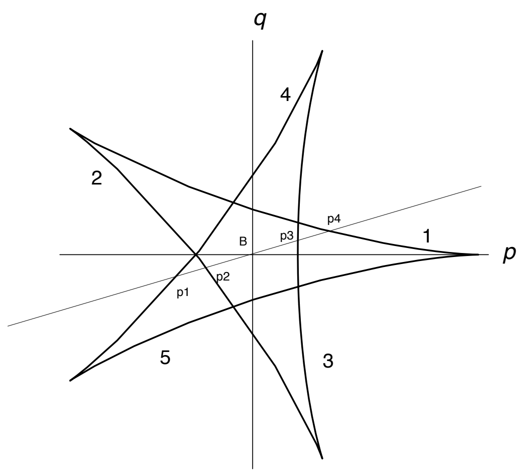

Fig. 2

According to Picard-Severi theorem, we now take a generic line C through the base point $B = (0,0)$ which intersects the real branch of the hypocycloid in four (finite) points P_i (Fig. 2). To each of such representative points, we attach a generator of $\pi_1(\mathbb{C} - \mathbb{C} \cap L; B)$ by constructing a loop which leaves B along a straight line, makes an infinitesimal loop around P_i counterclockwise in the complex plane $\mathbb{C}$ containing the real line, and comes back to B again in the opposite direction. If the straight line encounters a real branch of L, it will be taken to undercross it by describing a small semicircle in the complex plane. If by varying the angular coefficient of the straight line no critical point of L is encountered, the corresponding points give rise to equivalent loops. Generally inequivalent loops are obtained if two straight lines intersect L along points belonging to two different real branches of L emanating from the critical points. We thus obtain 15 loops, 5 of which go around the sides of the pentagonal figure described by L and 10 going around the branches emanating from the cusps (5 of them are shown in Fig. 3.).

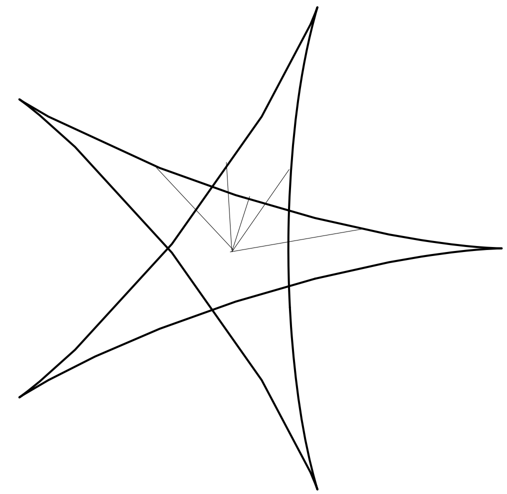

Fig. 3

We now quote the Van Kampen relations [57] between loops which, added to the free group generated by the above mentioned 15 generators, make it isomorphic to $\pi_1(CP_2 - L; B)$.

For the nodes corresponding to transversal intersections of the real branches of L, we have (see Fig. 4)

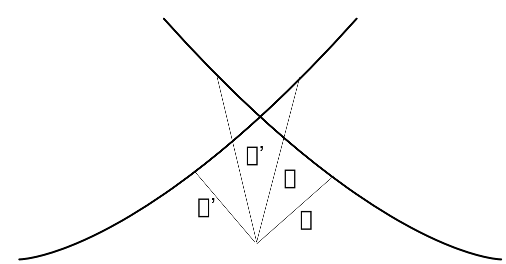

Fig. 4

$$\alpha\beta = \beta\alpha \qquad ; \qquad \alpha = \alpha' \qquad ; \qquad \beta = \beta' \tag{6.16}$$

i.e., we can slide the representative loops across the node without any change, and the loops around two branches commute. In this way, the 15 generators reduce to five independent ones.

For each cusp (see Fig. 5) we have the relation

$$\alpha\beta\alpha = \beta\alpha\beta \tag{6.17}$$

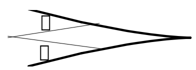

Fig. 5

Let us enumerate in increasing order the five branches of the hypocycloid described successively by the point of the small circle rolling inside the big circle (Fig. 2). Then, denoting by $\alpha_i\,, i = 1, \cdots, 5$ the loops winding around the five branches of Fig. 1, we have the following set of relations among the generators

$$\begin{array}{ll} \alpha_1\alpha_3 = \alpha_3\alpha_1 & \alpha_1\alpha_2\alpha_1 = \alpha_2\alpha_1\alpha_2 \\ \alpha_1\alpha_4 = \alpha_4\alpha_1 & \alpha_2\alpha_3\alpha_2 = \alpha_3\alpha_2\alpha_3 \\ \alpha_2\alpha_4 = \alpha_4\alpha_2 & \alpha_3\alpha_4\alpha_3 = \alpha_4\alpha_3\alpha_4 \\ \alpha_2\alpha_5 = \alpha_5\alpha_2 & \alpha_4\alpha_5\alpha_4 = \alpha_5\alpha_4\alpha_5 \\ \alpha_3\alpha_5 = \alpha_5\alpha_3 & \alpha_5\alpha_1\alpha_5 = \alpha_1\alpha_5\alpha_1 \end{array} \tag{6.18}$$

We note that the subset of relations (6.18) not involving α_5 coincide with the defining relations of the braid group B_5, with four generators $\alpha_i\,, i = 1, \ldots, 4$,

$$\begin{cases} \alpha_i\alpha_j = \alpha_j\alpha_i & |i-j| \geq 2 \\ \alpha_i\alpha_{i+1}\alpha_i = \alpha_{i+1}\alpha_i\alpha_{i+1} & i = 1, \ldots, 3 \end{cases} . \tag{6.19}$$

However, it may be easily verified that the element of B_5 exchanging the first and the fifth strand can be written in terms of the four generators α_i by the word

$$\alpha_5 = (\alpha_4\alpha_3\alpha_2)\alpha_1(\alpha_4\alpha_3\alpha_2)^{-1} \tag{6.20}$$

With a little effort, using the Van Kampen relations for $\alpha_1, \cdots, \alpha_4$ one checks that the extra relations of (6.18) involving α_5 are indeed verified. Therefore, we come to the conclusion that, if the relation (6.20) also holds among the monodromy generators α_i ($i = 1, \cdots, 5$) of the hypocycloid, then $\pi_1(C(2) - L; B)$ is isomorphic to B_5.

The reason why we do not find eq. (6.20) among the Van Kampen relations is that we have not considered the critical point of L at ∞ and the associated generator. Rather than studying such critical point, we shall give evidence in the sequel that eq. (6.20) must be satisfied by the monodromy generators, so that indeed B_5 coincides with the fundamental group associated to $\mathcal{W}$.

6.3 Behaviour of the periods around the singular curve

Till now we have only considered the abstract presentation of the fundamental group in terms of its generators α_i. To obtain an explicit realization on the periods of $\mathcal{W}$, it is necessary to consider their leading behaviour in the neighbourhood of the singularity locus $L(a,b)$. Let us evaluate the integral defined in (6.2) on a suitable contour. Setting $\mathrm{y}_3 = 1$, we may rewrite it in the following way

$$\omega_0 = 5 \oint d\mathrm{y}_2 d\mathrm{y}_4 d\mathrm{y}_5 \oint \frac{d\mathrm{y}_1}{\mathrm{y}_1^5 + f(a,b,\mathrm{y}_2,\mathrm{y}_4,\mathrm{y}_5)} \,, \tag{6.21}$$

where

$$f = \mathrm{y}_2^5 + \mathrm{y}_4^5 + \mathrm{y}_5^5 + 1 - 5a\,\mathrm{y}_4^3\mathrm{y}_5^2 - 5b\,\mathrm{y}_4^2\mathrm{y}_5^3 \,. \tag{6.22}$$

Performing the last integration on the cycle $|\mathrm{y}_1| = const$ (posing $\mathrm{y}_1^5 = t$), gives immediately $\frac{2\pi i}{5}(-f)^{4/5}$, so that we may write

$$\begin{aligned} \omega_0 &= 2\pi i \oint \frac{d\mathrm{y}_2 d\mathrm{y}_4 d\mathrm{y}_5}{(\mathrm{y}_2^5 + \mathrm{y}_4^5 + \mathrm{y}_5^5 + 1 - 5a\,\mathrm{y}_4^3\mathrm{y}_5^2 - 5b\,\mathrm{y}_4^2\mathrm{y}_5^3)^{4/5}} \\ &= 2\pi i \oint d\mathrm{y}_4 d\mathrm{y}_5 \oint \frac{d\mathrm{y}_2}{[\mathrm{y}_2^5 + g(a,b,\mathrm{y}_4,\mathrm{y}_5)]^{4/5}} \end{aligned} \tag{6.23}$$

where

$$g = \mathrm{y}_4^5 + \mathrm{y}_5^5 + 1 - 5a\mathrm{y}_4^3\mathrm{y}_5^2 - 5b\mathrm{y}_4^2\mathrm{y}_5^3 \,. \tag{6.24}$$

Again, with $y_2 = g^{1/5}u$, the last integral on the cycle $|y_2| = const$ yields

$$g^{-3/5} \oint \frac{du}{(u^5+1)^{4/5}} \tag{6.25}$$

which is a number independent of a and b, so that we arrive to

$$\omega_0 = const \times \oint \frac{dy_4 dy_5}{g^{3/5}} \tag{6.26}$$

After the further change of variables

$$y_4 = \sigma\tau^{1/2} \qquad ; \qquad y_5 = \sigma\tau^{-1/2} \tag{6.27}$$

we find

$$\omega_0 = const \times \oint \frac{d\tau}{\tau} \oint \frac{\sigma d\sigma}{[1+\sigma^5 h(\tau)]^{3/5}} \tag{6.28}$$

where $h(\tau) = \tau^{5/2} + \tau^{-5/2} - 5a\tau^{1/2} - 5b\tau^{-1/2}$. Setting at last $\xi = \sigma h^{1/5}$, the σ cyclic integral gives $h^{2/5}$ times a purely numerical integral. Therefore we obtain that the periods ω_0 can be written as a simple 1–dimensional integral

$$\omega_{ij} = const \times \oint_{\gamma_{ij}} \frac{d\tau}{(\tau^5 - 5a\,\tau^3 - 5b\,\tau^2 + 1)^{2/5}} \tag{6.29}$$

along a set of suitably chosen contours γ_{ij}. The quintic polynomial $P(\tau) = \tau^5 - 5a\tau^3 - 5b\tau^2 + 1$ has five roots τ_i, and the possible singularities of the integral (6.29) arise as pinching singularities due to the coincidence of two roots encircled by a figure-eight contour (see Fig. 6).

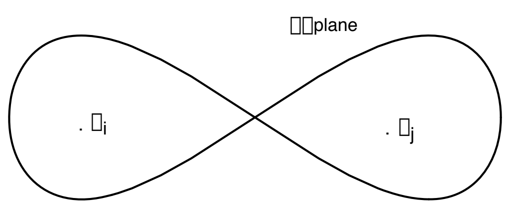

Fig. 6

One can check that the singular locus of the singularities of $P(\tau)$ is given again by the curve $L(a,b) = 0$ of eq. (6.10) by computing the resultant between $P(\tau)$ and $\frac{dP}{d\tau}$.

At this point, we may conclude that the monodromy group is exactly B_5. In fact we note that the braid group B_n can be identified with the fundamental group of the space of all unordered sets of n distinct complex numbers τ_i $i = 1\ldots,n$. More precisely, $B_n = \pi_1(\mathbb{C}^n/S_n \backslash S/S_n)$, where S_n is the permutation group on five elements and S is the union of the hyperplanes $\tau_i = \tau_j$ $\forall(i,j)$ [59]. Identifying the numbers τ_i with the roots of a polynomial $P_n(\tau)$ we see that in our example the monodromy group of the periods ω_0 must be a subgroup of B_5, since $P(\tau)$ is a quintic polynomial. On the other hand, B_5 contains the element given by (6.20) exchanging the strands 1 and 5. Since

$$\alpha_5 = \alpha_4\alpha_3\alpha_2\alpha_1\alpha_2^{-1}\alpha_3^{-1}\alpha_4^{-1} \tag{6.30}$$

satisfyes the relations (6.18) involving α_5, the group whose presentation is given by (6.18) must actually coincide with B_5.

There is yet another independent argument leading to the same conclusions. Given a general polynomial of 5–th degree, we may always fix 3 of its 5 coefficients by a Möbius transformation to arbitrary values. As $P(\tau)$ contains only two parameters, it is infact a gauge fixed form of a generic quintic polynomial, and therefore the associated subgroup of B_5 is really B_5 itself.

One can show that the set of all possible figure eight contours γ_{ij} encircling a couple of roots can be expressed linearly in terms of three of them, thus confirming that the number of linearly independent periods of $\mathcal{W}$ is indeed equal to three. To see this, let us denote by $\gamma_{i,i+1}$ the figure–eight contour encircling two consecutive roots τ_i, τ_{i+1} (loops encircling non consecutive roots are easily written as products of the $\gamma_{i,i+1}$'s, *e.g.*$\gamma_{i,i+2} = \gamma_{i,i+1} \circ \gamma_{i+1,i+2}$, etc.), and by $\omega_{i,i+1}$ the corresponding period. Only three of them are independent, since they satisfy the following two relations

$$\begin{aligned} &\sum_{k=1}^{5} \omega_{k,k+1} = 0 \\ &\sum_{k=1}^{5} e^{-ik4\pi/5}\omega_{k,k+1} = 0 \end{aligned} \tag{6.31}$$

The second relation easily follows from the fact that $\gamma_{1,2} \circ \gamma_{2,3} \circ \gamma_{3,4} \circ \gamma_{4,5} \circ \gamma_{5,1}$ is homotopic to a single loop encircling all the five roots of $P(\tau)$, and the integral is regular at ∞. The first relation can be obtained by observing that

$$\omega_{i,i+1} = I_{i+1} - I_i \tag{6.32}$$

where I_i is the integral around a loop winding counterclockwise around the simple root τ_i. Therefore, the representation of B_5 on the $\omega_{i,i+1}$ is 3–dimensional, thus confirming the observation made in the introduction that the vanishing of the Yukawa couplings reduces the 6–dimensional $Sp(6, \mathbb{Z})$–valued representation of the periods into the $3 + \overline{3}$- representation which will be later shown to belong to $U(1,2)$.

The behaviour of $\omega_{ij}(a,b)$ around the $L(a,b) = 0$ singularity can be now obtained by expanding $P(\tau)$ in the neighbourhood of a point τ_0 where $P(\tau)$ vanishes together with its first derivative:

$$\begin{aligned} P(\tau) &= (\tau - \tau_0)^2 + H(\tau, a, b) \\ \lim_{\tau \to \tau_0} H(\tau, a, b) &= L(a,b) \end{aligned} \tag{6.33}$$

If we now set $\tau - \tau_0 = H^{1/2}\eta$, then we find

$$\omega_{ij} = const\ L^{1/10} \oint \frac{d\eta}{(\eta^2+1)^{3/5}} \equiv const\ L^{1/10}(a,b) \tag{6.34}$$

thus finding that, upon performing a loop around a branch of the hypocycloid, the integral acquires a phase $e^{i\pi/5} \equiv z$.

An alternative way of reaching the same conclusion is to consider the effect of analytic continuation of $\omega_{i,i+1}$ around the loop $\alpha_i \in B_5$ in the (a,b)–plane,

$$\omega_{i,i+1} \xrightarrow{\alpha_i} e^{i\pi/5}\omega_{i,i+1} \tag{6.35}$$

Indeed, when the path α_i winds around the i–th branch of the hypocycloid, the roots $\tau_i(a,b), \tau_{i+1}(a,b)$ of the polynomial $P(\tau)$ are exchanged. In the τ plane, $\gamma_{i,i+1}$ gets deformed as in Fig. 7. The final result is a new circuit followed in opposite direction where the base point B is on a different Riemann sheet. This gives the same integral as before except for a phase $-(e^{2\pi i})^{-2/5} = z$.

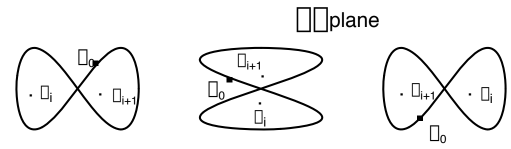

Fig. 7

The correctness of this result can also be ascertained by studying the leading behaviour of the period ω_0 from the Picard-Fuchs equations. Using the methods of [7,8], one derives the following set of three partial differential equations

$$\begin{aligned} \frac{\partial^2\omega_0}{\partial a^2} &= \frac{1}{1-4ab}\left[6b^2\frac{\partial^2\omega_0}{\partial a\partial b} + 3a\frac{\partial^2\omega_0}{\partial b^2} + 12b\frac{1-2ab}{1-4ab}\frac{\partial\omega_0}{\partial a} + \frac{12a^2}{1-4ab}\frac{\partial\omega_0}{\partial b}\right] \\ \frac{\partial^2\omega_0}{\partial b^2} &= \frac{1}{1-4ab}\left[6a^2\frac{\partial^2\omega_0}{\partial a\partial b} + 3b\frac{\partial^2\omega_0}{\partial a^2} + 12a\frac{1-2ab}{1-4ab}\frac{\partial\omega_0}{\partial b} + \frac{12b^2}{1-4ab}\frac{\partial\omega_0}{\partial a}\right] \\ \frac{\partial^2\omega_0}{\partial a\partial b} &= \frac{1}{1-13ab}\left[6a^2\frac{\partial^2\omega_0}{\partial a^2} + 6b^2\frac{\partial^2\omega_0}{\partial b^2} + 11a\frac{\partial\omega_0}{\partial a} + 11b\frac{\partial\omega_0}{\partial b} + \omega_0\right] \end{aligned} \tag{6.36}$$

The study of the singularities of eqs. (6.36) is better achieved by writing the associated linear system. Setting

$$\begin{aligned} \omega_1 &= \frac{\partial\omega_0}{\partial a} = \oint_\gamma \frac{\mathrm{y}_4^3\mathrm{y}_5^2}{\mathcal{W}^2}\omega \\ \omega_2 &= \frac{\partial\omega_0}{\partial b} = \oint_\gamma \frac{\mathrm{y}_4^2\mathrm{y}_5^3}{\mathcal{W}^2}\omega \end{aligned} \tag{6.37}$$

and by elimination of the mixed derivatives in (6.36) we find

$$\frac{\partial}{\partial a}\mathbf{\Pi} = A(a,b)\mathbf{\Pi} \quad , \quad \frac{\partial}{\partial b}\mathbf{\Pi} = B(a,b)\mathbf{\Pi} \quad , \quad \mathbf{\Pi} = \begin{pmatrix}\omega_0 \\ \omega_1 \\ \omega_2\end{pmatrix} \tag{6.38}$$

where the 3×3 matrices $A(a,b)$ and $B(a,b)$ are rational functions of a and b whose denominator contains the singular locus $L(a,b)$. Actually, most of the matrix elements of A and B also contain an extra factor of $(1-4ab)$ whose appearance would imply the extra singular locus $4ab=1$. However, such singularity is a gauge artifact, as the two systems in (6.38) are covariant under gauge transformations

$$A \to \mathcal{N}^{-1}A\mathcal{N} - \mathcal{N}^{-1}\partial\mathcal{N} \tag{6.39}$$

(and similarly for B), where $\mathcal{N}$ belongs to the Borel subgroup of lower triangular matrices of $GL(3,\mathbb{C})$. The exam of the Picard-Fuchs system in the neighbourhood of the variable $v = ab = 1/4$ gives in fact perfectly regular solutions in the new basis

$$\begin{aligned} \widetilde{\omega_0} &\equiv \omega_0 \\ \widetilde{\omega_1} &= \omega_1 + 8a^5\omega_2 \\ \widetilde{\omega_2} &= 2(64a^5-5)\omega_1 - (1+32a^5)\omega_2 \end{aligned} \tag{6.40}$$

The same conclusion is found by replacing the two linear systems in (6.38) by two 3rd order ordinary differential equations in a and b for ω_0, with coefficients depending on the other variable, where the singularity $ab = 1/4$ is absent. Furthermore, the Fuchsian analysis of the linear system (6.38) or of the two 3rd order differential equations in the neighbourhood of $L(a,b)$=0 gives again the behaviour $L^{1/10}$ around the singularity, thus confirming our previous analysis.

6.4 The monodromy generators

In this section we determine the representation $\mathcal{R}(\alpha_i) \subset GL(3,\mathbb{C})$ of the fundamental group acting on V, the vector space spanned by three independent periods. The computation is made by extending the representation of the group B_5 to a representation on the group ring over the complex field $\mathbb{C}$. Noting that α_i acts on the period $\omega_{i,i+1}$ as an analytic continuation around the i–th branch of $L(a,b)$, the corresponding discontinuities are given by

$$\begin{aligned} (\mathcal{R}(\alpha_i)-1)\omega_{i,i+1} &= (z-1)\omega_{i,i+1} \\ (\mathcal{R}(\alpha_i)-1)\omega_{j,j+1} &= 0 \qquad\qquad i \neq j, j+1 \end{aligned} . \tag{6.41}$$

Hence we introduce the (Picard–Lefschetz) discontinuity operators [59]

$$\mathcal{R}(\alpha_i) = (z-1)u_i + 1 \tag{6.42}$$

where u_i are 1-dimensional projection operators obeying $u_i^2 = u_i$. The first set of relations (6.18) imply

$$u_i u_j = u_j u_i \tag{6.43}$$

where i,j are non contiguous indices and we are considering $1,2,3,4,5$ cyclicly ordered so that 1 and 5 are contiguous. By right multiplication with u_j and left multiplication with u_i we get

$$\begin{aligned} u_i u_j &= u_j u_i u_j \equiv \lambda\, u_j \\ u_i u_j &= u_i u_j u_i \equiv \lambda\, u_i \end{aligned} \tag{6.44}$$

where we have used the fact that since the u_i are 1-dimensional projection operators, for any operator $\mathcal{O}$,

$$u_i \mathcal{O} u_i = \lambda_{\mathcal{O}}\, u_i \tag{6.45}$$

Eqs. (6.44) then imply

$$u_i u_j = u_j u_i = 0 \tag{6.46}$$

which is a relation much stronger than (6.43). Indeed, eq. (6.46) has an intuitive meaning since *e.g.* u_1, u_3 correspond to projection of the integral (6.29) around the disconnected figure eight circuits winding the roots τ_1, τ_2 and τ_3, τ_4 respectively. From the second set of (6.18) we get

$$(z-1)^2 u_i u_{i+1} u_i + z u_i = (z-1)^2 u_{i+1} u_i u_{i+1} + z u_{i+1} \tag{6.47}$$

Again, since the u_i are 1-dimensional projection operators, we have

$$\begin{aligned} u_i u_{i+1} u_i &= \rho\, u_i \\ u_{i+1} u_i u_{i+1} &= \sigma\, u_{i+1} \end{aligned} \tag{6.48}$$

Multiplying the two equations in (6.48) by u_{i+1} on the right and u_i on the left respectively, we find $\rho = \sigma$, so that (6.47) gives

$$u_i((z-1)^2\rho + z) = u_{i+1}((z-1)^2\rho + z) \tag{6.49}$$

or

$$\rho = -\frac{z}{(z-1)^2} \tag{6.50}$$

Using the relation (6.30) in the form $\alpha_5\alpha_1\alpha_3\alpha_2 = \alpha_4\alpha_3\alpha_2\alpha_1$, we find from (6.42)

$$\begin{aligned}(z-1)^3 u_5u_4u_3u_2 + (z-1)^2 u_5u_4u_3 + (z-1)u_5u_4 + u_5 = \\ = (z-1)^3 u_4u_3u_2u_1 + (z-1)^2 u_3u_2u_1 + (z-1)u_2u_1 + u_1\end{aligned} \tag{6.51}$$

By right multiplication with u_2 and left multiplication with u_5, using (6.48) – (6.50) we get

$$(z-1)z^2 u_5u_4u_3u_2 = u_5u_1u_2 \tag{6.52}$$

where the final form of the coefficient on the l.h.s. of (6.52) is obtained by using the relations obeyed by the tenth roots of unity $z = e^{i\pi/5}$ (*e.g.* $1 - z + z^2 - z^3 + z^4 \equiv 0$). Obviously, equations analogous to (6.52) are also obeyed by similar products of u_i's with indices cyclically permuted. Such relations allow us to replace a product of four contiguous u_i operators in decreasing order with (a coefficient times) the product of three contiguous u_i's in increasing order, the first and the last factors being the same in both expressions.

We are now ready to construct the explicit representation for the α_i's. We select three arbitrary linearly independent basis vectors $\omega_{51}, \omega_{12}, \omega_{45}$ defined as the eigenvectors of u_5, u_1, u_4 corresponding to eigenvalue one

$$\begin{aligned} u_5\,\omega_{51} &= \omega_{51} \equiv \Psi_{51} \\ u_1\,\omega_{51} &= \frac{1}{z-1}\omega_{12} \equiv \Psi_{12} \\ u_4\,\omega_{51} &= \frac{1-z}{z}\omega_{45} \equiv \Psi_{45} \end{aligned} \tag{6.53}$$

where the factors in front of ω_{ij} have been chosen in such a way that the action of the cyclic permutation Z of $\mathbb{Z}_5 \subset B_5$ gives $Z\omega_{i,i+1} = \omega_{i+1,i+2}$ with no extra phase. The application of $u_i\,, i = 1,\ldots,5$ to any basis vector gives a linear combination of them, namely

$$u_iu_k\Psi_{51} = p_i\Psi_{51} + q_i\Psi_{12} + r_i\Psi_{45} \qquad k = 5,1,4 \tag{6.54}$$

It is now easy to compute the coefficients p_i, q_i, r_i by repeated use of the formulae (6.48)-(6.51). For instance, if we take $i = 3$, then

$$\begin{aligned} u_3\Psi_{51} &= u_3u_5\Psi_{51} = 0 \\ u_3\Psi_{45} &= u_3u_4\Psi_{51} = p\Psi_{51} + qu_1\Psi_{51} + ru_4\Psi_{51} \end{aligned} \tag{6.55}$$

On the other hand,

$$u_3u_4\Psi_{51} = u_3u_4u_5\Psi_{51} \tag{6.56}$$

and using (6.52)

$$u_3u_4u_5\Psi_{51} = (z-1)z^2u_3u_2u_1u_5\Psi_{51} = p\Psi_{51} + qu_1\Psi_{51} + ru_4\Psi_{51} \tag{6.57}$$

Multiplying the last two sides by u_2 on the left we obtain

$$\begin{aligned} &-(z-1)z^2\frac{z}{(z-1)^2}u_2u_1\Psi_{51} = qu_2u_1\Psi_{51} \\ &\to q = -\frac{z^3}{z-1} \end{aligned} \tag{6.58}$$

Applying now u_3 on the left of (6.55), we find

$$u_3u_4\Psi_{51} = r\,u_3u_4\Psi_{51} \qquad \to r = 1 \tag{6.59}$$

Finally, acting with u_4 in (6.55) we have

$$\begin{aligned} &u_4u_3u_4\Psi_{51} = (p+r)u_4\Psi_{51} \\ &\to -\frac{z}{(z-1)^2}u_4\Psi_{51} = (p+r)u_4\Psi_{51} \qquad \to p = \frac{1-z^3}{(z-1)^3} \end{aligned} \tag{6.60}$$

In the same way one can compute the coefficients for the action of any other projection operator u_i. The final result for the monodromy operator α_i on the basis $\{\omega_{45}, \omega_{51}, \omega_{12}\}$ is

$$\begin{aligned} &\mathcal{R}(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & z \end{pmatrix} \quad , \quad \mathcal{R}(\alpha_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z^4 & -1+z-z^2 & z \end{pmatrix} \\ &\mathcal{R}(\alpha_3) = \begin{pmatrix} z & z^2(1-z) & z^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \mathcal{R}(\alpha_4) = \begin{pmatrix} z & 0 & 0 \\ -z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\mathcal{R}(\alpha_5) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & z & 0 \\ 0 & -z & 1 \end{pmatrix} \quad . \end{aligned} \tag{6.61}$$

From eqs. (6.61) we may verify that indeed α_5 satisfies the relation (6.20). Furthermore, we can use (6.61) to compute the monodromy generator around ∞, which we have disregarded until now. It can be shown that the generator α_∞ can be written by the following word

$$\alpha_\infty = \alpha_4\alpha_2\alpha_3\alpha_5\alpha_1\alpha_5 \tag{6.62}$$

and we obtain

$$\mathcal{R}(\alpha_\infty) = \begin{pmatrix} z^2 & 1+z^2 & z(1+z^2) \\ -z^2 & -1-z^2 & -z^3 \\ -1 & -1 & -1-z \end{pmatrix} \tag{6.63}$$

We note that the eigenvalues of $\mathcal{R}(\alpha_\infty)$ are $\{-1,-1,-z\}$ thus showing the presence of a singularity at ∞ with critical exponent$-\frac{2}{5}$. This can also be confirmed by the behaviour of the integral (6.29) for large values of a and b. We find

$$\omega_0 \overset{a,b\to\infty}{\sim} \oint \frac{d\tau}{(5a\tau^3+5b\tau^2)^{2/5}}\ , \tag{6.64}$$

and by the rescaling $a \to \lambda\xi\ , b \to \lambda\eta$ we find

$$\omega_0 \sim \oint \frac{d\tau}{(\xi\tau^3+\eta\tau^2)^{-2/5}}\ \lambda^{-2/5} = const\ \lambda^{-2/5} \tag{6.65}$$

thus confirming the critical behaviour computed from (6.63).

6.5 The duality group

It is known that the full duality group of the moduli space is given not only by the monodromy group of the periods, Γ_M, but also by the symmetry group of the defining polynomial $\mathcal{W}$, $\Gamma_\mathcal{W}$. We now want to show that the symmetries of the defining polynomial $\mathcal{W}(\mathrm{y};a,b)=0$ give at most a central extension for the monodromy group B_5 acting on the 3–dimensional basis of the periods.

It is easily seen that the transformations leaving invariant $\mathcal{W}$, are given by

$$\begin{cases} a & \to \rho\, a \\ b & \to \rho^{-1}\, b \end{cases} \tag{6.66}$$

with $\rho^5=1$, as they can be undone by the linear coordinate transformation

$$\begin{cases} \mathrm{y}_4 & \to \rho\, \mathrm{y}_4 \\ \mathrm{y}_5 & \to \rho^{-1}\, \mathrm{y}_5 \end{cases} \tag{6.67}$$

Since there is apparently no other action with this property, we conclude that the duality group of the superpotential is given by $\mathbb{Z}_5$.

In order to find the representation U of the transformations (6.66) on the periods, we observe that on any integral, say ω_{51}, the transformation (6.66) can be compensated in the integrand by the map

$$\tau \to \rho^3\tau \tag{6.68}$$

On the other hand, choosing $\rho = e^{4\pi i/5}$, the transformations (6.66) on the (p,q) real plane correspond to a rotation of an angle $4\pi/5$, mapping the 5–th branch of the hypocycloid into the 1–st, so that γ_{51} is mapped into γ_{12}. Taking into account that $d\tau \to \rho^3 d\tau$, we find

$$U\ :\ \omega_{51} \to e^{2\pi i/5}\omega_{12} \tag{6.69}$$

and analogous relations for cyclically permuted indices. We now observe that the transformation $\omega_{5,1} \to \omega_{1,2}$ is realized by the monodromy operator

$$Z = \mathcal{R}(\alpha_3\alpha_2\alpha_1\alpha_5) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z^4 & -1+z-z^2 & z-1 \end{pmatrix} \tag{6.70}$$

which corresponds to a generator of the cyclic subgroup $\mathbb{Z}_5 \subset B_5$. It follows that

$$UZ\ =\ e^{2\pi i/5}\mathbb{I} \tag{6.71}$$

on any period $\omega_{i,i+1}$ and therefore also on the selected basis $\{\omega_{45},\omega_{51},\omega_{12}\}$. Thus we conclude that, unless there is some element of B_5 represented by UZ, the U–transformation gives a central extension of the braid group B_5. The above central extension gives the full duality group of the moduli space of the Calabi–Yau manifold.

Notice that our result differs from the previously studied one–dimensional examples, where the full duality group Γ was given by the semidirect product $\Gamma_M \ltimes \Gamma_W$ of the monodromy group of the periods and the symmetry group of $\mathcal{W}$, as suggested in [8]. We further remark that, since the moduli space is 2–dimensional, we may take as coordinates the ratios $t_1 = \frac{\omega_{12}}{\omega_{51}}, t_2 = \frac{\omega_{45}}{\omega_{51}}$ which correspond to a linear combination of the "special" variables of special geometry. Hence, on t_1, t_2, the action of the full duality group is given by a faithful projective 3–dimensional representation of B_5.

Recalling that the full symmetry of the moduli space is given by modding out by Γ the local moduli space , we obtain that the geometry of $\mathcal{M}$ is given by

$$\mathcal{M} = \frac{U(1,2)}{U(1) \otimes U(2)} / \widehat{B}_5 \tag{6.72}$$

where $\widehat{B}_5$ is the previously introduced central extension of B_5.

Some comments are in order. We have found a 3-dimensional representation of the monodromy group for the three fundamental periods $\omega_{4,5}, \omega_{5,1}, \omega_{1,2}$ given by the three integrals associated to independent loops of the integral (6.29), or, equivalently, to the top solution of the system of differential equations (6.38). We know that B_5 must act as a group of discrete isometries on the local moduli space $\frac{U(1,2)}{U(1)\otimes U(2)}$ and therefore our matrices should belong to the $U(1,2)$ group. Infact, it can be shown that the matrices (6.61) satisfy

$$\mathcal{R}(a_i^\dagger)\, g\, \mathcal{R}(a_i) \;=\; g \tag{6.73}$$

where g is the metric given by

$$g = \begin{pmatrix} 1 & -z^2 & 1-z^3 \\ z^3 & 1-z+z^4 & -z^2 \\ 1+z^2 & z^3 & 1 \end{pmatrix} \tag{6.74}$$

Since g has one positive and two negative eigenvalues, indeed $\alpha_i \in U(1,2)$. As we have already remarked, there must exist a canonical basis for the $H_{(3)}$ homology of the Calabi–Yau , where the direct sum of the 3 and $\overline{3}$ representations of $U(1,2)$ given by (6.61) and their complex conjugate take values in $Sp(6,\mathbb{Z})$, six being the dimension of $H_{(3)}$.

Let us summarize our results. Starting with the family of manifolds given in eq. (6.1) we have been able to compute exactly the duality group of the periods associated to $\mathcal{W}(\mathrm{y}; a, b)$ by means of some very efficient and powerful techniques of algebraic geometry, without resorting to the explicit computation of the periods *e.g.* via solutions of the Picard-Fuchs equations. Our method is in principle applicable also to more complicated situations where more moduli are present and/or Yukawa couplings are non–vanishing. In fact, the computation of the fundamental group $\pi_1(CP_N - L^{N-1}; B)$ is always possible in virtue of the fundamental theorems of Picard–Severi and Zariski, together with the Van Kampen relations. The actual construction of the monodromy group relies however also on the knowledge of the behaviour of the periods around the singular locus of the defining polynomial. In our example, this computation was derived from the study of the 1–dimensional integral, which simplifies the actual task. It is clear that in general one cannot expect that the periods can always be reduced to such one–dimensional integrals, and the exam of the leading singularity can be more involved. Still, it is important to realize that, as mentioned in section 5, the analysis of the singularities can always be done in a systematic way from the linear system of Picard–Fuchs equations using standard techniques of fuchsian analysis.

Appendix A. Special Geometry

In this appendix we first briefly recall the properties of a Kähler–Hodge manifold as they are relevant for $N = 1$ supergravity. Then we turn to special Kähler manifolds on whose geometrical structure this paper is based upon. We briefly indicate how special geometry arises from $N = 2$ supergravity and assemble the main formulas used in the text.

A.1 Kähler–Hodge manifolds

The metric of an n–dimensional Kähler manifold $\mathcal{M}$ is given by

$$g_{\alpha\overline{\beta}}(z,\overline{z}) = \partial_\alpha \partial_{\overline{\beta}} K(z,\overline{z}) \,, \tag{A.1}$$

where $K(z,\overline{z})$ is the Kähler potential. Let us introduce the 1–form $\mathcal{Q}$ defined by

$$\mathcal{Q} = -\frac{i}{2}\left(\partial_\alpha K dz^\alpha - \partial_{\overline{\alpha}} K d\overline{z}^{\overline{\alpha}}\right) . \tag{A.2}$$

Under Kähler transformations

$$K \longrightarrow K + f(z) + \overline{f}(\overline{z}) \tag{A.3}$$

$g_{\alpha\overline{\beta}}$ is left invariant, while $\mathcal{Q}$ transforms as a $U(1)$ connection (Kähler connection):

$$\mathcal{Q} \longrightarrow \mathcal{Q} + d(Im f) \,. \tag{A.4}$$

Introducing the Kähler closed 2–form ω

$$\omega = i g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^{\overline{\beta}} \,, \qquad d\omega = 0 \,, \tag{A.5}$$

we find

$$d\mathcal{Q} = \omega \,. \tag{A.6}$$

Therefore, the first Chern class of the $U(1)$ bundle L whose connection is $\mathcal{Q}$ coincides with the Kähler class ω. A manifold with this property is called a Kähler–Hodge manifold.

A section $\psi(z,\overline{z})$ of L with Kähler weight $(p,\overline{p})$ is defined by the transformation law

$$\psi(z,\overline{z}) \longrightarrow \psi(z,\overline{z}) \quad e^{-\frac{p}{2}f} \quad e^{-\frac{\overline{p}}{2}\overline{f}} \,. \tag{A.7}$$

Accordingly, we define $U(1)$ covariant derivatives by

$$\begin{aligned} D_\alpha \psi &= (\partial_\alpha + \tfrac{p}{2}\partial_\alpha K)\psi \,, \\ D_{\overline{\alpha}} \psi &= (\partial_{\overline{\alpha}} + \tfrac{\overline{p}}{2}\partial_{\overline{\alpha}} K)\psi \,. \end{aligned} \tag{A.8}$$

A covariantly holomorphic section, satisfying $D_{\overline{\alpha}}\psi = 0$, is related to a purely holomorphic field $\widetilde{\psi}$ by

$$\widetilde{\psi} = e^{\frac{\overline{p}}{2}K} \psi \,. \tag{A.9}$$

$\widetilde{\psi}$ has weight $(p-\overline{p},0)$ and satisfies $\partial_{\overline{\alpha}}\widetilde{\psi} = 0$. The Levi-Civita connections and their curvatures are defined by

$$\Gamma^\alpha_{\beta\gamma} = g^{\alpha\overline{\delta}} \partial_\beta g_{\gamma\overline{\delta}} \,, \qquad R^\alpha_{\beta\overline{\gamma}\delta} = \partial_{\overline{\gamma}} \Gamma^\alpha_{\beta\delta} \,. \tag{A.10}$$

(Analogous formulas hold for the barred quantities $\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}$ and $R^{\overline{\alpha}}_{\overline{\beta}\gamma\overline{\delta}}$.) Thus for a vector ϕ_α of weight $(p,\overline{p})$ the covariant derivatives read

$$\begin{aligned} D_\alpha \phi_\beta &= \left(\partial_\alpha + \tfrac{p}{2}\partial_\alpha K\right)\phi_\beta - \Gamma^\gamma_{\alpha\beta}\phi_\gamma \,, \\ D_{\overline{\alpha}} \phi_\beta &= \left(\partial_{\overline{\alpha}} + \tfrac{\overline{p}}{2}\partial_{\overline{\alpha}} K\right)\phi_\beta \,. \end{aligned} \tag{A.11}$$

A.2 Special Kähler manifolds

The notion of special Kähler geometry first arose in the context of coupling vector multiplets to $N=2$ supergravity (in four space–time dimensions). It was shown that the Kähler manifold spanned by the scalar fields z^α of the vector multiplets must be suitably restricted as a consequence of $N=2$ supersymmetry [20]. A coordinate free characterization of such restricted geometry was given in [24] in the context of $N=2$ supergravity and in [22,23] for a Calabi-Yau moduli space.

In order to understand how special geometry arises from $N=2$ supersymmetry let $(\lambda^{I\alpha}, \lambda^{\overline{\alpha}}_I)$ be the chiral–antichiral components of the gaugino field ($I=1,2$ being an $O(2)$ index), and $\mathcal{A}^A_\mu$ ($A=0,1,\ldots,n$) the vector superpartners and the graviphoton. On general grounds, their supersymmetry transformation laws are

$$\begin{aligned} \delta_\epsilon \mathcal{A}^A_\mu &= f^A_\alpha \overline{\lambda}^{\alpha I}\gamma_\mu \epsilon^J \epsilon_{IJ} - 2\epsilon_{IJ} L^A \epsilon^I \psi^J_\mu + h.c. \,, \\ \delta_\epsilon \lambda^{\alpha I} &= -2i g^{\alpha\overline{\beta}} \epsilon_J \left[C_{\overline{\beta}\gamma\delta} \overline{\lambda}^{\gamma I} \lambda^{\delta J} + C_{\overline{\beta}\overline{\gamma}\overline{\delta}} \overline{\lambda}^\gamma_L \lambda^\delta_M \epsilon^{IL}\epsilon^{JM}\right] + \ldots \end{aligned} \tag{A.12}$$

where the dots stand for terms that are irrelevant for now. Here ϵ^I and ψ^I_μ are the (chiral) supersymmetry parameter and gravitino field respectively, and

ϵ_{IJ} is the $O(2)$ antisymmetric symbol. The sections L^A, f^A_α, $C_{\overline{\alpha}\beta\gamma}$, $C_{\overline{\alpha}\overline{\beta}\overline{\gamma}}$ and their chiral partners $\overline{L}^A$, $\overline{f}^A_{\overline{\alpha}}$, $C_{\alpha\overline{\beta}\overline{\gamma}}$, $C_{\alpha\beta\gamma}$ are a priori unrestricted scalars and tensors whose Kähler weight is fixed by Kähler covariance. The restrictions on the Kähler geometry arise from the on shell closure of the above supersymmetry transformation rules. In the superspace approach, this corresponds to imposing the Bianchi identities on the supercurvatures. One finds that the closure on $\mathcal{A}^A_\mu$ implies

$$\begin{aligned} D_{\overline{\alpha}}L^A &= 0\,, & D_\alpha \overline{L}^A &= 0\,, \\ D_\alpha L^A &= f^A_\alpha\,, & D_{\overline{\alpha}}\overline{L}^A &= \overline{f}^A_{\overline{\alpha}}\,, \\ D_\alpha \overline{f}^A_{\overline{\beta}} &= g_{\alpha\overline{\beta}}\overline{L}^A\,, & D_{\overline{\alpha}} f^A_\beta &= g_{\alpha\overline{\beta}} L^A\,. \end{aligned} \tag{A.13}$$

(Note that the last set of equations is just the integrability condition of the second set.) Closure of the gaugino transformation implies

$$\begin{aligned} C_{\overline{\alpha}\beta\gamma} = C_{\alpha\overline{\beta}\overline{\gamma}} &= 0\,, \\ D_\alpha f^A_\beta &= -iC_{\alpha\beta\gamma}g^{\gamma\overline{\delta}}\overline{f}^A_{\overline{\delta}}\,, \\ D_{\overline{\alpha}}\overline{f}^A_{\overline{\beta}} &= -iC_{\overline{\alpha}\overline{\beta}\overline{\gamma}}g^{\overline{\gamma}\delta}f^A_\delta\,, \\ D_{\overline{\alpha}}C_{\beta\gamma\delta} &= D_{[\alpha}C_{\beta]\gamma\delta} = 0\,, \\ D_\alpha C_{\overline{\beta}\overline{\gamma}\overline{\delta}} &= D_{[\overline{\alpha}}C_{\overline{\beta}]\overline{\gamma}\overline{\delta}} = 0\,, \end{aligned} \tag{A.14}$$

as well as $C_{\alpha\beta\gamma}$ being a completely symmetric tensor. As an integrability conditions of eq. (A.14) one finds that the curvature satisfies the following constraint

$$R_{\overline{\alpha}\beta\overline{\gamma}\delta} = g_{\overline{\alpha}\beta}g_{\delta\overline{\gamma}} + g_{\overline{\alpha}\delta}g_{\beta\overline{\gamma}} - C_{\beta\delta\mu}g^{\mu\overline{\mu}}C_{\overline{\mu}\overline{\alpha}\overline{\gamma}}\,. \tag{A.15}$$

From eq. (A.14) we also learn that $C_{\alpha\beta\gamma}$ obeys

$$C_{\alpha\beta\gamma} = D_\alpha D_\beta D_\gamma S\,, \tag{A.16}$$

where S has weight $(2,-2)$.

The above properties lead to the following definition of a special Kähler manifold: A special Kähler manifold is a Kähler–Hodge manifold for which there exists a set of $n+1$ sections $L^A(z,\overline{z})$ and $\overline{L}^A(z,\overline{z})$ of weight $(1,-1)$ and $(-1,1)$ respectively, satisfying (A.13) and (A.14), and a section of weight $(2,-2)$ ($(-2,2)$) $C_{\alpha\beta\gamma}$ ($C_{\overline{\alpha}\overline{\beta}\overline{\gamma}}$) which is completely symmetric in its indices and satisfies (A.14).

Equivalently, a special Kähler manifold can be defined by introducing a 3–index symmetric tensor $C_{\alpha\beta\gamma}$ on a Kähler–Hodge manifold with the properties (A.14) and furthermore restricting the curvature by the constraint (A.15). The existence of the sections L^A and their properties then follow.

The Kähler potential itself is most easily expressed in terms of holomorphic sections. By using (A.9) one defines $X^A(z)$ and $W_{\alpha\beta\gamma}(z)$ of Kähler weight $(2,0)$ and $(4,0)$ respectively:

$$\begin{aligned} X^A(z) &= e^{-\frac{K}{2}}L^A(z,\overline{z})\,, & \partial_{\overline{\alpha}}X^A &= 0\,, \\ W_{\alpha\beta\gamma}(z) &= e^{-K}C_{\alpha\beta\gamma}(z,\overline{z})\,, & \partial_{\overline{\alpha}}W_{\beta\gamma\delta} &= 0\,. \end{aligned} \tag{A.17}$$

We also need to introduce a functional $F(X^A)$ which is holomorphic and homogeneous of degree 2 in the X^A:

$$2F = X^A F_A(X)\,, \qquad F_A \equiv \frac{\partial}{\partial X^A}F\,. \tag{A.18}$$

In terms of X^A and F_A the Kähler potential which solves the constraints (A.14) and (A.15) reads

$$K(z,\overline{z}) = -\ln iY\,, \qquad Y = X^A N_{AB} X^B = X^A\overline{F}_A - \overline{X}^A F_A\,, \tag{A.19}$$

where

$$N_{AB} = F_{AB}(X) - \overline{F}_{AB}(\overline{X})\,, \qquad F_{AB} = \partial_A\partial_B F\,. \tag{A.20}$$

Furthermore, $C_{\alpha\beta\gamma}$ is given by

$$\begin{aligned} C_{\alpha\beta\gamma} &= D_\alpha D_\beta D_\gamma S = e^K \partial_\alpha X^A \partial_\beta X^B \partial_\gamma X^C F_{ABC}\,, \\ S &= -\frac{1}{2}e^K X^A N_{AB} X^B\,. \end{aligned} \tag{A.21}$$

From eqs. (A.13), (A.14) and (A.17)-(A.21) it is straightforward to verify that X^A and F_A satisfy the same set of constraints. Therefore we introduce the $(2n+2)$ dimensional row vectors[⋆]

$$V \;=\; (X^A, F_A) \;:\equiv\; (X^0, X^\alpha, F_\alpha, -F_0)\;, \qquad (\alpha = 1,\ldots,n)\;. \tag{A.22}$$

Using (A.17)–(A.21) we rewrite the identities (A.13) and (A.14) as follows

$$\begin{aligned} D_\alpha V &= U_\alpha \\ D_\alpha U_\beta &= -iC_{\alpha\beta\gamma} g^{\gamma\overline{\delta}} \overline{U}_{\overline{\delta}} \\ D_\alpha \overline{U}_{\overline{\beta}} &= g_{\alpha\overline{\beta}} \overline{V} \\ D_\alpha \overline{V} &= 0\,. \end{aligned} \tag{A.23}$$

It is this set of constraints we use in the main text. Similarly, one derives the constraints including the anti–holomorphic derivative $D_{\overline{\alpha}}$.

The Kähler potential can be expressed in terms of V and $V^\dagger$ as follows:

$$K = -\ln\left(V(-iQ)V^\dagger\right)\,, \tag{A.24}$$

which makes its $Sp(2n+2,\mathbb{R})$ symmetry manifest. Above, Q is a symplectic metric which satisfies $Q^2 = -1\,, Q = -Q^T$. Our convention is

$$Q \;=\; \begin{pmatrix} & & & 1 \\ & & -\mathbb{1}_n & \\ & \mathbb{1}_n & & \\ -1 & & & \end{pmatrix}\,. \tag{A.25}$$

Note that the vector V in (A.22) is symplectic with respect to this metric.

An important property which follows from eqs. (A.23) is that the connections of special geometry defined in eq. (A.11) naturally decompose into holomorphic and non-holomorphic parts [14]. This fact can be displayed by defining

$$t^a(z) = \frac{X^a}{X^0}\,. \tag{A.26}$$

⋆ We take the expression (X^A, F_A) always as an abbreviation for $(X^0, X^a, F_a, -F_0)$.

In terms of t^a and X^0 one finds

$$\begin{aligned} K_\alpha(z,\overline{z}) &= \widehat{K}_\alpha(z) + \mathcal{K}_\alpha(z,\overline{z})\,, \\ \Gamma^\gamma_{\alpha\beta}(z,\overline{z}) &= \widehat{\Gamma}^\gamma_{\alpha\beta}(z) + T^\gamma_{\alpha\beta}(z,\overline{z})\,, \end{aligned} \tag{A.27}$$

where

$$\begin{aligned} \mathcal{K}_\alpha(z,\overline{z}) &= e^a_\alpha(z) K_a(z,\overline{z}) \equiv e^a_\alpha(z) \frac{\partial}{\partial t^a} K(t(z), \overline{t}(\overline{z})) \\ \widehat{K}_\alpha(z) &= -\,\partial_\alpha \ln X^0(z) \\ e^a_\alpha(z) &= \partial_\alpha t^a(z) \\ T^\gamma_{\alpha\beta}(z,\overline{z}) &= e^a_\alpha e^b_\beta \partial_b g_{a\overline{d}} g^{-1\overline{d}c} e^{-1\gamma}_c \\ \widehat{\Gamma}^\gamma_{\alpha\beta}(z) &= (\partial_\beta e^a_\alpha) e^{-1\gamma}_a\,. \end{aligned} \tag{A.28}$$

The holomorphic objects $\widehat{K}_\alpha$ and $\widehat{\Gamma}^\alpha_{\beta\gamma}$ transform as connections under Kähler and holomorphic reparametrizations respectively; moreover $T^\alpha_{\beta\gamma}$ is a tensor under holomorphic diffeomorphisms and $\mathcal{K}_\alpha$ is Kähler invariant. As a consequence one can define holomorphic covariant derivatives in analogy with (A.11) by

$$\widehat{D}_\alpha \phi_\beta \;=\; (\partial_\alpha + \tfrac{p}{2}\partial_\alpha \widehat{K})\phi_\beta - \widehat{\Gamma}^\gamma_{\alpha\beta}\phi_\gamma\;. \tag{A.29}$$

The covariant Picard–Fuchs equations precisely use this holomorphic derivative.

Moreover, $\widehat{\Gamma}$ is a flat connection, i.e. satisfies

$$\widehat{R}^\gamma_{\delta\alpha\beta} \equiv \partial_\delta \widehat{\Gamma}^\gamma_{\alpha\beta} - \partial_\alpha \widehat{\Gamma}^\gamma_{\delta\beta} + \widehat{\Gamma}^\mu_{\alpha\beta}\widehat{\Gamma}^\gamma_{\mu\delta} - \widehat{\Gamma}^\mu_{\delta\beta}\widehat{\Gamma}^\gamma_{\mu\alpha} = 0\,. \tag{A.30}$$

The holomorphic metric for which $\widehat{\Gamma}$ is a connection reads

$$\widehat{g}_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab} \tag{A.31}$$

where η_{ab} is a constant (invertible) symmetric matrix. (Note that $\widehat{g}_{\alpha\beta}$ has two holomorphic indices in contrast to the Kähler metric $g_{\alpha\overline{\beta}}$.)

The flat coordinates are exactly the "special coordinates" $t^a = z^\alpha$. In these coordinates we find

$$e^a_\alpha = \delta^a_\alpha, \qquad \widehat{\Gamma}^\delta_{\alpha\beta} = 0, \qquad \widehat{g}_{\alpha\beta} = \eta_{\alpha\beta} \tag{A.32}$$

(The gauge choice $X^0 = 1$ implies $\widehat{K}_\alpha = 0$.)

In terms of t^a one defines the Kähler invariant function

$$\mathcal{F}(t^a) \equiv (X^0)^{-2} F(X^A) \,. \tag{A.33}$$

The Kähler potential can then be expressed as (up to Kähler freedom)

$$K \;=\; -\ln i\big[2(\mathcal{F} - \overline{\mathcal{F}}) + (\mathcal{F}_a + \overline{\mathcal{F}}_a)(t^a - \bar{t}^a)\big] \,. \tag{A.34}$$

The special coordinates t^a play the double role of flat coordinates for the holomorphic geometry with flat connection $\widehat{\Gamma}$ and of "free falling frame" coordinates for (non-holomorphic) special geometry. The analogous of local Lorentz transformations in the free falling frame is given in our case by the symplectic transformations that relate equivalent patches of special coordinates.

Appendix B. Remarks on $w_3 = 0$ and covariantly constant w_4.

We first show that $w_3 = 0$ does not imply that the solutions of eq. (2.1) are equivalent to (2.10). Let us start from an arbitrary solution $V = (v_1, v_2, v_3, v_4)$. It is always possible to rescale the entire vector V by $1/v_1$. This leads to $V \to \widetilde{V} = (1, v_2/v_1, v_3/v_1, v_4/v_1)$ where $\widetilde{V}$ satisfies an equation (2.1) with $a_0 = 0$. In the next step we perform the coordinate transformation $z \to \widetilde{z} = v_2/v_1$. In these coordinates eq. (2.1) turns into

$$\left(\widetilde{\partial}^4 + \widetilde{a}_3\widetilde{\partial}^3 + \widetilde{a}_2\widetilde{\partial}^2\right)\widetilde{V} = 0\,, \qquad \widetilde{V} = (1, \widetilde{z}, f_1(\widetilde{z}), f_2(\widetilde{z})) \tag{B.1}$$

(Again, we have scaled out a_4). The two steps so far can be done for any fourth order equation. It is equivalent to fixing the scale (Kähler)–freedom and the coordinate frame. In these new coordinates w_3 is given by (we drop the tilde)

$$w_3 = -\partial a_2 - \frac{1}{2}a_2 a_3 + \frac{1}{2}\partial^2 a_3 + \tfrac{3}{4}a_3\partial a_3 + \tfrac{1}{8}a_3^3 \,. \tag{B.2}$$

By writing $a_3 = -2\partial \ln W$ and $a_2 = b_2 W$, w_3 simplifies to

$$w_3 = W(\partial b_2 - \partial^3 W^{-1}) \,. \tag{B.3}$$

Thus $w_3 = 0$ implies the relation

$$b_2 = \partial^2 W^{-1} + c_1 \,, \tag{B.4}$$

where c_1 is a constant. Inserting (B.4) into (B.1) we find

$$\left(\partial^2 W^{-1}\partial^2 + c_1\partial^2\right)\widetilde{V} = 0 \,. \tag{B.5}$$

For $c_1 = 0$ this is solved by

$$\partial^2 f_1 = W, \qquad \partial^2 f_0 = zW \tag{B.6}$$

which implies

$$f_0 = z f_1 - 2f_1 + c_2 z + c_3 \,. \tag{B.7}$$

This is precisely what is true in special geometry. However, one can easily check (by series expansion) that this does not apply any more if $c \neq 0$. This means that $w_3 = 0$ does not fully characterize the differential equation (2.6) of special geometry.

We finally discuss briefly solutions of the generic fourth order equation (2.5) with covariantly constant w_4:

$$\widehat{D}\, w_4 = 0 \,. \tag{B.8}$$

This can easily be solved in special coordinates, where (B.8) reduces to $\partial w_4(t) = 0$, by setting

$$W(t) = e^{\sqrt{5}\alpha t} \,. \tag{B.9}$$

From this we obtain $w_2 = -\frac{5}{2}\alpha^2$ and $w_4 = \alpha^4$, and (2.5) is solved by first changing to the coordinate system $u(t)$ where $w_2 = 0$, that is, where

$$\{u, t\} = -\tfrac{1}{2}\alpha^2 \,. \tag{B.10}$$

Then one solves the associated second order linear differential equation

$$\theta'' - \alpha^2\theta = 0 \Rightarrow \theta_1 = e^{\alpha t} \quad , \quad \theta_2 = e^{-\alpha t} \qquad u(t) \equiv \frac{\theta_1}{\theta_2} = e^{2\alpha t} \,. \tag{B.11}$$

In the coordinates u we have

$$w_4(u) = \left(\frac{dt}{du}\right)^4 w_4(t) = \left(\alpha\frac{dt}{du}\right)^4 = \frac{1}{16u^4}\ , \tag{B.12}$$

and the fourth order differential equation (2.5) becomes:

$$\widetilde{V}^{iv} + \frac{1}{16u^4}\widetilde{V} = 0 \qquad\qquad \widetilde{V} = \big(u'\big)\big)^{3/2}V \tag{B.13}$$

It has solutions $\widetilde{V} = u^{\beta_i}$, where β_i are the roots of

$$\beta(\beta-1)(\beta-2)(\beta-3) + \frac{1}{16} = 0\ . \tag{B.14}$$

Altogether we find:

$$V(t) = e^{3\alpha t}\big(e^{2\alpha\beta_1 t}, e^{2\alpha\beta_2 t}, e^{2\alpha\beta_3 t}, e^{4\alpha\beta_4 t}\big)\ . \tag{B.15}$$

This is similar to the instanton–corrected solution of [6], and more specifically suggests that a covariantly constant w_4 characterizes single instantons, in accordance with our considerations in sect. 2.1.

Appendix C. Differential equations for cubic F-functions

It is helpful to first reconsider the first order system for one variable. We have seen in sect. 2.2 that for constant Yukawa coupling and in special coordinates, where $W = 1$ and $F = \frac{1}{6}t^3$, the matrix connection is given by the step generator

$$\Gamma_w \ = \ \mathbb{C} \ \equiv \ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ = \ J_-\ , \tag{C.1}$$

of the principal $SL(2)$ subgroup $\mathcal{K} \subset Sp(4)$. The diagonal generator J_0 belongs to the gauge group. Thus, in accordance with our considerations in sect 3.1, the moduli space is $SL(2)/U(1)$.

More generally, consider special geometries that have a cubic F–function

$$F = \frac{1}{3!}W_{abc}\frac{X^aX^bX^c}{X^0}$$

They correspond to special, homogeneous Kählerian manifolds, G/H provided W_{abc} satisfy suitable restrictions[20][58][60]They typically describe moduli spaces of orbifolds.

Let us fix the gauge $X^a \equiv t^a, X^0 \equiv 1$. Then the flat coordinates t^a are associated to G/H. More precisely, they are associated with the (mutually commuting) broken raising generators of G in the Cartan-Weyl basis, and thus they are coordinates of G^c/B (which is, essentially, isomorphic to G/H). Furthermore, the subgroups H act linearly on the coordinates. One may view the maximal compact subgroups H as being gauged by the connections $\widehat{\Gamma}$ in (3.10). Thus the generalization to many variables is the system of coupled matrix differential equations

$$\Big[\mathbb{1}\partial_a \ - \ \mathbb{C}_a\Big]\mathbf{V} \ = \ 0, \tag{C.2}$$

where $\mathbb{C}_a$ are the generators of G/H appropriately embedded into $sp(2n+2)$ (with $n = dim_c G/H$). These equations are solved by each column of the symplectic matrix

$$\mathbf{V} \ = \ e^{t^a\mathbb{C}_a} \ = \ \begin{pmatrix} 1 & t^a & F_a & F \\ 0 & 1 & F_{ab} & t^a F_{ab} - F_b \\ 0 & 0 & 1 & t^b \\ 0 & 0 & 0 & 1 \end{pmatrix}\ , \tag{C.3}$$

whose first row gives the "period" vector*, $V = (1, t^a, \frac{1}{2}W_{abc}t^bt^c, \frac{1}{6}W_{abc}t^at^bt^c)$. Observe that $\mathbf{V} \in G^c/B \cong G/H$ reflecting the fact that the moduli space is given by G/H. Furthermore, the Yukawa couplings are just the top-bottom components of the triple product of coset generators (cf. (4.7)):

$$W_{abc} \ = \ \big(\mathbb{C}_a\mathbb{C}_b\mathbb{C}_c\big)_1^{\ (2n+2)}$$

$\star$ Note that the components of V are like elements of some local ring $\mathcal{R}^{(3)}$, the structure constants of which are given by the coset generators $\mathbb{C}_\alpha$.

The symplectic embeddings† of G generalize the principal embedding of $\mathcal{K}$. Note that in order for F to be cubic, the representation of V must be irreducible with respect to G:

$$R = \underline{2n+2} \text{ of } Sp(2n+2,\mathbb{R}) \longrightarrow r = \underline{2n+2} \text{ of } G\ ,$$

so that the top and bottom rows of $\mathbf{V}$ are highest and lowest weights of r. Otherwise, the action of $\mathbb{C}_a$ vanishes on some intermediate components of $\mathbf{V}$ (the highest weights), which implies that $W_{abc} \equiv 0$. For instance, for $\frac{SU(n,1)}{U(n)}$ with $n > 1$, $r = \underline{n+1} \oplus \overline{\underline{n+1}}$ is reducible and accordingly, F is not cubic but only quadratic.

As a further example, consider

$$G/H = \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}\ , \qquad dim_c G/H = n = 9\ . \tag{C.4}$$

Here, $G = SU(3,3)$ is maximally embedded in $Sp(20)$ according to $20 = (6\times 6\times 6)_{antisym}$. The variables t^a correspond to the broken generators $\mathbb{C}_a$, which transform as $(3,\overline{3})$ under $SU(3)\otimes SU(3)$. These matrices are the following nine commuting generators of $G = SU(3,3)$ in the 20−dimensional, threefold totally symmetric representation:

$$\begin{pmatrix} 0 & \delta_i^j\delta_{\overline{i}}^{\overline{j}} & 0 & 0 \\ 0 & 0 & \epsilon_{ijk}\epsilon_{\overline{ijk}} & 0 \\ 0 & 0 & 0 & \delta_i^j\delta_{\overline{i}}^{\overline{j}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.5}$$

with $i,\overline{i}, = 1,\cdots 3$. The local symmetry group $H = U(3)\times U(3)$ is embedded in $U(9)\times U(1)$.

† Such symplectic embeddings have also been considered in [61].

References

[1] T. Banks, L. Dixon, D. Friedan and S. Shenker, Nucl. Phys. B299 (1988) 613.

[2] E. Martinec, Phys. Lett. B217 (1989) 431; C. Vafa and N.P. Warner, Phys. Lett. B218 (1989) 51; W. Lerche, C. Vafa and N. Warner, Nucl. Phys. B324 (1989) 427; D. Gepner, Phys. Lett. B222 (1989) 207; P. Howe and P. West, Phys. Lett. B223 (1989) 377; S. Cecotti, L. Girardello and A. Pasquinucci, Nucl. Phys. B328 (1989) 701; Int. J. Mod. Phys. A6 (1991) 2427; C. Vafa, Int. J. Mod. Phys. A6 (1991) 2829; K. Intrilligator and C. Vafa, Nucl. Phys. B339 (1990) 95; C. Vafa, Mod. Phys. Lett. A4 1615 and Mod. Phys. Lett. A4 (1989) 1169; S. Cecotti, Int. J. Mod. Phys. A6 (1991) 1749 and Nucl. Phys. B355 (1991) 755; A. Giveon and D.J. Smit, Mod. Phys. Lett. A6 24 (1991) 2211.

[3] E. Witten, Comm. Math. Phys. 117 (1988) 353; 118 (1988) 411 and Nucl. Phys. B340 (1990), 281; T. Eguchi and S.K. Yang, Mod. Phys. Lett. A5 (1900) 1693; C. Vafa, Mod. Phys. Lett. A6 (1991) 337; K. Li, Nucl. Phys. B354 (1991) 711; B. Blok and A. Varchenko, Int. J. Mod. Phys. A7 (1992) 1467; R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435 and Nucl. Phys. B352 (1991) 59; A. Giveon and D.J. Smit, Progr. Theor. Phys. Suppl. 102 (1990) 351; Mod. Phys. Lett. A6 (1991) 2211; Int. J. Mod. A7 (1992) 973; S. Cecotti and C. Vafa, Nucl. Phys. B367 (1991) 359.

[4] P. Candelas, G. Horowitz, A. Strominger and A. Witten, Nucl. Phys. B268 (1985) 46; see also M. Green, J. Schwarz and E. Witten, *Superstring Theory* (Cambridge, Univ. Press, 1987).

[5] P. Candelas and X.C. de la Ossa, Nucl. Phys. B355 (1991) 455.

[6] P. Candelas, X.C. de la Ossa, P.S. Green and L. Parkes, Phys. Lett. 258B (1991) 118; Nucl. Phys. B359 (1991) 21.

[7] A. Cadavid and S. Ferrara, Phys. Lett. B267 (1991) 193.

[8] W. Lerche, D. Smit and N. Warner, Nucl. Phys. B372 (1992) 87.

[9] D. Morrison, in *Essays on Mirror manifolds* , , S.T. Yau Editor, Intenational Press (1992); A. Font, Nucl. Phys. B391 (1993) 358; A. Klemm and S. Theisen, Nucl. Phys. B389 (1993) 153.

[10] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59.

[11] E. Verlinde and N.P. Warner, Phys. Lett. 269B (1991) 96; Z. Maassarani, Phys. Lett. 273B (1991) 457; A. Klemm, M. G. Schmidt and S. Theisen, Int. J. Mod. Phys. A7 (1992) 6215.

[12] P. Aspinwall and D. Morrison,Comm. Math. Phys. 151 (1993) 245; E. Witten, in *Essays on Mirror Manifolds,* edited by S.T. Yau, Intenational Press (1992).

[13] S. Cecotti and C. Vafa, Nucl. Phys. B367 (1991) 359.

[14] S. Ferrara and J. Louis, Phys. Lett. B278 (1992) 240.

[15] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche and J. Louis, Int. J. Mod. Phys. A8 (1993) 79.

[16] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Harvard preprints HUTP-93/A008, HUTP-93/A025.

[17] P. Candelas, X.C. de la Ossa, A. Font, S. Katz and D. Morrison, *Mirror Symmetry for Two Parameter Models 1,* preprint CERN-TH.6884/93, UTTG-15-93, NEIP-93-005, OSU-M-93-1.

[18] P. Berglund, P. Candelas, X. de la Ossa, A. Font, T. Hübsch, D. Jančić, F. Quevedo, *Periods for Calabi–Yau and Landau–Ginzburg Vacua,* preprint CERN-TH. 6865/93, HUPAPP-93/3, NEIP 93-004 , NSF-ITP-93-96, UTTG-13-93.

[19] S. Hosono, A. Klemm, S. Theisen and S.T. Yau *Mirror Symmetry, Mirror Map and Applications to Calabi–Yau Hypersurfaces,* preprint HUTMP-93/0801 and LMU-TPW-93-22.

[20] B. de Wit and A. van Proeyen, Nucl. Phys. B245 (1984) 89; B. de Wit, P. Lauwers and A. van Proeyen, Nucl. Phys. B255 (1985) 569; E. Cremmer, C. Kounnas, A. van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B250 (1985) 385.

[21] S. Ferrara and A. Strominger, *N=2 spacetime supersymmtry and Calabi-Yau moduli space* , in Proceedings of College Station Workshop (1989) 245 ; S. Cecotti, Comm. Math. Phys. 124 (1989) 23; A. Cadavid, M. Bodner and S. Ferrara, Phys. Lett. B247 (1991) 25.

[22] A. Strominger, Comm. Math. Phys. 133 (1990) 163.

[23] P. Candelas and X.C. de la Ossa, Nucl. Phys. B355 (1991) 455.

[24] L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. B241 (1990) 57; Class. Quant. Grav. 1 (1990) 317; R. D'Auria, S. Ferrara and P. Fré, Nucl. Phys. B359 (1991) 705.

[25] N. Seiberg, Nucl. Phys. B303 (88) 206.

[26] S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A4 (1989) 2475, Phys. Lett. B213 (1988) 443.

[27] L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. B329 (1990) 27.

[28] B. Blok and A. Varchenko, Int. J. Mod. Phys. A7 (1992) 1467.

[29] L. Dixon and D. Gepner, unpublished; W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 427; B. Greene and M. Plesser, Nucl. Phys. B338 (1990) 15; P. Candelas, M. Lynker and R. Schimmrigk, Nucl. Phys. B341 (1990) 383; P. Berglund and T. Hübsch, Nucl. Phys. B393 (1993) 377; P. Berglund and S. Katz, *Mirror Symmetry for Hypersurfaces in Weighted Projective Space and Topological Couplings*, preprint IASSNS-HEP-93/65, NSF-ITP-93-128, OSU-M-93-2, UTTG-24-93.

[30] R. D'Auria and S. Ferrara, *String Quantum Symmetries from Picard–Fuchs Equations and their Monodromy* , preprint CERN-TH.6777/93, POLFIS-TH.24/93 and hep-th@xxx.lanl.gov 9306020, to appear on Ann. of Phys. .

[31] A. Ceresole, R. D'Auria and T. Regge, *Duality Group for Calabi–Yau two–Moduli Space,* preprint POLFIS-TH. 05/93, DFTT 34/93, in press Nucl. Phys. B.

[32] K. Kikkawa and M. Yamasaki, Phys. Lett. 149B (1984) 357; N. Sakai and L. Senda, Progr. Theor. Phys. 75 (1986) 692; V.P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. B287 (1987) 402; A. Giveon, E.

Rabinovici and G. Veneziano, Nucl. Phys. B322 (1989) 167; A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 167; M. Dine, P. Huet and N. Seiberg, Nucl. Phys. B322 (1989) 301; J. Molera and B. Ovrut, Phys. Rev. D40 (1989) 1150; J. Lauer, J. Maas and H.P. Nilles, Phys. Lett. B226 (1989) 251 and Nucl. Phys. B351 (1991) 353; W. Lerche, D. Lüst and N.P. Warner, Phys. Lett. B231 (1989) 418; M. Duff, Nucl. Phys. B335 (1990) 610; A. Giveon and M. Porrati, Phys. Lett. B246 (1990) 54 and Nucl. Phys. B355 (1991) 422; Giveon, N. Malkin and E. Rabinovici, Phys. Lett. B238 (1990) 57; J. Erler, D. Jungnickel and H.P. Nilles, preprint MPI-Ph/91-90; S. Ferrara, D. Lüst, A. Shapere and S. Theisen, Phys. Lett. B233 (1989) 147; J. Schwarz, Caltech preprint CALT-68-1581 (1990), Phys. Lett. B272 (1991) 239 and in *Strings: Stony Brook 1991* , World Scientific.

[33] E. Cremmer, S. Ferarara, C. Kounnas, and D. V. Nanopoulos, Phys. Lett. 133B (1983) 61; J. Ellis, A. B. Lahanas, D. V. Nanopoulos and K. Tamvakis, Phys. Lett. 134B (1984) 429; J. Ellis, C. Kounnas, and D. V. Nanopoulos, Nucl. Phys. B247 (1984) 373; R. Barbieri, E. Cremmer and S. Ferrara, Phys. Lett. 163B (1985) 143.

[34] E. Witten, Phys. Lett. B155 (1985) 151.

[35] T. Regge, *The Fundamental Group Of Poincaré And The Analytic Properties Of Feynman Relativistic Amplitudes* , Nobel Symposium Series VIII (1968); G. Ponzano and T. Regge, Proceedings of the Varna Conference (1968).

[36] G. Ponzano, T. Regge, E. R. Speer and M. J. Westwater, Comm. Math. Phys. 15 (1969) 83 and Comm. Math. Phys. 18 (1970) 1.

[37] A. Forsyth, *Theory of Differential Equations*, Vol. 4, Dover Publications, New-York (1959).

[38] P. Di Francesco, C. Itzykson and J.-B. Zuber, Comm. Math. Phys. 140 (1991) 543.

[39] S. Ferrara, D. Lüst and S. Theisen, Phys. Lett. 242B (1990) 39.

[40] Drinfel'd and V. G. Sokolov, Jour. Sov. Math. **30** (1985) 1975.

[41] B. Kostant, Am. J. Math. 81 (1959) 973.

[42] J. Balog, L. Feher, L. O'Raifeartaigh, P. Forgaćs and A. Wipf, Phys. Lett. 244B (1990) 435; Ann. Phys. 203 (1990) 194.

[43] W. Lerche, Nucl. Phys. B238 (1984) 582; W. Lerche and W. Buchmüller, Ann. Phys. 175 (1987) 159.

[44] See eg., P. Griffiths, Ann. Math. 90 (1969) 460.

[45] P. Candelas, Nucl. Phys. B298 (1988) 458.

[46] S. Cecotti, Comm. Math. Phys. 131 (1990) 517, Nucl. Phys. B355 (1991) 755, Int. J. Mod. Phys. A6 (1991) 1749.

[47] P. Berglund, E. Derrick, T. Hübsch and D. Jancic, *On Periods for String Compactifications*, preprint HUPAPP-93/6, IASSNS-HEP-93/80, UTTG-27-93.

[48] B. Greene, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 371.

[49] C. Vafa, Mod. Phys. Lett. A6 (1991) 337.

[50] A. Giveon and D.-J. Smit, Mod. Phys. Lett. A **6** No. 24 (1991) 2211.

[51] A. Erdélyi, F. Oberhettinger, W. Magnus and F.G. Tricomi, *Higher Trascendental Functions*, Mac Graw–Hill, New York, 1953.

[52] P. Frè and P. Soriani, private communication; see also P. Soriani, Sissa Ph. D. Thesis.

[53] W. Lerche, D. Lüst and N.P. Warner,Phys. Lett. 231 (1989) 417.

[54] M. Villasante, Phys. Rev. D45 (1992) 1831

[55] See O. Zariski, *Algebraic Surfaces* , , II ed., Springer-Verlag (1971).

[56] O. Zariski, Ann. for Math 38 (1937) 131.

[57] E. R. van Kampen, Amer. J. Math. 55 (1933) 255.

[58] E. Cremmer and A. Van Proyen, Class. Quantum Grav. 2 (1985) 445.

[59] V. I. Arnold, S. M. Gusein–Zade and A. N. Varchenko, *Singularities of Differentiable Maps* , Vol bf II (Birkhäuser).

[60] S. Cecotti, Comm. Math. Phys. 124 (1989) 23; B. de Wit and A. van Proeyen, Comm. Math. Phys. 149 (1992) 307 .

[61] P. Frè and P. Soriani, Nucl. Phys. B371 (1992) 659 .