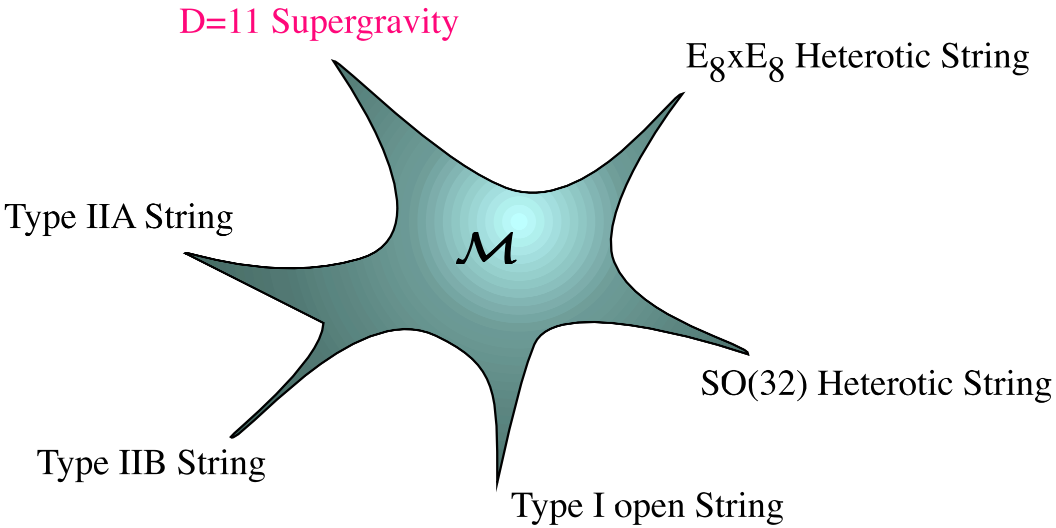
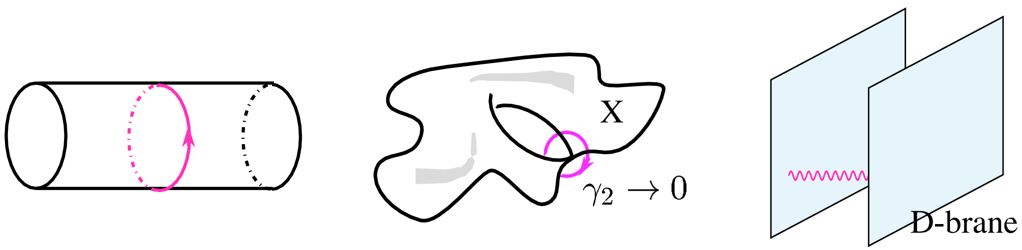
Quantum Geometry of D-Branes

W.L. Vienna Nov. 2004



- In stringy geometry, geometrical notions are in general ambiguous ...

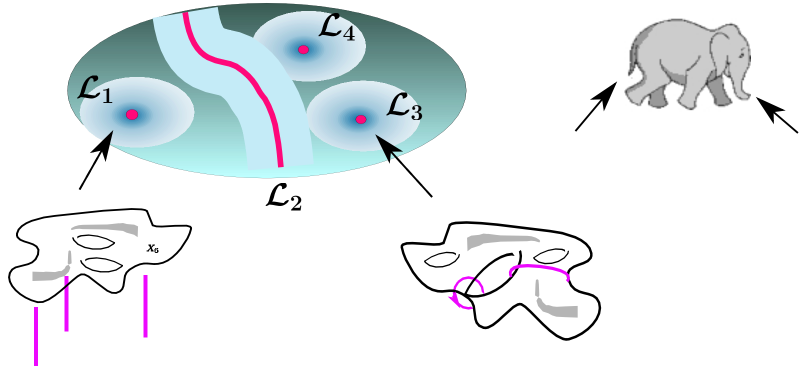
  One and the same theory may have many different dual geometric interpretations. For example gauge theory:



i) heterotic ii) type II iii) type I

Moduli Space = Space of Vacuum States, VEV's

- Lagrangian description makes sense only in "local coordinate patches" covering the parameter space $\mathcal{M}$:



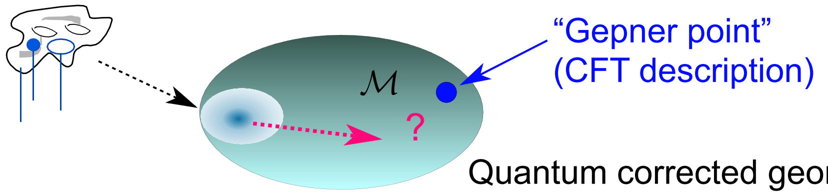
These describe **different local approximations** of the same theory in terms of different weakly coupled physical degrees of freedom.

The perturbative physics (local QFT) may look very different in the various local patches (eg, different gauge groups, different brane configurations)

- As a general rule, there is **no global description** that would be valid throughout the whole parameter space; no particular theory is more fundamental than the other ones.

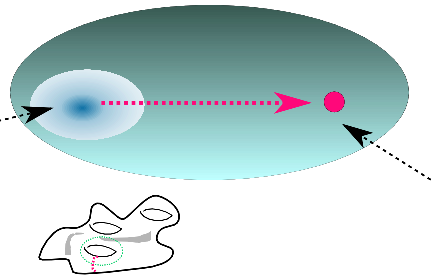
Quantum Properties of D-Branes

- Classical geometry ("branes wrapping p-cycles") makes sense only at weak coupling/large radius:



Quantum corrected geometry: (instanton) corrections wipe out notion of classical geometry

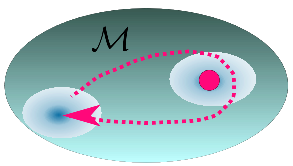
- Example: "**quantum volume**"



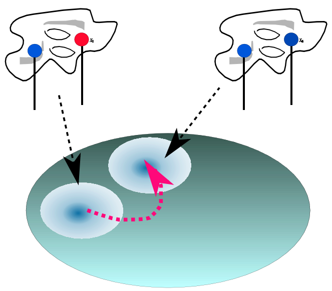
Calabi-Yau = 6-cycle is large

6-cycle -> zero size however: "embedded" 2,4 cycles have non-zero size !

- Example: "**monodromy**"

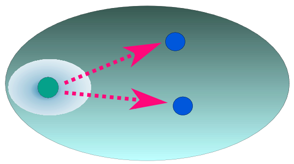


Looping around singularities returns "different" brane configuration



A brane-anti brane pair with apparent SUSY breaking, turns into a SUSY preserving brane-

- Example: "**orientifold plane**"



In the quantum theory, it splits into a pair of non-perturbative D-branes; no CFT description !

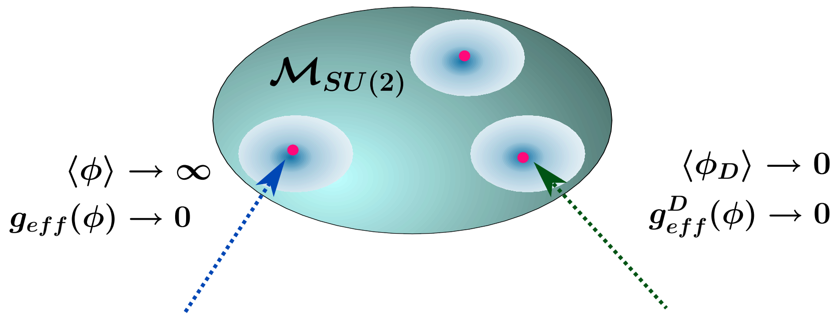
N=2 SUSY Gauge Theory

- Moduli (parameter) space $\mathcal{M} = \mathcal{M}(u) \quad u \sim < Tr\phi^2 >$

  ϕ ...complex adjoint Higgs field

Seiberg-Witten: Effective gauge coupling gets renormalized, and depends on the Higgs VEV:

$$\tau_{eff}(\phi) \;=\; \frac{1}{2\pi}\theta_{eff}(\phi) + 2\pi i \frac{1}{g^2_{eff}(\phi)}$$



gauge fields weakly coupled; monopoles strongly coupled

$$\tau(\phi) = \frac{i}{\pi}\log\left[\frac{\phi^2}{\Lambda^2}\right] - \frac{i}{\pi}\sum_{\ell=1}^{\infty} c_\ell \left(\frac{\Lambda}{\phi}\right)^{4\ell}$$

1-loop — instanton corr

SU(2) gauge theory with instanton corrections

gauge fields strongly coupled; massless monopoles weakly coupled, look like electrons:

$$\tau_D(\phi_D) = \frac{-1}{2\pi}\log\left[\frac{\phi_D^2}{\Lambda^2}\right] - \frac{1}{\pi}\sum_{\ell=1}^{\infty} c^D_\ell \left(\frac{\phi_D}{\Lambda}\right)^{2\ell}$$

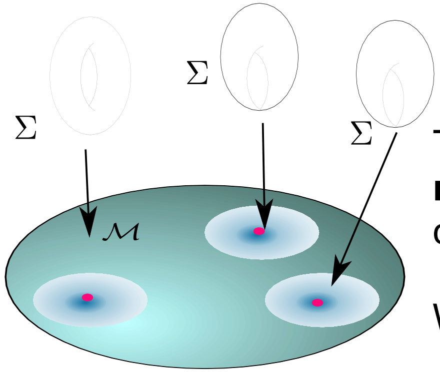
1-loop — non-pert corr

U(1) gauge theory with extra electrons

Resummation of non-perturbative corrections

Quantum Curves and Calabi-Yau Manifolds

- S&W: Interpretation of SYM parameter space as **moduli space** of an **elliptic curve** Σ :



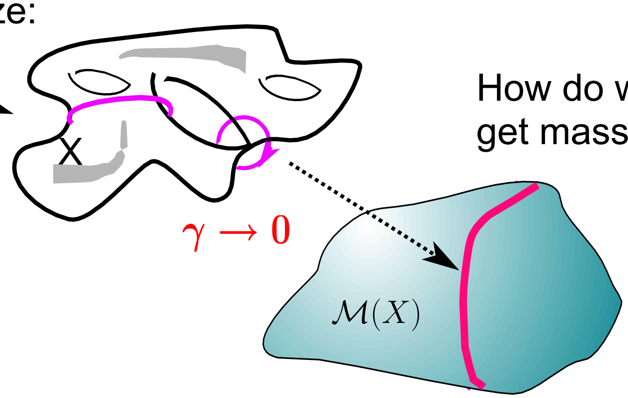
The loci where **massless non-perturbative states** appear correspond to singular curves.

What is their meaning ?

- This has a natural interpretation in string theory compactified on some Calabi-Yau manifold X (suitable field theory limit reproduces Σ).

There is a **proliferation of physical degrees of freedom**, obtained from wrapping strings (p=1), membranes (p=2), general p-branes around non-contractible p-cycles of X.

At a given singularity in the parameter space $\mathcal{M}(X)$, a compactification manifold X becomes singular in that some p-dimensional "vanishing cycle γ" shrinks to zero size:



How do we know that we get massless states?

Central Charges and Period Integrals

Supersymmetric ("BPS") configurations saturate the **BPS bound:**

$$m^2 \geq |Z|^2 \qquad \{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta}\, p_\mu + \delta_{\alpha\beta}\, Z$$

- Central charges have a **topological character**:

$$Z(u) = N_A\, \Pi^A(u)$$

$$\Pi^A(u) = \int_{\gamma_A} \omega(u)$$ ... **period integrals**, "quantum volumes"

N_A ... el, mag U(1) RR charges of D-brane

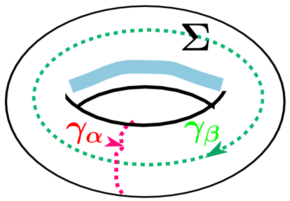
u ... modulus of CY, massless scalar VEV

- Eg., in SU(2) N=2 SUSY gauge theory, the mass of a BPS state is governed by

$$Z(u) = N\langle \phi \rangle(u) + M\langle \phi_D \rangle(u)$$

electric magnetic

$$= \int_\gamma \omega_\Sigma(u)$$

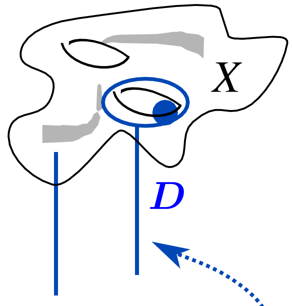


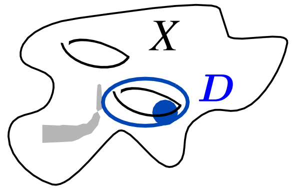
wrapped string

$$\gamma \equiv N\, \gamma_\alpha + M\, \gamma_\beta$$

... if the volume of this cycle vanishes, we get an extra massless state in the theory

- Eg, brane configurations D in Type II string theories:





world volume
3+1d N=1 SUSY "brane world"

p-branes wrapping p-cycles $\gamma_A^{(p)}$ appear as particle excitations in N=2 eff theory

Consider here only **internal**, wrapped piece of D-brane

There are **two kinds** of supersymmetric (BPS) branes:

"**A-type**" branes: wrap special lagrangian cycles $\gamma_A^{(p=3)}$
"**B-type**" branes: wrap holomorphic cycles $\gamma_A^{(p=0,2,4,6)}$

- Central charge (tension) of **A-type** brane:

$$Z^{(A)} = \int_{\gamma_A^{(3)}} \Omega^{(3,0)}$$

holomorphic 3-form

This is an exact result !

- Central charge (tension) of **B-type** brane

gauge bundle data V encoded in RR charge vector:

$$Q = tr\, e^F \sqrt{\widehat{A}(R)}$$

$$\int_{\gamma^{(2i)}} Q = \left\{ tr1 \equiv N_6,\ trF = c_1(V) \equiv N_4,\ N_2,\ N_0 \right\}$$

$$Z^{(B)}(t) = \int_X e^J Q + \ldots$$

$$= N_0 + N_2 \int_{\gamma^{(2)}} J + N_4 \int_{\gamma^{(4)}} J \wedge J + N_6 \int_{\gamma^{(6)}} J \wedge J \wedge J + \ldots$$

$$= N_0 + N_2 t + N_4 t^2 + N_6 t^3 + \mathcal{O}(e^{-t})$$

Instanton corrections from world-sheets wrapping 2-cycles !

$$\exp(-S_{inst}) = \exp(-\int_{P^1} J) \equiv \exp(-t)$$

- Fortunately, there is **mirror symmetry**:

B-branes wrapped over holom. (0,2,4,6) cycles of Calabi-Yau X ⟺ A-branes wrapped over special lagrangian 3-cycles of mirror CY $\widehat{X}$

We can thus equate central charges:

$$Z^{(B)}_{X;D}(t) = Z^{(A)}_{\widehat{X};\widehat{D}}(u(t))$$

and quantitatively study the implications of the instanton corrections

Monodromy of RR Charges

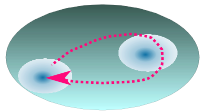
From mirror symmetry, we have for the BPS tension

$$Z(u(t)) \;=\; N_A \Pi^A \;=$$

$$N_A \int_{\gamma_A^{(3)}} \Omega^{(3,0)}(u(t)) \;=\; N_0 + N_2 t + N_4 \partial_t \mathcal{F}(t) + N_6 \mathcal{F}_0(t)$$

- Periods $\Pi_A = (X_a, \mathcal{F}^b)$ are multi-valued sections

Non-trivial loops in the moduli space will thus induce monodromy



$$\Pi^A \;\longrightarrow\; R \cdot \Pi^A, \qquad R \in Sp(2h^{2,1}+2, Z)$$

$$N_A \;\longrightarrow\; N_A \cdot R^{-1}$$

which for generic paths will completely mix up the RR-charges = brane wrapping numbers N_A !

The notion of a p-dimensional cycle, perhaps with a gauge bundle configuration V on top of it, looses its geometric meaning away from the semi-classical large radius limit !

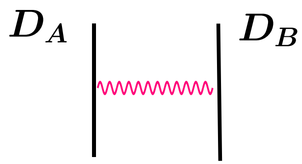
Flow of Gradings

A related phenomenon is tied to the phase of the central charge, the **grading**:

$$\phi^{(D)}(u) \;=\; \frac{1}{\pi} Im \; \ln(Z_{(D)}(u))$$

- For a single brane D it plays no role, but for two branes, it determines the mass and charge of an open string stretched between the branes:

$$q_{AB} \;=\; \phi^{(D_A)} - \phi^{(D_B)} \qquad {m_{AB}}^2 \;=\; \frac{1}{2}(q_{AB} - 1)$$



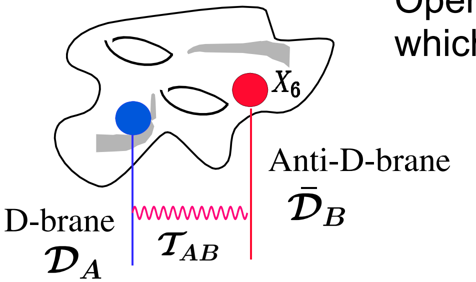
- Note that the open string can become **tachyonic**, signalling bound state formation

- These quantities depend continuously on the Kahler moduli, and it is important to follow them over the full moduli space.

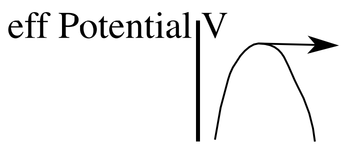
- In general, there won't be a globally valid notion of what a brane and an anti-brane is!

Stability and SUSY Breaking

- In the open string sector, SUSY is typically broken



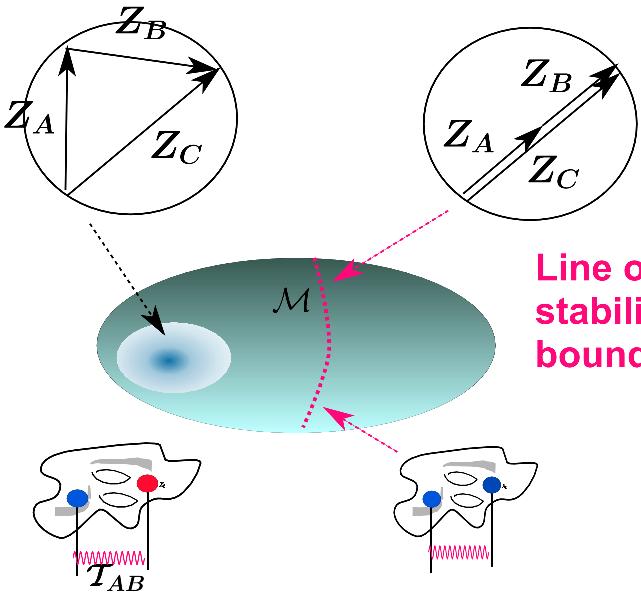
Open string is a tachyon $\mathcal{T}_{AB}$ which implies an unstable vacuum



Problem of SUSY vacuum structure is equivalent to **bound state problem** for wrapped branes

- Global flow of gradings: $m^2_{AB} \sim Im \ln[Z_A/Z_B]$

$m_C < m_A + m_B$ $\qquad m_C = m_A + m_B$



Line of marginal stability: bound state decays

SUSY broken $\qquad$ SUSY restored

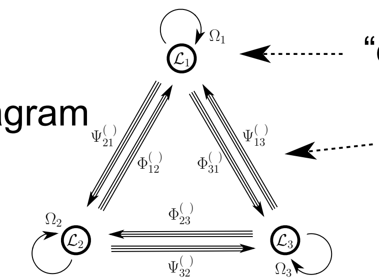
The Derived Category

- Have seen: geometrical notions such as the dimension of a p-cycle, bundle configurations, RR charges become blurred once we leave the large radius/weak coupling limit.

... need to develop formalism capable of describing the physics of general D-brane configurations (here: topological B-type D-branes)

→ **derived category** (of coherent sheaves on CY)

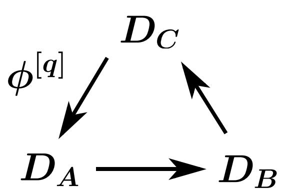
Quiver diagram



"objects": D-branes

"morphisms": open strings

- more general than K-theory (RR U(1)charges)
- keeps track of brane locations
- treats branes and anti-branes on equal footing
- easily describes bound state formation/tachyon condensation:



If $\phi^{[q]} \in Ext^q(D_A, D_B)$ is tachyonic, A and B will form a bound state C (analogous for A,C and B,C)

Boundary Landau-Ginzburg Theory

- The caterogy-theoretical framework seems very abstract, and one may ask for what it is good for in practice ?

It turns out that a certain open string topological field theory, namely boundary LG theory, provides a **very concrete physical realization** of it.

- Action:

$$S = \int_D d^2z d^2\theta \, W_{LG}(x) + \int_{\partial D} d\tau d\theta \, \Lambda J(x)$$

$$D\Lambda = E(x)$$

fermionic boundary superfield

- BRST operator/ supercharge: $Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}$

- B-type BPS branes are characterized by

$$W_{LG} = \frac{1}{2} Q^2 = J \, E$$

that is, by all polynomial **matrix factorizations** of the LG superpotential!

Minimal models

...may be viewed as building blocks of more complicated TFT's, like ones describing Calabi-Yau's.

Bulk (closed string) sector is described by superpotential ("level k"):

$$W_{LG}(x) = \frac{x^{k+2}}{k+2}$$

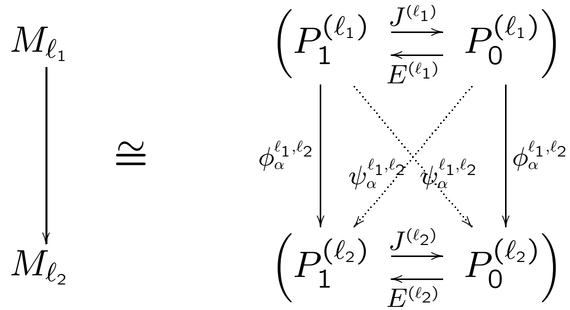
B-type D0-branes M_ℓ are described by the factorizations:

$$J(x) = x^{\ell+1}\,, \quad E(x) = \frac{x^{k-\ell+1}}{k+2}\,, \quad \ell = -1, 0, \ldots, \left[\frac{k}{2}\right]$$

- It turns out that this LG model realizes **precisely** a certain Z2 graded category defined by Kontsevich

Objects = D0 branes $M_\ell \cong \left(P_1^{(\ell)} \underset{E^{(\ell)}}{\overset{J^{(\ell)}}{\rightleftarrows}} P_0^{(\ell)} \right)$

Morphisms = boundary BRST cohomology



- One can study explicitly and exactly all details of bound state formation (cone construction), and determine the effective action (in terms of deformation parameters s,t):

$$W_{eff}(s,t) = \oint W_{LG}(x,t) \, \log\det J(x,s)$$

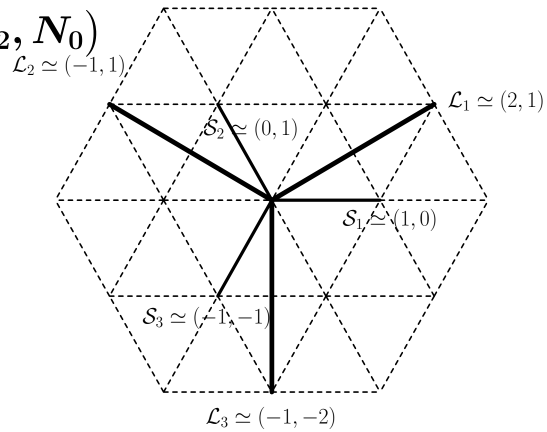
D-branes on the Elliptic Curve

- Simplest Calabi-Yau: the cubic torus

$$T_2 : \quad W_{LG} \equiv x_1{}^3 + x_2{}^3 + x_3{}^3 + a\, x_1 x_2 x_3 = 0$$

B-type D-branes: $(N_2, N_0) = (rank(V), c_1(V))$

..are mirror to A-type D1-branes with wrapping numbers $(p, q) = (N_2, N_0)$



- Simplest are matrix factorizations $J_i E_i = W_{LG}$ corresponding to branes $\mathcal{L}_i$ with (p,q)=(-1,1),(2,1),(-1,1)

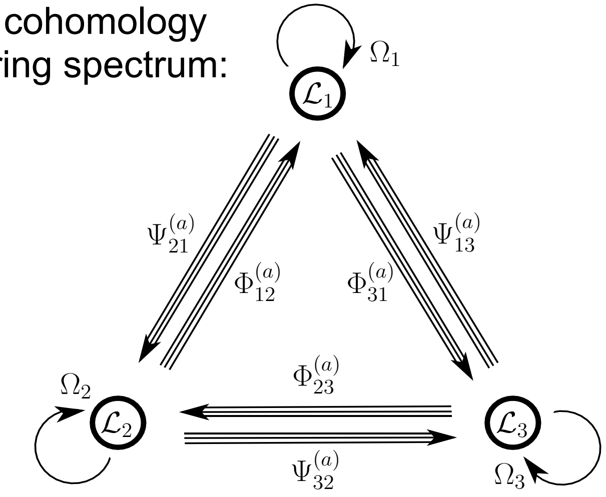
$$E_i = \begin{pmatrix} \frac{1}{\alpha_1^{(i)}} x_1{}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_2 x_3 & \frac{1}{\alpha_3^{(i)}} x_3{}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_1 x_2 & \frac{1}{\alpha_2^{(i)}} x_2{}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_1 x_3 \\ \frac{1}{\alpha_2^{(i)}} x_3{}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_1 x_2 & \frac{1}{\alpha_1^{(i)}} x_2{}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_1 x_3 & \frac{1}{\alpha_3^{(i)}} x_1{}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_2 x_3 \\ \frac{1}{\alpha_3^{(i)}} x_2{}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_1 x_3 & \frac{1}{\alpha_2^{(i)}} x_1{}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_2 x_3 & \frac{1}{\alpha_1^{(i)}} x_3{}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_1 x_2 \end{pmatrix}$$

$$J_i = \begin{pmatrix} \alpha_1^{(i)} x_1 & \alpha_2^{(i)} x_3 & \alpha_3^{(i)} x_2 \\ \alpha_3^{(i)} x_3 & \alpha_1^{(i)} x_2 & \alpha_2^{(i)} x_1 \\ \alpha_2^{(i)} x_2 & \alpha_3^{(i)} x_1 & \alpha_1^{(i)} x_3 \end{pmatrix}$$

$$\alpha_\ell^{(i)} \sim \Theta\Big[\frac{1-\ell}{3} - \frac{1}{2} - \frac{1}{2}\Big|\, 3u_i, 3\tau\Big]$$

u... brane locations,
Wilson lines

- Solving the BRST cohomology yields the open string spectrum:



Explicit matrix representation of the morphisms, eg for the fermionic open strings:

$$\Psi_{21}^{(i)} = \begin{pmatrix} 0 & F_{21}^{(i)} \\ G_{21}^{(i)} & 0 \end{pmatrix}, \quad i = 1, 2, 3.$$

with

$$F_{21}^{(1)} = \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & \zeta_2 \\ 0 & \zeta_3 & 0 \end{pmatrix}$$

$$G_{21}^{(1)} = -\begin{pmatrix} \frac{\zeta_1}{\alpha_1^{(1)}\alpha_1^{(2)}} x_1 & \frac{\zeta_3}{\alpha_1^{(1)}\alpha_2^{(2)}} x_2 & \frac{\zeta_2}{\alpha_1^{(1)}\alpha_3^{(2)}} x_3 \\ \frac{\zeta_2}{\alpha_1^{(2)}\alpha_3^{(1)}} x_2 & \frac{\zeta_1}{\alpha_2^{(2)}\alpha_3^{(1)}} x_3 & \frac{\zeta_3}{\alpha_3^{(1)}\alpha_3^{(2)}} x_1 \\ \frac{\zeta_3}{\alpha_1^{(2)}\alpha_2^{(1)}} x_3 & \frac{\zeta_2}{\alpha_2^{(1)}\alpha_2^{(2)}} x_1 & \frac{\zeta_1}{\alpha_2^{(1)}\alpha_3^{(2)}} x_2 \end{pmatrix}$$

$$\zeta_\ell \sim \Theta\Big[\frac{1-\ell}{3} - \frac{1}{2} - \frac{1}{2}\Big|\, 3u_2 - 3u_1, 3\tau\Big]$$

- compute disk correlation functions:
  = Yukawa couplings on intersecting branes

$$C_{ijk} = \langle \Psi_{13}^{(i)} \Psi_{32}^{(j)} \Psi_{21}^{(k)} \rangle$$

$$= \frac{1}{2\pi i} \oint \mathrm{Str}\Big[\frac{\partial_1 Q \wedge \partial_2 Q \wedge \partial_3 Q}{\partial_1 W \partial_2 W \partial_3 W} \Psi_{13}^{(i)} \Psi_{32}^{(j)} \Psi_{21}^{(k)} \Big]$$

Result:

$$C_{111}(\tau,\xi) = e^{6\pi i \xi_1 \xi_2} q^{3{\xi_2}^2/2} \sum_m q^{3m^2/2} e^{6\pi i m \xi}$$

$$C_{123}(\tau,\xi) = e^{6\pi i \xi_1 \xi_2} q^{3{\xi_2}^2/2} \sum_m q^{3(m+1/3)^2/2} e^{6\pi i (m+1/3)\xi}$$

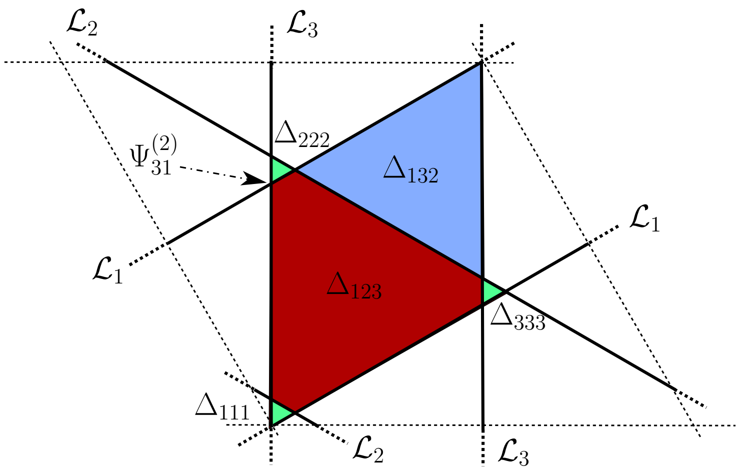
$$C_{132}(\tau,\xi) = e^{6\pi i \xi_1 \xi_2} q^{3{\xi_2}^2/2} \sum_m q^{3(m-1/3)^2/2} e^{6\pi i (m-1/3)\xi}$$

$(\xi \equiv u_1 + u_2 + u_3)$

Interpretation:

In A model mirror language, these are contributions from disk instantons whose world-sheets are bounded by the three D-branes.

$$q \sim \exp(-S_{inst}) \sim \exp(-Area)$$



Works analogously for Calabi-Yau threefolds!