Non-Perturbative Aspects of Supersymmetric Field and String Theories

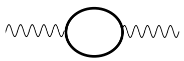
W.Lerche 1997

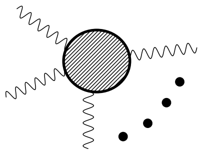
- **Seiberg and Witten 1994:**

Exact effective action (p=0) for N=2, d=4 supersymmetric gauge theory
effective gauge coupling:

$$\left(\frac{2\pi i}{g^2_{\rm eff}(\phi)} + \frac{\theta_{\rm eff}(\phi)}{2\pi}\right) = \underbrace{\frac{2\pi i}{{g_0}^2}}_{\text{bare coupling}} + \underbrace{\frac{i}{\pi}\log\left[\frac{\phi^2}{\Lambda^2}\right]}_{\text{one-loop}} - \underbrace{\frac{i}{\pi}\sum_{\ell=1}^{\infty} c_\ell \left(\frac{\Lambda}{\phi}\right)^{4\ell}}_{\text{instanton corrections}}$$

determine (c_ℓ)





- **Important concept: "Duality"**

Describe non-perturbative soliton-sectors (magn monopoles)
in terms of local QFT

- Self-consistent scheme based on some assumptions !

- Conceptual progress: can learn a lot about string theory, and stringy grand unification

Outline

- **N=2 supersymmetric gauge theory**
  - why supersymmetry
  - classical moduli space
  - quantum moduli space
  - monodromy problem and Riemann surface

- **N=2 supersymmetric string theory**
  - heterotic-type II string duality
  - derive SW theory from string duality

- **Self-dual strings on Seiberg-Witten Riemann surfaces:**

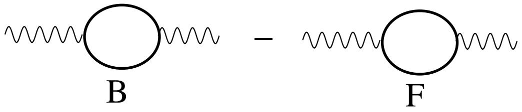
  ⇒ a new non-perturbative formulation of gauge theory

Why Supersymmetry ?

Supersymmetry = symmetry between fermions and bosons
Hope: not fundamentally important, but facilitates treatment

- **Non-renormalization properties:**

  keep perturbative corrections under tight control



= 0 (or finite)

- **Holomorphic structure** of Lagrangians, potentials:

  allows powerful methods of complex analysis and algebraic geometry

- **Duality symmetries** are more or less manifest

  eg., N=4 SUSY "BPS mass formula":

$$M^2 = |q + (i/g^2)m|^2 \, |\langle \phi \rangle|^2$$

is invariant under

	$\langle \phi \rangle \longleftrightarrow \langle \phi \rangle / g^2$	
perturbative	$g^2 \longleftrightarrow 1/g^2$	non-perturbative
electric charge	$q \longleftrightarrow m$	magnetic charge
elementary fields	$\longleftrightarrow$	solitons (mag. monopoles)

Gauge theories with N supersymmetries

N = 4: Theory is "self-dual": electric and magnetic sectors are equiv.;
has no quantum corrections: almost trivial

N = 1: probably not fully solvable; radiative corrections only partly
under control (for "chiral"=holomorphic superpotential etc)

N = 2: just solvable (in low-energy limit);
radiative corrections under full control !

N=2 Supersymmetric Gauge Theory for SU(2)

- Fields: constitute supermultiplets composed out of bosons and fermions
  bosonic: Higgs field ϕ , Gauge field A_μ

- Gauge invariant order parameter: vacuum value of Higgs field

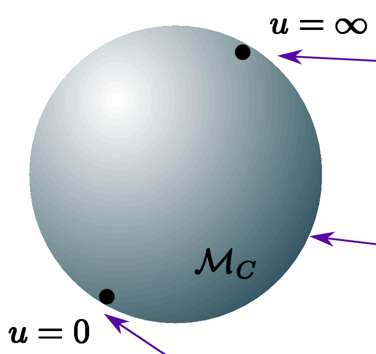
$$u \equiv \frac{1}{2}\mathrm{Tr}\langle \phi^2 \rangle$$

Spontaneous symmetry breaking of gauge symmetry G:

$$u = 0 \longrightarrow G = SU(2)\,, \qquad u \neq 0 \longrightarrow G = U(1)$$

- Parameter space = space of vacuum configurations = **"Moduli Space"**

classical moduli space $\mathcal{M}_C$: does not take quantum corrections into account



semi-classical, weakly coupled regime
perturbation theory is reliable

generically, the unbroken gauge
symmetry is U(1)

naively unbroken SU(2) gauge symmetry
but strongly coupled: non-perturbative quantum effects dominant

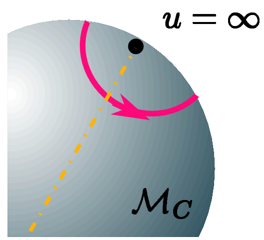
➡ Need to study global properties of the moduli space.

Monodromy of Gauge Coupling Constant

In the semi-classical regime, where u is large, the complexified gauge coupling can be evaluated in perturbation theory to leading order:

$$\tau(\phi) \equiv \left(\frac{2\pi i}{g^2_{\text{eff}}(\phi)} + \frac{\theta_{\text{eff}}(\phi)}{2\pi}\right) = \tau_0 + \underbrace{\frac{i}{\pi}\log\left[\frac{u(\phi)}{\Lambda^2}\right]}_{\text{one-loop with cutoff }\Lambda} + \text{non-pert corr}$$

Consider path in $\mathcal{M}_C$ around semi-classical limit $u=\infty$:



The cut in the logarithm induces a shift

$$\log[u] \rightarrow \log[u] + 2\pi i \quad \text{or} \quad \tau \rightarrow \tau - 2$$

This corresponds to a physically irrelevant shift of the theta-angle,

$$\theta \longrightarrow \theta - 4\pi$$

What we learn is that the complexified gauge coupling is ambiguous and multi-valued !

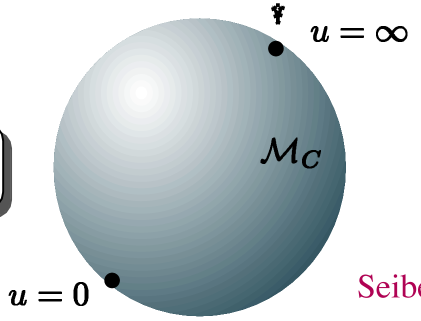
Not even that, it turns out that also the imaginary part, the true gauge coupling, is multi-valued......

Global Properties of the exact quantum moduli space

- gauge coupling $Im(\tau)$ = metric on $\mathcal{M}$; unitarity ?
  since $Im(\tau)$ harmonic: cannot have minimum if globally defined

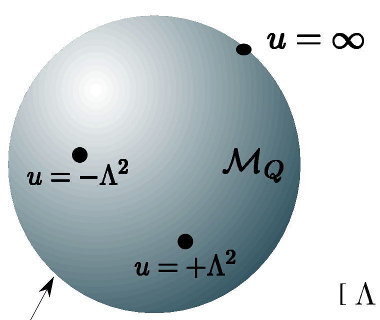
τ cannot be globally, but only locally defined on $\mathcal{M}$!

classical moduli space



Seiberg&Witten: nonperturbative corrections lead to different singularity structure in strong-coupling region !

quantum moduli space



[Λ = dynamically generated scale]

No singularity at u=0: no unbroken SU(2)

The extra "quantum" singularities are due to magnetic monopoles becoming massless at the corresponding parameter values u !

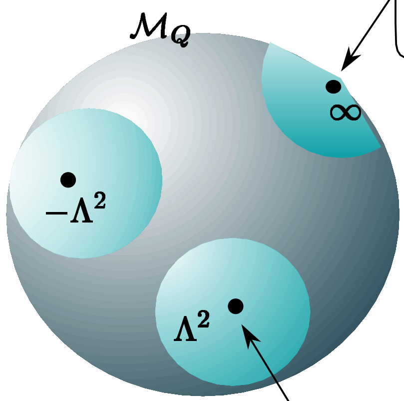
- This is a self-consistent assumption, a postulate !

Global Picture of the Quantum Moduli Space

Introduce a "magnetic dual" Higgs field, and consider section:

$\begin{pmatrix} \phi \\ \phi_D \end{pmatrix}$
- ϕ - - - good coordinate in semi-classical region near $u = \infty$
- ϕ_D - - good coordinate in strong coupling region near $u = +\Lambda^2$

Effective action is defined only in **local coordinate patches**, describing **completely different perturbative physics** !



$$\tau(\phi) = \frac{i}{\pi} \log\left[\frac{\phi^2}{\Lambda^2}\right] - \frac{i}{\pi} \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\Lambda}{\phi}\right)^{4\ell}$$

describes spontaneously broken SU(2) gauge theory; converges well for large ϕ

analytic continuation

$$\tau_D(\phi_D) = \frac{-i}{2\pi} \log\left[\frac{\phi_D^2}{\Lambda^2}\right] - \frac{i}{\pi} \sum_{\ell=1}^{\infty} c_\ell^D \left(\frac{\phi_D}{\Lambda}\right)^{2\ell}$$

describes massless magnetic monopoles with U(1) gauge symmetry; duality equivalent to U(1) electromagnetism with electrons;
converges well for small ϕ_D

Global Consistency Condition

Encircling any singularity induces non-trivial monodromy transformation:

$$\begin{pmatrix}\phi\\ \phi_D\end{pmatrix}(u) \longrightarrow M\cdot\begin{pmatrix}\phi\\ \phi_D\end{pmatrix}(u)$$

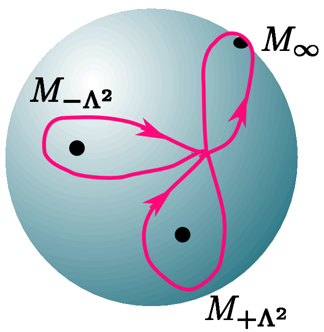
- around $u=\infty$:

$$\begin{pmatrix}\phi\\ \phi_D\end{pmatrix}(u) \sim \begin{pmatrix}\frac{i}{\pi}\sqrt{u}\log[u]\\ \sqrt{u}\end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix}-1 & 2\\ 0 & -1\end{pmatrix}}_{\equiv: M_\infty}\cdot\begin{pmatrix}\phi\\ \phi_D\end{pmatrix}(u)$$

- around $u=\pm\Lambda^2$: for massless monopole of charges (g,q), the monodromy matrix is

$$M_{(g,q)} = \begin{pmatrix}1+g\,q & q^2/2\\ -2g^2 & 1-g\,q\end{pmatrix}$$

Non-trivial consistency condition:
Patch together local information (from perturbation theory)
in a globally consistent way !



$$M_{+\Lambda^2}\cdot M_{-\Lambda^2} \equiv M_\infty$$

is effectively a condition on beta-functions, ie, on massless spectra at strong-coupling singularities

Solution:

$$M_{+\Lambda^2} = M_{(1,0)}$$
$$M_{-\Lambda^2} = M_{(1,2)}$$

gives quantum numbers (g,q) of massless magnetic monopoles !

Solving the Monodromy Problem

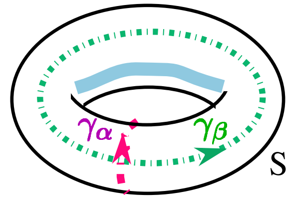
So far, we have found the group of monodromy transformations from perturbation theory plus global consistency.

Remaining task:
Solve the monodromy problem, ie., find functions (sections) $\{\phi(u),\phi_D(u)\}$ that display the right monodromy behavior around the singularities $u=\pm\Lambda^2,\infty$

→ Standard mathematical ("Riemann-Hilbert") problem, which is known to have unique solution !

Trick:
Introduce "auxiliary" **Riemann surface S** (here torus), whose moduli space is precisely the same as $\mathcal{M}_Q$; it is described by

$$S:\quad y^2=(x+\Lambda^2)(x-\Lambda^2)(x-u)$$



Then the Higgs field and its dual are given by "**period integrals**", because these have, by construction, the right properties:

$$\phi(u) \sim \oint_{\gamma_\alpha}\frac{(x-u)}{y(x)}dx \sim {}_2F_1(1/4,1/4,1/2,1/(\Lambda^4-u^2))$$

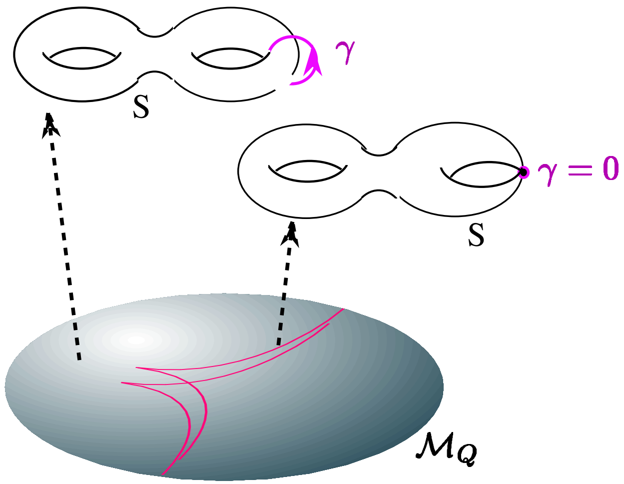
$$\phi_D(u) \sim \oint_{\gamma_\beta}\frac{(x-u)}{y(x)}dx \sim {}_2F_1(1/4,3/4,2,\Lambda^4-u^2)$$

$$\longrightarrow \quad \tau(u(\phi)) = \frac{\partial\phi_D(u)/\partial u}{\partial\phi(u)/\partial u}$$

This then yields finally for the instanton correction coefficients c1=1/4, c2=5/2, ... etc in the effective action.

Recap: Generalities of N=2 Gauge Theory

- The theory is solved in terms of an auxiliary Riemann surface S; for SU(n), it is of genus $g = n-1$.
- Over singular regions in the moduli space, this surface degenerates in that certain 1-cycles γ shrink to zero.
  The corresponding periods $\{\phi(u), \phi_D(u)\} \sim \oint_\gamma \lambda$ vanish, signalling the appearence of extra massless states ("monopoles").

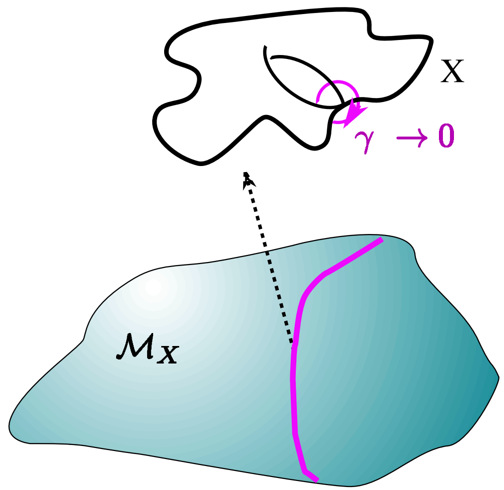


- Local coordinate patches on $\mathcal{M}_Q$ describe different local approximations in terms of different, weakly coupled physical degrees of freedom; perturbative physics looks different in the various patches.
- The monodromies around the singularities act on physical fields and represent the exact quantum, "duality" symmetries of the theory.

> All of these concepts play a natural role in string theory !
> ⇒ Is there a physical meaning of the surface S ?

Moduli Space of N=2 String Theories

- Complex manifolds X typically arise in string compactifications from d=10 to d<10.
  The deformation parameters of these manifolds correspond to parameters of the effective string theory, and thus are coordinates on the "string moduli space" $\mathcal{M}_X$.
- For certain values of the moduli, a compactification manifold X may become singular, in that certain p-dimensional "vanishing cycles γ " shrink to zero size:



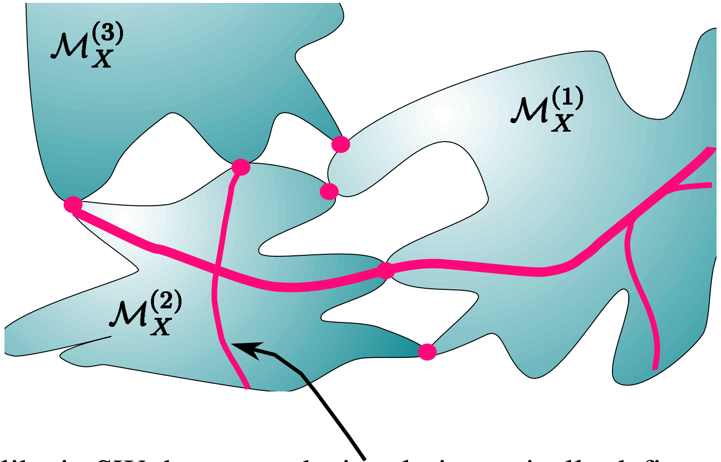
- This typically implies certain "p-branes" (inherent in the string theory) to become massless, when wrapped around γ :

$$M^2_{\mathrm{p-brane}} = |\int_\gamma \Omega(X)|^2 \longrightarrow 0 \quad \text{if} \quad \gamma \longrightarrow 0$$

In an appropriate situation, the remnant of this in d=4 space-time are simply massless particles of various kinds (gauge bosons, quarks, Higgs fields...).

Coordinate patches on the space of theories

- For N=2 supersymmetric strings, all such vacua are connected and form a complicated web with (10000?) components that have in general different dimensions (~100) :



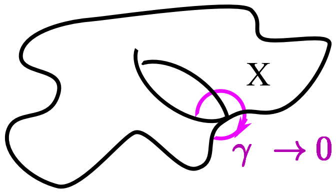
- Just like in SW theory, each singularity typically defines a local coordinate patch that describes certain physical excitations as elementary and weakly coupled;
  viewed from this coordinate patch, other excitations look non-local (solitonic) and strongly coupled.

- No local lagrangian exists that would be globally valid throughout the whole moduli space.
  .... rethink the concept of "grand unification" !

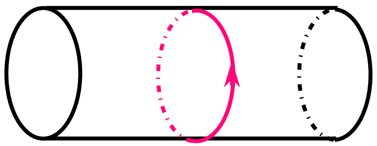
- In regions where patches overlap, there often exist several distinct, but equivalent descriptions of the same physical degrees of freedom; these are related by "duality" transformations.

For example, one and the same massless gauge boson may have the following representations in terms of (only apparently different) string theories:

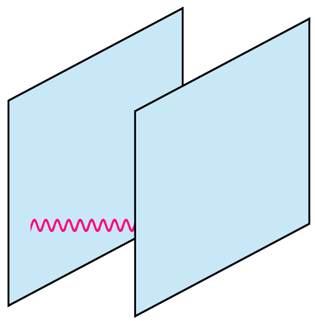
- In type IIB string theory compactified on a Calabi-Yau 3-fold X ;
  as a 3-brane wrapped around a vanishing 3-cycle γ :



- In the heterotic string compactified on $K3 \times T_2$;
  as the fundamental heterotic string wrapped around a cylinder of radius 1 :



- In type I string theory;
  as an open string stretched between D-branes, in the limit of coinciding D-branes :



Recovering Seiberg-Witten Theory from String Duality

Starting point:
Non-perturbative equivalence of type II string, compactified on "Calabi-Yau" manifold X, with heterotic string compactified on $K3 \times T_2$

Type IIA/B String — d=10 — Heterotic String

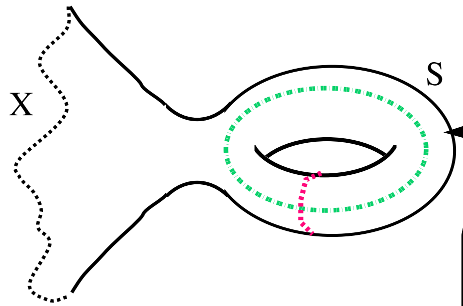
X — $K3 \times T_2$

Identical N=2 SUSY theory in d=4

3-branes wrapping 3-cycles	⟷	fundamental gauge bosons
3-branes wrapping dual 3-cycles	⟷	SW monopoles
Tree-level physics in gauge sector	⟷	Non-perturbative quantum theory
world-sheet instanton effects	⟷	space-time gauge instanton effects
classical computation	➡	exact non-perturbative SW theory, incl gravity corrections !

Stringy Interpretation of Seiberg-Witten Curve S

- In type IIB string theory on a Calabi-Yau X, matter fields (monopoles) and gauge fields arise from 3-branes wrapped around 3-cycles of X.
- In the relevant region of the parameter space $\mathcal{M}_X$, X degenerates in a special way and looks, roughly, locally like:



"1-brane leftover" of 3-brane wraps Seiberg-Witten curve !

This 1-brane leftover is nothing but a string, but a very special one; it is not the fundamental type IIB string !

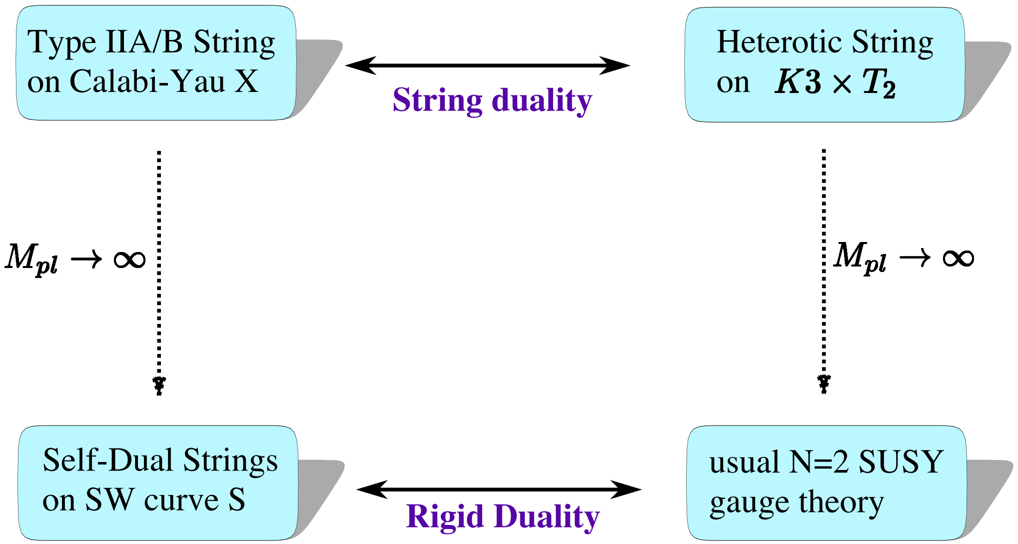
➡ **"Self-Dual String"**

- non-critical: does not involve gravity
- lives naturally in d=6
- no known perturbative lagrangian formulation !

Forget gravity and ten dimensions, and try to interpret Seiberg-Witten theory directly in terms of self-dual strings wrapped around the Riemann surface S !

This gives a natural dual reformulation of N=2 supersymmetric Yang-Mills theory !

It is nothing but the "rigid remnant" of the type II/heterotic string duality, that remains after decoupling gravity:



Gauge bosons and monopoles are on equal footing ! They simply correspond to γ_α and γ_β cycles on the curve S.

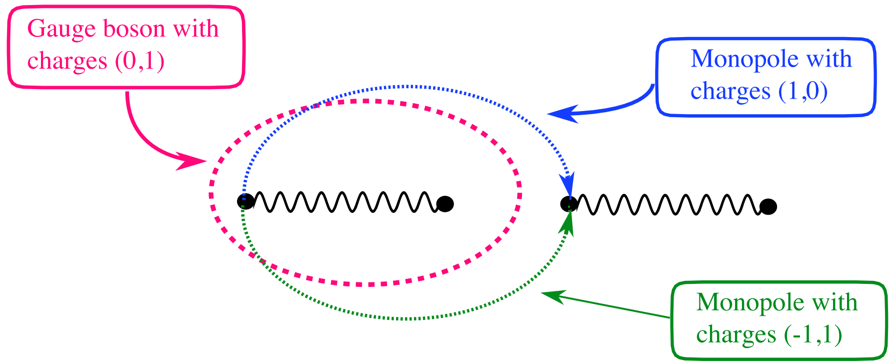
Elementary gauge bosons, but solitonic monopoles (or vice versa)

Can study non-perturbative properties of the N=2 gauge theory, that are extremely hard to get at in ordinary local QFT !

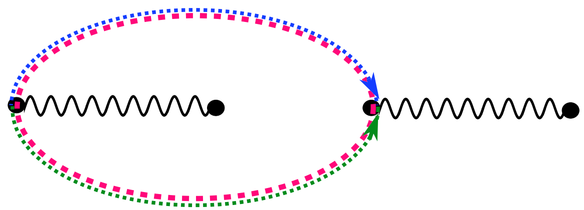
Decay of Gauge Boson into Monopoles

The self-dual string formulation allows to represent the physical states in the gauge theory in terms of homology cycles on the SW Riemann surface S.

Represent S best in terms of branched complex plane; for generic, large vacuum values u, the basic states of the theory with (mag,ele) charges are then represented as follows:



However, for special values of modulus u, the string representation degenerates:



The quantum state of the gauge boson becomes indistinguishable of the 2-particle state composed out of two magnetic monopoles!

This kind of features is extremely hard to see, if at all, in ordinary local QFT ...

Summary

- We learn from Seiberg-Witten theory that one and the same theory can have many different local (perturbative) descriptions; these descriptions can be associated with coordinate patches on the parameter space. The map between these descriptions is given by duality transformations.
  - There is no unique, globally valid description, or lagrangian.
  - Glueing the local (perturbative) realizations together in a globally consistent way, fixes (solves) the theory completely. For this, one introduces an auxiliary Riemann surface S, whose period integrals carry the non-trivial information.
  - Massless states are associated with singularities in the moduli space, over which the auxiliary Riemann surface S degenerates.

- These ideas generalize to string theory; here (for N=2 SUSY theories) the moduli space is vastly more complicated.
  - The analogs of S in string theory are certain complex manifolds X. Massless states are associated with degenerations of X, and arise from wrapping higher dimensional branes around the vanishing cycles of X.
  - In the appropriate field theory limit, where one recovers N=2 gauge theory, one is left with non-critical strings wrapping around S inside of X.
  - The gives a novel, dual reformulation of N=2 gauge theory, in which one can address non-perturbative questions.

Plus much more ....