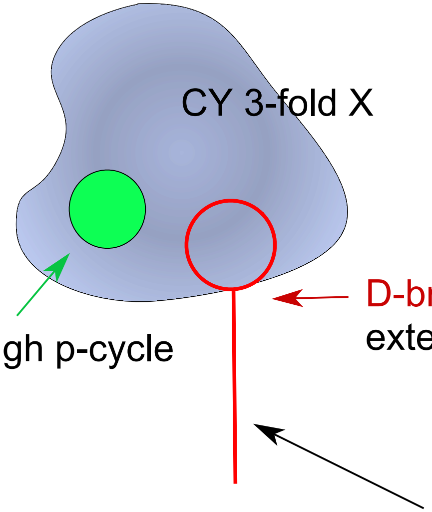
Fluxes and D-branes on Calabi-Yau manifolds

Part 2

W.Lerche, Trieste Spring School 2003

- Physical motivation:
  reduce SUSY from N=2 to N=1



Flux though p-cycle

D-brane wraps p-cycle and extends over space-time

N=1 SUSY on 3+1d world volume

What are the effective superpotential W, and the effective gauge couplings ?

- New feature: open string instantons

Turning on fluxes

- The 10d Type II strings have various massless antisymmetric, (p-1)-form tensor fields $C^{(p-1)}$, coupling to (p-2)-branes.

Field strengths: $H^{(p)} = dC^{(p-1)}$

	$H^{(p)}_{NSNS}$	$H^{(p)}_{RR}$
Type IIA: p=	3,7	2,4,6,8
Type IIB: p=	1,3,7	1,3,5,7,9

- In a CY compactification, various H's can be “turned on”, ie, the H-flux through a p-cycle is non-zero:

$$\int_{\gamma^p} H^{(p)} \neq 0$$

We will mainly consider only (quantized) RR-fluxes, corresponding to D-branes

- 10d action: non-vanishing flux will typically induce non-zero potentials and SUSY breaking

$$S \sim \int H^{(p)} \wedge {}^* H^{(p)}$$

Type IIB string on three-fold $\widehat{X}$ with 3-form flux

- It can be shown that upon turning on H$^{(3)}$ flux, N=2 SUSY is broken to N=1 SUSY with superpotential:

$$\mathcal{W}_{IIB/\widehat{X}} = \int_{\widehat{X}} \Omega^{(3,0)} \wedge \tilde{H}^{(3)}$$

$$\tilde{H}^{(3)} \equiv \tau\, H^{(3)}_{NSNS} + H^{(3)}_{RR}$$

Type IIB coupling: $\tau \equiv C^{(0)} + i\, e^{-\varphi}$

set in the following $H^{(3)}_{NSNS} \to 0$

- Denote 3-cycle dual to flux H$^{(3)}$ by Γ^3 and expand in integral symplectic basis of 3-cycles:

$$\Gamma^3 = N^a \gamma^3_a + N^b \gamma^3_b \qquad N^a \in Z$$

Then

$$\mathcal{W}_{IIB/\widehat{X}}(z) = \int_{\Gamma^3} \Omega^{(3,0)}(z)$$
$$= N^a X_a + N_b \mathcal{F}^b \equiv N^A \Pi_A(z)$$

where $\Pi_A = (X_a, \mathcal{F}^b)$ are nothing but the period integrals !

Type IIA string on three-fold X with fluxes

- Rule: replace period by volume integrals
  ... will be corrected by world-sheet instantons

$$\mathcal{W}_{IIA/X}(t) = \int_X \sum_{k=1}^{3} H^{(2k)}_{RR} \left(\wedge J^{(1,1)} \right)^{3-k} + \ldots$$
$$= N^{(6)} + N^{(4)} t + N^{(2)} t^2 + N^{(0)} t^3 + \mathcal{O}(e^{-t})$$

flux numbers

- A priori, it would be hard to compute the instanton corrections, but mirror symmetry predicts

$$\boxed{\mathcal{W}_{IIA/X}(t) = \mathcal{W}_{IIB/\widehat{X}}(z) = \sum N^A \Pi_A(z(t))}$$

$$\Pi_A(z(t)) = (X_a, \mathcal{F}^b) = (1, t_i, \partial_i \mathcal{F}, 2\mathcal{F} - t_i \partial_i \mathcal{F})$$

- Thus, the superpotential is completely determined by the “bulk” geometry: spont. broken N=2 SUSY

  Note that flux appears as auxiliary field in N=2 eff action

$$\Phi = t + \theta^2 H^{(2)} + \ldots$$

  Thus, if $\langle H^{(2)} \rangle = N^{(2)} \neq 0$

  then $\int d^4\theta \mathcal{F}(\Phi) \to \int d^2\theta N^{(2)} \frac{\partial}{\partial \Phi} \mathcal{F}(\Phi) \equiv \mathcal{W}$

as above !

A first glimpse of Quantum Geometry: monodromy

Periods $\Pi_A = (X_a, \mathcal{F}^b)$: sections valued in $Sp(2h^{2,1}+2, Z)$

Non-trivial loops in the moduli space $\mathcal{M}_{CS}(\widehat{X})$
will thus induce monodromy

$$\Pi_A \;\rightarrow\; \Pi_A \cdot R, \quad R \in Sp(2h^{2,1}+2, Z)$$

- Consider eg looping around $z \sim e^{2\pi i t} \rightarrow 0$
in the semi-classical, large volume regime:

$$t \sim \tfrac{1}{2\pi i} \ln z \;\rightarrow\; t + 1$$

Thus

$$\begin{aligned} Z \;&=\; N^{(6)} + N^{(4)} t + \dots \\ &\rightarrow\; (N^{(6)} + N^{(4)} + \dots) + (N^{(4)} + \dots) t + \dots \end{aligned}$$

- Looping generic (non-perturbative) singularities will typically mix all fluxes which each other:

$$N^A \;\rightarrow\; R \cdot N^A$$

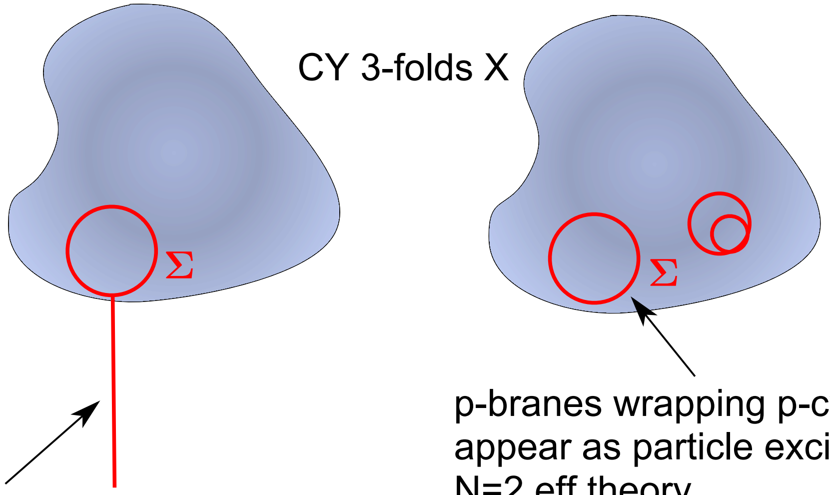
Since

$$N^A \;=\; \int_{\gamma^{p_A}} H^{(p_A)}$$

the dimensions of p-cycles loose their invariant meaning !

D-Branes on Calabi-Yau manifolds

Various manifestations:



CY 3-folds X

p-branes wrapping p-cycles
appear as particle excitations
N=2 eff theory

world volume
3+1d N=1 SUSY “brane world”

- The eff space-time physics depends on the properties of the wrapped internal part of the brane
- We are interested in BPS configurations that break 1/2 of the SUSY (N=2 -> N=1)

Condition for “SUSY p-cycles”:
covariantly constant spinor η, with $(1-\Gamma)\eta = 0$

$$\Gamma \equiv \frac{1}{\sqrt{h}} \epsilon^{\alpha_1 \dots \alpha_{p+1}} \partial_{\alpha_1} X^{m_1} \dots \partial_{\alpha_{p+1}} X^{m_{p+1}} \Gamma_{m_1 \dots m_{p+1}}$$

induced metric

pull-back to
world-vol

10d Gamma
matrices

- Two classes of solutions:

“A-type” branes: wrap special lagrangian cycles $\Sigma_A^{(p=3)}$
“B-type” branes: wrap holomorphic cycles $\Sigma_B^{(p=0,2,4,6)}$

A-type branes

- Wrap “special lagrangian” cycles Σ_A

$$dim(\Sigma_A) \;=\; 1/2dim(X) \;=\; 3$$

- $f^* J^{(1,1)} \;=\; 0$

  Pull-back of Kahler form vanishes; $f : \Sigma_A \longrightarrow X$

- $f^*(Im\, e^{i\theta}\Omega^{(3,0)}) \;=\; 0$
  Pull-back of holom 3-form vanishes

- $F = 0$
  U(1) gauge field on world-volume must be flat

- What are the moduli of the brane ?
  A priori:

$$dim_R(\mathcal{M}_{\Sigma_A}) \;=\; b_1(\Sigma_A)$$

which can be odd ...
but we need complex fields for SUSY reasons

➡ Pair up with “Wilson line” moduli of the flat U(1) gauge connection to get complexified moduli fields:

$$\hat{t}_i, \;\; i = 1, ..., dim_C(\mathcal{M}_{\Sigma_A}, WL) \;=\; b_1(\Sigma_A)$$

B-type D-branes

- Wrap holomorphic submanifolds: $\Sigma_B^{(p)}$, p=0,2,4,6

- Apart from the holomorphic embedding geometry, $f : \Sigma_B \longrightarrow X$, there is more structure:
  the gauge field configuration, “U(N) bundle V”
  (if N branes coincide)

  Eg for D6 branes (wrapping all of X), SUSY requires that the gauge bundle V is holomorphic:

$$F_{i\bar{j}} \;=\; 0$$

  (NB: further “stability” requirements)

- Important correspondence:

Gauge field configuration V ⇔ brane bound states

...due to anomalous world-volume couplings:

$$S_{WZ} \;=\; \int_{\Sigma_B^{(p)}\times R} C \wedge Tr[e^F] \wedge \sqrt{\hat{A}(R)}\Big|_{p+1form}$$

RR tensor fields

Dirac genus

$$C \equiv \bigoplus_k C^{(k)} \begin{cases} \text{Type IIA: k=odd} \\ \text{Type IIB: k=even} \end{cases}$$

$\hat{A}(R) = 1 + 1/24R^2 + ...$

Chern
character of V

- Example: D4-brane

$$S_{WZ} \;=\; \int_{\Sigma_B^{(4)}\times R} \tfrac{1}{2}\, C^{(1)} \wedge F \wedge F + \dots$$

so if there is an instanton configuration V such that $\frac{1}{2}\int F\wedge F = n$ then there is an induced coupling

$$n \int C^{(1)}$$ = source term for n D0-branes !

- More generally:

n gauge instantons on p-brane

⟺ bound state of the p- with n (p-4)-D-branes

- Even more generally:

A brane configuration of r D6 branes on CY X is characterized by the “generalized Mukai” charge vector Q:

$$Q \;=\; Tr[e^F] \wedge \sqrt{\hat{A}(R)}$$

$$= \;(Tr1,\, TrF,\, \tfrac{1}{2}(TrF)^2 - TrF^2 + \tfrac{1}{24}TrR^2, \dots)$$

Thus $\int_X Q \;=$

$$= (r(V), c_1(V), ch_2(V) + \tfrac{r}{24}c_2(T_{\Sigma_B}), ch_3(V) + \tfrac{r}{24}c_1(V)c_2(T_{\Sigma_B}))$$

$$= \;(\,M^{(6)},\, M^{(4)},\, M^{(2)},\, M^{(0)}\,)$$ D-brane RR charges

This gives direct translation between gauge bundle data (Chern classes of V) and D-brane charge content

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Mirror symmetry and D-branes

- Recap mirror map:

$$Type\ IIA/X \quad\longleftrightarrow\quad Type\ IIB/\widehat{X}$$

RR fields:

$$\{C^{(1)}, C^{(3)}, C^{(5)}, \dots\} \;\longleftrightarrow\; \{C^{(0)}, C^{(2)}, C^{(4)}, \dots\}$$

Dp(=even) branes | Dp(=odd) branes

Equivalence of non-perturbative theories implies equivalence of

B-branes wrapped over holom. (0,2,4,6) cycles of X ⟺ A-branes wrapped over special lagrangian 3-cycles of $\widehat{X}$

- This is reflected in the 2d string world-sheet boundary conditions of the N=(2,2) superconformal currents:

B-type branes	A-type branes
$J_L \;=\; J_R$	$J_L \;=\; -J_R$
$G_L^{\pm} \;=\; \pm G_R^{\pm}$	$G_L^{\pm} \;=\; \mp G_R^{\pm}$
$T_L \;=\; T_R$	$T_L \;=\; T_R$

Mirror symmetry just switches $J_R \leftrightarrow -J_R$!

Tension of wrapped D-branes

(particles in 4d N=2 SUSY)

- Recall BPS mass formula: $m_{BPS} = |Z|$

Central charge Z in N=2 SUSY algebra

$$\{Q^+, Q^-\} = p\cdot\gamma + Z$$

essentially given by volume of wrapped cycle

- Recall factorization of CY moduli space:

$$\mathcal{M}_X = \mathcal{M}_{KS}(t)\times\mathcal{M}_{CS}(z)$$

~ ...even ...odd cycles

The mass of wrapped B-branes depends only on the Kahler moduli t, while the mass of the A-branes depends only on the complex structure moduli z.

- A-branes in Type IIB:

$$\mathrm{Z}_{A/IIB}(z) = M^A\int_{\gamma_A^3}\Omega^{(3,0)}(z) = M^A\Pi_A(z)$$

- B-branes in Type IIA:

$$\mathrm{Z}_{B/IIA}(t) = \int_X e^{J^{(1,1)}}\wedge Q + \mathcal{O}(e^{-t}) \quad \text{(instanton corr)}$$

$$= \mathrm{Q}_0 + \int J^{(1,1)}\wedge Q_2 + \tfrac{1}{2}\int J^{(1,1)}\wedge J^{(1,1)}\wedge Q_4 +$$

$$= M^{(0)} + M^{(2)}t + M^{(4)}t^2 + M^{(6)}t^3 + \mathcal{O}(e^{-t})$$

- Mirror symmetry: $\boxed{Z_{B/IIA}(t) = Z_{B/IIA}(z(t))}$

$$= M^{(0)} + M^{(2)}t + M^{(4)}\partial_t F(t) + M^{(6)}(2\mathcal{F} - t\partial_t\mathcal{F})(t)$$

Quantum Volume

- Non-trivial identification:

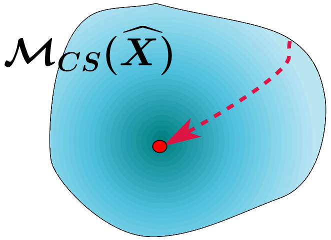
$$\mathrm{M}^A\int_{\gamma_A^3}\Omega^{(3,0)}(z) = M^{(0)}+M^{(2)}t+M^{(4)}\partial_tF(t)+M^{(6)}\mathcal{F}_0(t)$$

3-cycles on X on equal footing ⟹ 0,2,4,6-cycles on $\widehat{X}$ on equal footing too !

- Massless state in 4d:

$Z = 0 : \quad \Pi_A \to 0$ for some A

Example:
conifold singularity (strong coupling region)



Type IIB: 3-cycle $\gamma_A^3 \to 0$ ⟹ Type IIA: $\mathcal{F}_0(t)\to 0$
6-cycle quantum volume
(whole CY) X shrinks to nothing!

However, the “embedded” 0,2,4 cycles do not have vanishing quantum volume:
$(1, t, \partial_t F(t)) \not\to 0$

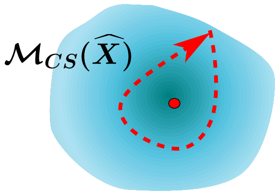
The classical geometric picture is swamped out by instanton corrections

Monodromy of RR charges

- Recall that when encircling singularities in $\mathcal{M}_{CS}(\widehat{X})$, monodromies will be induced on the periods:

$$\Pi_A \;\to\; \Pi_A \cdot R, \quad R \in Sp(2h^{2,1}+2, Z)$$

Thus, just as before the flux numbers N^A, now the D-brane charges M^A will get mixed.



- Eg., encircling $z \sim e^{2\pi i t} \longrightarrow 0$ in $\mathcal{M}_{CS}(\widehat{X})$ induces t -> t+1, and

$$Z \;=\; M^{(0)} + M^{(2)} t + ...$$

$$\to (M^{(0)} + M^{(2)} + ...) + (M^{(2)} + ...)t + ...$$

ie., the D0 brane number jumps

roughly: "tensoring V by a line bundle": Z$\sim$ $\int e^{J^{(1,1)}} \wedge e^{F}$

Again we see that the notion of p-cycles, and gauge bundle configurations V on top of them, has no good meaning away from the semi-classical large radius limit !

Central charge and domain walls

- We have seen that in type IIB compactifications, 3-fluxes $H^{(3)}$ induce an N=1 superpotential:

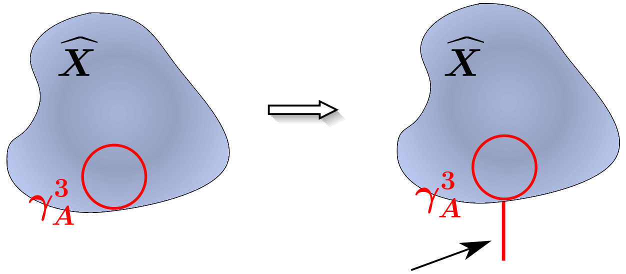
$$\mathcal{W}_{N=1}(z) \;=\; N^A \Pi_A(z)$$

However the same expression gave the central charge of a wrapped D3 A-type brane:

$$Z(z) \;=\; M^A \Pi_A(z)$$

What is the significance ?

- Replace fully wrapped D3 brane by a D5 brane:



Domain wall in 3+1d

Central charge of a DW is known to be $Z = \Delta \mathcal{W}_{N=1}$

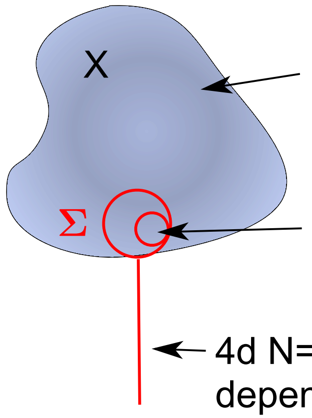
However, the D5 brane tension is still

$$Z = M^A \int_{\gamma_A^3} \Omega^{(3,0)} = M^A \Pi_A$$

and it generates M^A units of $H^{(3)}$ flux across the domain wall

Moduli of D-brane configurations

- Consider 1/2 BPS configurations breaking to N=1 SUSY:



CY moduli t, z

brane (and bundle) moduli $\hat{t}, \hat{z}$

4d N=1 SUSY eff action $S_{eff}(t, \hat{t}, z, \hat{z})$
depends how on moduli ?

- Focus on

  complex structure moduli:

  $z \sim \gamma_A^3$ sizes of 3-cycles

  $\hat{z} \sim \hat{\gamma}_N^3$ sizes of 3-chains

  Kahler moduli:

  $t \sim \gamma_i^2$ sizes of P^1's

  $\hat{t} \sim \hat{\gamma}_n^2$ sizes of disks ending on D-brane

- Decoupling theorems (from CFT):

B-branes
$$\begin{cases} W(z, \hat{z}),\ \tau(z, \hat{z}) & \text{holom. potentials} \\ D(t, t^*, \hat{t}, \hat{t}^*) & \text{FI D-term potential} \end{cases}$$

A-branes
$$\begin{cases} W(t, \hat{t}),\ \tau(t, \hat{t}) & \text{holom. potentials} \\ D(z, z^*, \hat{z}, \hat{z}^*) & \text{FI D-term potential} \end{cases}$$

Preview

- Next time: use mirror symmetry

$$W_{A/IIA}(t, \hat{t}) \;=\; W_{B/IIB}(z(t), \hat{z}(t, \hat{t}))$$

and set up math framework for systematically computing superpotentials for a large class of D-brane geometries