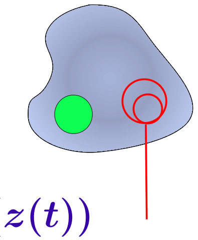
Open/closed string mirror symmetry

- Recap: reduce SUSY from N=2 to N=1 in Type II compactifications on CY threefolds, by



  - Switching on fluxes

$$\mathcal{W}(t) \;=\; \int \Omega^{(3,0)} \wedge H \;=\; \sum N^A \Pi_A(z(t))$$

flux numbers, periods

$$\int_{\gamma^3_A} \Omega(z) = (1, t_a, \partial^a \mathcal{F}, F^0)(t)$$

...superpotential depends only on “bulk” geometry

  - Putting in extra D-branes

new ingredient: brane moduli $\hat{t}, \hat{z}$
parametrizing open string (”boundary”) geometry

- How do these ingredients fit together ?

Seek: uniform description of open/closed string backgrounds labeled by

$$\{X, N^A; M^A\}$$

closed ; open string sector

and make use of mirror symmetry:

$$\{X, N^A; M^A\}(t,\hat{t}) \;\cong\; \{\widehat{X}, \widehat{N}^A; \widehat{M}^A\}(z,\hat{z})$$

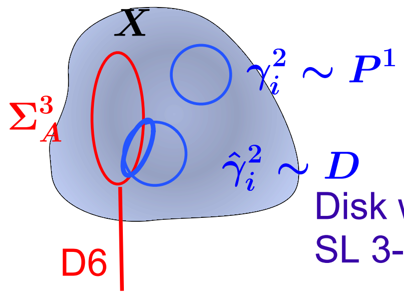
A-type branes in Type IIA compactification

- relevant moduli: Kahler deformations

closed sector: $t_i \;=\; \int_{\gamma_i^2} J^{(1,1)}\,, \qquad i = 1,\ldots,h^{1,1}(X)$
size of P^1

open sector: $\hat{t}_i \;=\; \int_{\hat{\gamma}_i^2} J^{(1,1)}\,, \qquad i = 1,\ldots,h^{1}(\Sigma_A)$

position of brane in homology class ~ size of disk



Disk with boundary on SL 3-cycle Σ_A^3

These volume integrals give contributions of the world-sheet instantons to the disk amplitude $\mathcal{F}_{g,h} = \mathcal{F}_{0,1}$; (which coincides with the superpotential):

$$\mathcal{F}_{0,1}(t,\hat{t}) \;\equiv\; \mathcal{W}(t,\hat{t}) \;=$$

$$= 0\cdot t\,\hat{t} + \sum_{\substack{n_1\ldots n_r;\\ m_1\ldots m_s}} N_{n_1\ldots n_r;m_1\ldots m_s} Li_2({q_1}^{n_1}\ldots{q_r}^{n_r};\hat{q}_1^{n_1}\ldots\hat{q}_s^{n_s})$$

math theorem: classically, deformations of SL cycles are unobstructed

superpotential is entirely non-perturbative ($q \equiv e^{-t}$)

$(\vec{n},\vec{m})$ labels a “**relative**” homology class in $(H_2(X), H_1(\Sigma_A))$

B-type branes in Type IIB compactification

- relevant moduli: complex structure deformations

closed sector: $\Pi_A(z) \;=\; \int_{\gamma_A^3} \Omega^{(3,0)}(z)$ volumes of 3-cycles in $\widehat{X}$

open sector: (?)

- Consider holom. Chern-Simons action (describing open strings for D6-brane on $\widehat{X}$):

$$S_{CS} \;=\; \int_{\widehat{X}} \Omega^{(3,0)} \wedge Tr[A \wedge \bar{\partial} A + \tfrac{2}{3} A \wedge A \wedge A]$$

We will be interested only in (complex) one dimensional cycles: $\Sigma_B \sim \gamma^2$;

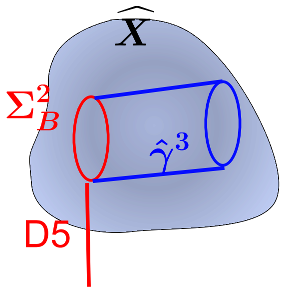
Dimensionally reducing $A \longrightarrow \phi$ yields

$$\mathcal{W} \;=\; \int_{\Sigma_B} \Omega^{(3,0)}_{ijz} \phi^i \bar{\partial}_z \phi^j dz d\bar{z}$$

Rewriting locally using $\Omega_{ijz} = \partial_z \omega_{ij}$ gives:

$$\boxed{\mathcal{W}(z,\hat{z}) \;=\; \hat{\Pi} \;=\; \int_{\hat{\gamma}^3(\hat{z})} \Omega^{(3,0)}(z)}$$

where the integral is over the **3-chain** $\partial\hat{\gamma}^3$: $\partial\hat{\gamma}^3 \equiv \Sigma_B$ whose boundary is the holomorphic B-type cycle



So the relevant 3-volumes are that of 3-chains ending on D5-branes

Mirror symmetry for D-brane configurations

- Recall N=2 decoupling property (similar for IIB):

$$\mathcal{M}_{IIA}(X) \;\cong\; \mathcal{M}^{[t]}_{KS}(X) \times \mathcal{M}^{[z]}_{CS}(X)$$

Open string sector:

$$\mathcal{M}(X, D6)_{A/IIA} \;\cong\; \mathcal{M}^{[t,\hat{t}]}_{KS}(X) \times \mathcal{M}^{[z,\hat{z}]}_{CS}(X)$$

Reflected in decoupling theorems:

B-branes $\begin{cases} W(z,\hat{z}),\ \tau(z,\hat{z}) \\ D(t,t^*,\hat{t},\hat{t}^*) \end{cases}$ holom. potentials / FI D-term potential

A-branes $\begin{cases} W(t,\hat{t}),\ \tau(t,\hat{t}) \\ D(z,z^*,\hat{z},\hat{z}^*) \end{cases}$ holom. potentials / FI D-term potential

- Invoke mirror symmetry:

$$\boxed{\mathcal{W}_{A/IIA}(t,\hat{t}) \;=\; M^L \hat{\Pi}_L \;=\; \mathcal{W}_{B/IIB}(z(t),\hat{z}(t,\hat{t}))}$$

A-branes in Type IIA/X B-branes in Type IIB/$\widehat{X}$

$$\hat{\Pi}_L(t,\hat{t}) \;=\; \begin{cases} \hat{t} \\ \sum_{n,m} N_{n,m} Li_2(q^n \hat{q}^m) \end{cases} \qquad \hat{\Pi}_L(z,\hat{z}) \;=\; \int_{\hat{\gamma}^3_L(\hat{z})} \Omega^{(3,0)}(z)$$

... exact result !

...corrections by sphere and disk instantons

Unifying flux and D-brane potentials

- Aim: obtain an uniform description of generic superpotentials

Recall fluxes: $\mathcal{W}_{flux} = N^A \Pi_A$

Recall D-branes: $\mathcal{W}_{D-brane} = M^A \hat{\Pi}_A$

Write general potential:

$$\mathcal{W} = M^\Lambda \Pi_\Lambda = M^\Lambda \int_{\Gamma^3_\Lambda} \Omega^{(3,0)}$$

where

$$\Gamma^3_\Lambda = \{\gamma^3_A, \hat{\gamma}^3_L\} \in H_3(\widehat{X}, Y; Z)$$

are “**relative**” homology cycles on $\widehat{X}$ which are closed only up to the boundary $Y \equiv \partial\hat{\gamma}^3$

- The corresponding “relative” period vector

$$\Pi_\Lambda \equiv (\Pi_A, \hat{\Pi}_L) = (1, t_\lambda, \mathcal{W}^\mu, ...)$$

$$\{t_i, \hat{t}_k\} \quad \{\mathcal{F}^i, \mathcal{W}^k\}$$

contains the
”**holomorphic potentials of N=1 Special Geometry**”

for bulk (closed str) subsector: $\mathcal{W}^i = \partial_i \mathcal{F}$

for boundary (open str) subsector: $\mathcal{W}^k$ do not integrate!

> The existence of many independent potentials reflects that N=1 SUSY theories are less constrained than their N=2 counterparts

The Geometry of $\mathcal{W}$

- Just like for the N=2 prepotential $\mathcal{F}$, the N=1 superpotential $\mathcal{W}$ (given by periods and semi-periods) can be interpreted from three inter-related viewpoints:

A) Space-time effective action:
holom. superpotential
(note: superpot has special features as compared to generic supergravity superpotentials, eg integral instanton expansion)

B) Correlation functions and ring structure constants of open string TFT

C) Boundary (open string) variation of Hodge structures, in relative cohomology

Open string topological field theory (B-model)

- Recall observables in bulk B-model:

$$O_B^{(p,q)} = \omega^{(p,q)i_1\ldots i_p}_{\bar{j}_1\ldots\bar{j}_q}\lambda_{i_1}\ldots\lambda_{i_p}\psi^{\bar{j}_1}\ldots\psi^{\bar{j}_q} \in H^{0,q}_{\bar\partial}(\widehat{X},\wedge^p T^{1,0})$$

Complex structure deformations are associated with

$$O_B^{(-1,1)} = \omega^{(-1,1)i}_{\bar{j}}\lambda_i\psi^{\bar{j}} \in H^{-1,1} \cong H^{2,1}$$

which generate the (a,c) chiral ring:

$$\mathcal{R}^{(a,c)}: \quad O_{B,a}^{(-1,1)} \cdot O_{B,b}^{(-1,1)} = \sum_c c_{ab}{}^c\, O_{B,c}^{(-2,2)}$$

- Now in the open string B-model, we consider B-type (Dirichlet) boundary conditions along a sub-manifold Y:

$$\psi^{\bar{i}} = 0 \ \ (D) \qquad \lambda_i = 0 \ \ (N)$$

The observables are like above, however now elements of $H^{0,q}(Y,\wedge^p N_Y)$ (normal bundle to Y)

The "boundary" moduli are associated with 1-forms:

$$\hat{O}_\alpha^{(1)} = \omega_\alpha^{(1),i}\lambda_i \in H^0(Y,N_Y)$$

which generate the boundary (open string) and bulk-boundary chiral rings:

$$\hat{O}_\alpha^{(1)} \cdot \hat{O}_\beta^{(1)} = \sum_\gamma c_{\alpha\beta}^\gamma \hat{O}_\gamma^{(2)}$$

$$O_a^{(-1,1)} \cdot \hat{O}_\beta^{(1)} = \sum_\gamma c_{a\beta}^\gamma \tilde{O}_\gamma^{(2)}$$

The "relative" (open string) cohomology ring

The upshot is that we can pull through program of N=2 Special Geometry, but for "relative cohomology"

- We get an extension of the chiral ring by boundary operators:

$$\vec{\mathcal{O}}_\Lambda = (O_a^{(-1,1)},\hat{O}_\alpha^{(1)}) \in H^*(\widehat{X},Y)$$

$$\mathcal{R}^{oc}: \quad \vec{\mathcal{O}}_\Lambda \cdot \vec{\mathcal{O}}_\Sigma = \sum_\Delta c_{\Lambda\Sigma}{}^\Delta \vec{\mathcal{O}}_\Delta$$

where the relative cohomology group is defined as the dual to the relative homology $H_*(\widehat{X},Y)$ group discussed before.

This mirrors the structure of differentials in relative cohomology:

$$\vec{\Theta} = (\theta_X,\theta_Y),\ \theta_X \in H^*(\widehat{X}),\theta_Y \in H^*(Y)$$

equivalence rel: $\vec{\Theta} \cong \vec{\Theta} + (d\omega, i^*\omega - d\eta)$

Thus a form that is exact on $\widehat{X}$ and thus trivial in $H^*(\widehat{X})$ may be non-trivial in relative cohomology, and equivalent to some form on the sub-manifold Y.

...loosely speaking: total derivatives can become non-trivial once we have boundaries: $\int_\gamma d\lambda = \int_{\partial\gamma}\lambda$

- Physics interpretation:

Operators that are BRST exact in the bulk TFT, can become non-trivial in the open string sector !

The relative period matrix

- The natural pairing between relative homology cycles and cohomology elements is:

$$\Pi_{\Lambda\Sigma} \;\equiv \langle \Gamma_\Lambda, \Theta_\Sigma \rangle \;=\; \int_{\Gamma_\Lambda} \theta_X - \int_{\partial\Gamma_\Lambda} \theta_Y$$

$$= \begin{pmatrix} 1 & (t_i, \hat{t}_i) & (\mathcal{F}^j, \mathcal{W}^j) & \dots \\ 0 & \delta_{\Lambda\Sigma} & \partial_\Sigma(\mathcal{F}^j, \mathcal{W}^j) & \dots \\ 0 & \dots & \dots & \dots \end{pmatrix}$$

This relative period matrix contains all the building blocks of N=1 Special Geometry, and uniformly combines period and chain integrals; ie., closed (flux) and open string (D-brane) sectors.

Its first row is nothing but the rel. period vector we had before, which gives the total superpotential

$$\mathcal{W} \;=\; M^\Lambda \Pi_{\Lambda 1}$$

- Show: rel. period matrix satisfied a system of DEQs:

... analogous to ordinary period matrix

Variation of Hodge structures

- The variation of Hodge structures for the relative cohomology takes care of the boundary terms in a systematic way; schematically:

$$\begin{tikzcd}
(\Omega_X^{(3,0)},0) \arrow[rr, green] \arrow[dr, red] & & (\omega_X^{(2,1)},0) \arrow[rr, green] \arrow[dr, red] & & (\omega_X^{(1,2)},0) \arrow[rr, green] \arrow[dr, red] & & (\Omega_X^{(0,3)},0) \arrow[dr, green] & \\
 & (0,\omega_Y^{(2,0)}) \arrow[rr, green, shift left] \arrow[rr, red, shift right] & & (0,\omega_Y^{(1,1)}) \arrow[rr, green, shift left] \arrow[rr, red, shift right] & & (0,\omega_Y^{(0,2)}) \arrow[rr, green, shift left] \arrow[rr, red, shift right] & & 0
\end{tikzcd}$$

$\color{green}\longrightarrow$ $\sim$ $\partial/\partial z$ closed string deformation (N=2 bulk)

$\color{red}\longrightarrow$ $\sim$ $\partial/\partial \hat{z}$ open string deformation (N=1 boundary)

(This picture applies to a particular brane configuration, and becomes more complicated for several branes.)

- In effect one obtains a linear matrix system

$$\nabla_I \Pi_{\Lambda\Sigma}(z,\hat{z}) \;\equiv\; (\partial_I - \Gamma_I - C_I)\cdot \Pi_{\Lambda\Sigma}(z,\hat{z}) \;=\; 0$$

...which equivalent to a system of coupled, higher order generalized Picard-Fuchs operators.

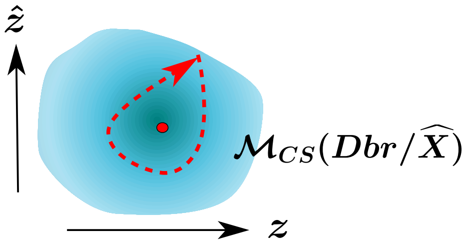
- Can show:

$$[\nabla_I, \nabla_J] \;=\; 0$$

Combined open/closed moduli space is flat.

... seems mathematically quite non-trivial !

Physics: open and closed string moduli fit consistently together in one combined moduli space.



- Thus there exist flat coordinates $t_i, \hat{t}_j$ on the combined moduli space.

For these, the ring structure constants obey

$$c_{ij}{}^{k}(t,\hat{t}) \;=\; \partial_i\partial_j\mathcal{W}^k(t,\hat{t})$$

$$\sim \langle \mathcal{O}_i \mathcal{O}_j \rangle^{(k)}$$

k-th flux or D-brane sector

N=1 “Special Geometry”

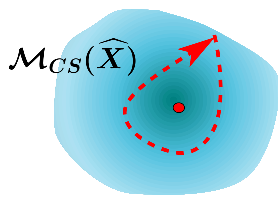
- Basic object: relative period vector

$$\hat{\Pi}_\Lambda \;=\; \int_{\Gamma_\Lambda} \Omega^{(3,0)} \;\sim\; (1, t_i, \hat{t}_k, \mathcal{F}^i, W^k, \ldots)$$

gives general flux and brane-induced N=1 superpot:

$$\mathcal{W}_{tot}(z(t), \hat{z}(t,\hat{t})) \;=\; \sum N^\Lambda \Pi_\Lambda$$
$$=\; N^{(0)} + N_i^{(2)} t_i + N_i^{(4)} \mathcal{F}^i(t) + M^{(k)}\hat{t}_k + M^{(\ell)} W^\ell(t,\hat{t})$$

- Monodromy:
mixes flux and brane numbers



(note: brane->brane+flux, not v.v)

“Non-renormalization” property:
boundary (open string) quantities can get modified/ corrected by bulk (closed) string quantities, but not vice versa: $z = z(t), \;\; \hat{z} = \hat{z}(t,\hat{t})$.

- The bulk (flux) sector is secretly N=2: the $\mathcal{F}^i = \partial_i \mathcal{F}$ integrate to the N=2 prepotential.
This is not so for the brane potentials, $\mathcal{W}^k$.
The ring coupling constants obey nevertheless:

$$c_{ij}{}^{k}(t,\hat{t}) \;=\; \partial_i\partial_j\mathcal{W}^k(t,\hat{t})$$

- The relative homology lattice $H_3(\widehat{X}, Y; Z)$ is the BPS charge lattice of the domain walls in the N=1 theory

Example: on blackboard