



INDEX THEOREMS IN $N=2$ SUPERCONFORMAL THEORIES

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ABSTRACT

Having emphasized the role of $N=2$ superconformal invariance in various string theories, we compute the 'index' $\text{Tr}(-1)^F$ for the $N=2$ supercharge on orbifold and manifold backgrounds. Resolving a quantization ambiguity in the appropriate way, we show that it generalizes the Dolbeault index $\text{ind}\bar{\partial}$. It has however, in general, no well-defined modular properties. The lack of modular invariance can be expressed in terms of the integral characteristic class $\frac{1}{2}c_1$, that is also related to the condition for space-time supersymmetry.

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1. Introduction

There exists now a variety of constructions of lower dimensional heterotic string theories. One type of constructions deals with compactification of higher dimensions [1], another type works directly in lower dimensions [2]. The overlap between these two approaches appears to be essentially the symmetric orbifold construction.

Many properties of a compactified theory can be interpreted in terms of topological properties of the compact space. It is certainly important to investigate whether some of these topological concepts can be translated or generalized to more general conformal field theories that cannot even be regarded as limiting cases of manifold compactifications.

We will consider here mainly a certain class of (abelian) orbifolds and the theories described by the covariant lattice approach [3]. The overlap between the latter theories and orbifolds is given by at least the set of asymmetric orbifolds with inner automorphism twists and with rank 22 gauge groups. A particular property of *all* of these theories is a global $N=2$ right-moving superconformal symmetry, extending the local $N=1$ world-sheet supersymmetry. In fact, most of all known theories exhibit this symmetry. For example, it was shown [4-9] that $N=1$ space-time supersymmetry implies $N=2$ sheet supersymmetry. We want to stress however that in the above-mentioned class of models the occurrence of $N=2$ supersymmetry is *independent* of any space-time supersymmetry.

It is thus interesting to study this $N=2$ superconformal invariance in more detail. In particular, we are interested in the index $\text{Tr}(-1)^F$ that one can define for the two supercharges. We will first compute this quantity for asymmetric orbifold and covariant lattice theories. In general such theories cannot be interpreted as compactifications of higher dimensional theories on manifolds, and the supercharges have no representation in terms of differential operators. However, when such an interpretation is possible, we show that $\text{Tr}(-1)^F$ generalizes the Dolbeault index $\text{ind}\bar{\partial}$. We also find that even though $\text{Tr}(-1)^F$ can have this interpretation as an index, it is not a well-defined function in loop space in that it has no good modular properties, unless a certain cohomological condition is met. This condition is equivalent to presence of space-time supersymmetry, and if it is satisfied, $\text{Tr}(-1)^F$ becomes identical to $\text{Tr}\Gamma_*$, the index of the Dirac-Ramond operator.

2. The N=2 Superconformal algebra

In conformal field theory, the $N=2$ superconformal algebra [10,11] can be represented in the operator product form:

$$\begin{aligned}
 T(z) \cdot T(w) &\sim \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \\
 T(z) \cdot G^\pm(w) &\sim \frac{\frac{3}{2}G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{(z-w)}, \\
 T(z) \cdot J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)}, \\
 J(z) \cdot J(w) &\sim \frac{\frac{1}{3}c}{(z-w)^2}, \quad J(z) \cdot G^\pm(w) \sim \pm \frac{G^\pm(w)}{(z-w)}, \\
 G^+(z) \cdot G^-(w) &\sim \frac{\frac{1}{3}c}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w) + \frac{1}{2}\partial J(w)}{(z-w)}, \\
 G^\pm(z) \cdot G^\pm(w) &\sim 0.
 \end{aligned} \tag{1}$$

Here, $T(z)$ represents the stress energy tensor, $G^\pm(z)$ the supercharges and $J(z)$ an $\widehat{U}(1)$ Kac-Moody current; c is the central charge. All these operator products have well-known representations in terms of (anti-)commutation relations of their mode components. These components are defined by $T(z) = \sum z^{-2-n} L_n$, $G^\pm(z) = \sum z^{-3/2-n \mp a} G_{n \pm a}^\pm$ and $J(z) = \sum z^{-1-n} J_n$, where $n \in \mathbf{Z}$ and a is arbitrary. One such commutation relation is

$$\{G_{n+a}^+, G_{m-a}^-\} = L_{n+m} + \frac{1}{2}(n-m+2a)J_{n+m} + \left(\frac{1}{6}c(n+a)^2 - \frac{1}{24}c\right)\delta_{n+m}. \tag{2}$$

This, together with the other corresponding (anti-)commutation relations it defines an $N=2$ superconformal algebra, A_a , for any given value of a [7][11] [12]. As the algebras A_a and A_{a+1} are isomorphic [7][11], the parameter, a , is restricted to the range $0 \leq a < 1$. We will however concentrate on the algebra A_0 .

In the representation theory of this $N=2$ superconformal algebra [5], the highest weight states are states $|h, q\rangle$ annihilated by L_n, G_n^\pm, J_n for $n > 0$, and have definite eigenvalues under a maximal set of commuting generators: $L_0 |h, q\rangle = h |h, q\rangle$, $J_0 |h, q\rangle = q |h, q\rangle$. The complete highest weight representation is then obtained by applying the corresponding creation operators ($n < 0$). The NS -sector is defined by $a = \frac{1}{2}$, and the R -sector by $a = 0$. In

the R -sector, the representation space splits further into two sectors $P^+ \oplus P^-$, characterized by the action of the two supercharges

$$G_0^\mp |h, q \mp \frac{1}{2}\rangle_\pm = 0, \quad |h, q \mp \frac{1}{2}\rangle_\pm \in P^\pm. \quad (3)$$

Furthermore, $G_0^+ : P^+ \rightarrow P^-$, $G_0^- = (G_0^+)^\dagger : P^- \rightarrow P^+$ and $\{(-1)^{J_0}, G_0^\pm\} = [L_0, G_0^\pm] = 0$. It follows that except for states annihilated by G_0^+ or G_0^- , all other states come in pairs with same mass and with $U(1)$ charges differing by one unit. The states which are not paired are massless, *i.e.*, have $h = \frac{c}{24}$. Thus one is tempted to define as 'index' for an arbitrary $N=2$ superconformal theory $\text{ind}G_0^+ = \text{Tr}(-1)^{J_0}$. This however can only represent a meaningful quantity if we can regulate the trace such that the regulator commutes with J_0 and the mass operator. For closed string theories with a right-moving $(2, 0)$ supersymmetry, the appropriate object to consider is

$$\text{ind}G_0^+ = \text{Tr} \left[(-1)^{J_0} \bar{q}^{(G_0^+ + (G_0^+)^\dagger)^2} q^{\tilde{L}_0 - \frac{c}{24}} \right] = \text{Tr} \left[(-1)^{J_0} \bar{q}^{L_0 - \frac{c}{24}} q^{\tilde{L}_0 - \frac{c}{24}} \right], \quad (4)$$

where the trace runs over the R -sector, \tilde{L}_0 is the hamiltonian for the left-moving sector, and $q = e^{2\pi i\tau}$, $\text{Im}\tau > 0$. As only states with $L_0 = \frac{c}{24}$ contribute, (4) is a function of q only. A priori, the value of the trace is not guaranteed to be integral as J_0 need not take integer eigenvalues on all states. This is however only a sufficient, but not necessary condition: for example, in orbifold theories, integrality of (4) need only occur after summing over all twisted sectors. If $(-1)^{J_0} = \pm 1$, then one can indeed identify the right-hand side of (4) with the usual notion of index: $\text{ind}G_0^+ = \dim \ker G_0^+ - \dim \ker (G_0^+)^\dagger$.

We like to point out that one can also consider the one-parameter family of indices $\text{ind}G_a^+ = \text{Tr}[(-1)^{J_0} \bar{q}^{L_0(a) - \frac{c}{24}} q^{\tilde{L}_0 - \frac{c}{24}}]$, with $L_0(a) = L_0 + aJ_0 + \frac{c}{6}a^2$ and where the trace runs over states created from a shifted Dirac sea $|\Omega_a\rangle$ ('spectral flow' [7]). This allows in particular to define also index theorems in the NS -sector ($a = \frac{1}{2}$). Naively, as a moves over one period (*e.g.*, from zero to one), one expects that $\text{ind}G_a^+$ returns to its original value. It turns out, however, that this is in general not true; rather, there occurs a global anomaly if the first Chern class c_1 (as defined below) does not vanish. Similarly, $\text{ind}G_{\frac{1}{2}}^+ = \text{ind}G_0^+$ if $\frac{1}{2}c_1 = 0$; this is relevant in supersymmetric theories. In the following, we will however confine ourselves to $a = 0$ and compute (4) for several classes of string theories.

3. N=2 Superconformal Symmetry in Specific String Theories

We need to explain some details of the $N=2$ superconformal symmetry arising in various orbifold and lattice string theories. Consider first an untwisted theory, that is, a ten dimensional heterotic string compactified on some $2d$ dimensional torus \mathcal{T} . The local right-moving $N=1$ symmetry is generated by $T_F(\bar{z}) = -\frac{1}{2}\psi^\mu\bar{\partial}X_\mu(\bar{z})$, $\mu = 1, \dots, 10$. Adopting a complex basis, this can trivially be rewritten $T_F(\bar{z}) = -\frac{1}{2}\psi^a\bar{\partial}X_a(\bar{z}) - \frac{1}{2}\psi^{\bar{a}}\bar{\partial}X_{\bar{a}}(\bar{z}) = G^+(\bar{z}) + G^-(\bar{z})$, and together with $J(\bar{z}) =: \psi^a\psi^{\bar{a}} : (\bar{z})$ we immediately can extend the $N=1$ algebra to an $N=2$ algebra with $c = 15$ ^{*}. The zero mode of J can thus be identified with the fermion number operator F .

Consider now Z_N twists g ($g^N = 1$) acting on some of the compactified right-moving coordinates, and choose a complex basis that diagonalizes these twists:

$$\begin{aligned} g : X^a(\bar{z}) &\rightarrow e^{2\pi i k_a/N} X^a(\bar{z}), & g : \psi^a(\bar{z}) &\rightarrow e^{2\pi i k_a/N} \psi^a(\bar{z}), \\ g : X^{\bar{a}}(\bar{z}) &\rightarrow e^{-2\pi i k_a/N} X^{\bar{a}}(\bar{z}), & g : \psi^{\bar{a}}(\bar{z}) &\rightarrow e^{-2\pi i k_a/N} \psi^{\bar{a}}(\bar{z}). \end{aligned} \tag{5}$$

Obviously, T_F and J defined above are preserved under these twists, and the resulting theory based on the (not necessarily symmetric) orbifold $\mathcal{O} \simeq \mathcal{T}/Z_N$ is $N=2$ invariant.

Thus, merely defining complex coordinates allows to extend the symmetry. This is quite general: it applies to any theory with complex fermions. In particular, one can easily show that any conformal field theory based on a covariant lattice Λ (consisting of the momenta of the bosonized fermions, superghost ϕ and the 'compactified' bosons) has $N=2$ supersymmetry: one can always cut T_F into two pieces with charges ± 1 under the zero mode of

$$J(\bar{z}) = i v \cdot \bar{\partial} H(\bar{z}), \tag{6}$$

by choosing an appropriate vector \vec{v} satisfying $v^2 = \frac{c}{3} = 5$ (note that \vec{v} does not lie on the lattice). The bosons denoted by H can be defined to be subset of compact bosons which describes the NSR fermions[†].

^{*} One can combine this algebra with the $N=2$ algebra of the superconformal ghost system [13] to obtain a $N=2$ symmetry acting on the complete right-moving sector with $c = 0$.

[†] In general, given a lattice, there is a priori no distinction between 'compactified' bosons and fermions. There can exist several (inequivalent) choices for \vec{v} , and accordingly several ways of realizing $N=2$ superconformal symmetry. In theories based on D_n -lattices, i.e., in theories with only periodic and antiperiodic fermionic boundary conditions, \vec{v} is a five dimensional vector with components ± 1 .

For our purposes, we can put orbifold and covariant lattice theories on equal footing, by bosonizing the fermions $\psi^a, \psi^{\bar{a}}$ above. In the fermionic sector, the orbifold twist operation (5) is then equivalent to a certain shift operation on the lattice describing the *NSR* fermions. Likewise, any (odd self-dual) covariant lattice Λ can be obtained from any other one, and in particular from a lattice Λ^T describing a torus-compactified supersymmetric heterotic string, by shift operations and appropriate projections [14,15]. More precisely, the untwisted sector corresponds to the lattice Λ_0^T of vectors of Λ^T with integer inner product with the shift vector $\vec{\delta}$, while the twisted sectors correspond to the conjugacy classes $[m] = [\Lambda_0^T + m\vec{\delta}]$, $m = 1, \dots, N-1$ (with $N\vec{\delta} \in \Lambda^T$, possibly having non-zero components also in the space-time and ϕ -ghost lattice sectors). The new covariant lattice Λ is a sublattice of the dual of Λ_0^T with the above conjugacy classes.

Define a vector $\vec{V} = (\vec{v}, +1)$, where $+1$ denotes the ϕ -ghost charge. The vector $(-\vec{V})$ is then associated with a vertex operator $e^{-iv \cdot H} e^{-i\phi}$ in the canonical ghost picture. It is the lattice analogue ($p = 5$) of the constant antiholomorphic p -form $\epsilon_{\bar{a}_1 \dots \bar{a}_p} \psi_0^{\bar{a}_1} \dots \psi_0^{\bar{a}_p}$, which exists on any $2p$ -dimensional Kähler manifold with vanishing first Chern class [16][5][6]. The vector, \vec{V} , belongs to the lattice Λ of the new theory precisely if it has integral (lorentzian) inner product with all other lattice vectors; as $\vec{V} \in \Lambda^T$, this is the case if $(V \cdot \delta) \in \mathbf{Z}$. We take this^{*} as the condition for vanishing of a generalized first Chern class, defined also for theories not admitting a compactification interpretation. Indeed, for orbifold twists of the form (5), it becomes the same as the condition for vanishing torsion first Chern class in orbifold theories, $\sum k_a = 0 \pmod{N}$, and we can write

$$c_1^{(tor)} = \sum k_a \cdot \hat{x} = N(V \cdot \delta) \cdot \hat{x} , \quad (7)$$

where \hat{x} is the generator of a torsion subgroup of $H^2(\mathcal{O}, \mathbf{Z})$ with $N \cdot \hat{x} = 0$ [17,18][‡]. This generalizes $c_1^{(free)} = \frac{i}{2\pi} \text{Tr} R = \frac{1}{2\pi} J_b^a R_a^b \in H^2(\mathcal{M}, \mathbf{R})$ of complex manifolds \mathcal{M} . Thus, \vec{V} and $\vec{\delta}$ play the role of complex structure J_b^a and curvature 2-form R_a^b , respectively. More precisely, the zero mode of the current $J = iv \partial H$ generates precisely the same $U(1)$ rotations $\delta\psi^a = i\psi^a$, $\delta\psi^{\bar{a}} = -i\psi^{\bar{a}}$ as the complex structure J_b^a . Note that the definition of c_1 above involves only the shift vector components in the *fermionic* (possibly also ϕ -ghost) sector of the theory, which is defined by \vec{V} .

* More generally, the existence of an operator with $(h, q) = (\frac{c}{6}, \frac{c}{3})$ in the matter sector.

‡ We implicitly assume an appropriate equivariant definition of \mathcal{O} .

The vector \vec{V} plays an interesting role for space-time supersymmetry: $(-\frac{1}{2}\vec{V})$ is the lattice representation of (a particular component of) the supersymmetry charge. The foregoing integrality condition is however not sufficient for the presence of this vector on the lattice Λ ; rather, only if there exists some choice of complex structure \vec{v} such that $(V \cdot \delta) = 0 \pmod{2}$, space-time supersymmetry arises (this is related to the 'charge integrality condition' [12]). That is, invariance of the constant $(0, p)$ form under a *discrete* holonomy group does not imply the invariance of the supercharge. One can express this by

$$\frac{1}{2}c_1^{(tor)} \equiv \frac{N}{2}(V \cdot \delta) \cdot \hat{x} = 0, \quad (8)$$

which, in integral cohomology, is a stronger statement than $c_1^{(tor)} = 0$. This generalizes the familiar condition for supersymmetry in manifold-compactified theories, $c_1^{(free)} = 0$. The integral class $\frac{1}{2}c_1^{(tor)}$ is well-defined if we assume the vanishing of the second Stiefel-Whitney class $w_2 \in H^2(\mathcal{O}, Z_2)$, which is the mod 2 reduction of c_1 . Indeed, $w_2 = 0$ iff $N(V \cdot \delta) = 0 \pmod{2}$; this is one of the level-matching conditions [19].

4. Computation of $\text{Tr}(-1)^F$

Following the discussion of [20], we recall some facts about spinors and holomorphic p -forms on Kähler manifolds. On a $2d$ dimensional Kähler manifold \mathcal{M} one can always find a complex basis for the gamma matrices such that $\{\gamma^{\bar{a}}, \gamma^b\} = \delta^{\bar{a}b}$, $\bar{a}, b = 1..d$. This then allows one to interpret these as creation and annihilation operators. Define now a (spinorial) Fock vacuum by $\gamma^a |\Omega\rangle = 0 \forall a$. Then the other spinor states are obtained by acting with products $\gamma^{\bar{a}_1} \gamma^{\bar{a}_2} \dots \gamma^{\bar{a}_p}$ on it. A general spinor field on \mathcal{M} can therefore be expanded as

$$\Psi(x_a, x_{\bar{a}}) = \bigoplus_p \Psi^{(p)}(x_a, x_{\bar{a}}) = \bigoplus_{p=0}^n \Phi_{\bar{a}_1 \dots \bar{a}_p}(x_a, x_{\bar{a}}) \gamma^{\bar{a}_1} \dots \gamma^{\bar{a}_p} |\Omega\rangle. \quad (9)$$

If $c_1 = 0$, $\Phi_{\bar{a}_1 \dots \bar{a}_p}$ are equivalent to antiholomorphic p -forms. Thus there is a one-to-one correspondence between spinors and $(0, p)$ -forms on \mathcal{M} . Since the application of $\gamma^{\bar{a}}$ switches chirality, the eigenvalue of γ_* acting on $\Psi^{(p)}$ is $(-1)^p$. Moreover, the action of the Dirac operator on Ψ is equivalent to the action of the Dolbeault operator on antiholomorphic p -forms, $\bar{\partial} : (0, p) \rightarrow (0, p + 1)$. The Dirac index on \mathcal{M} is therefore the same as the

Dolbeault index $\text{ind}\bar{\partial}$, which counts the difference between the numbers of even and odd antiholomorphic harmonic forms, respectively:

$$\text{ind}\mathcal{D} \equiv \text{Tr}\gamma_* = \sum_{p=0}^d (-1)^p \dim H_{\bar{\partial}}^{(0,p)}(\mathcal{M}, R) \equiv \text{ind}\bar{\partial}. \quad (10)$$

If however $c_1 \neq 0$, $\Phi_{\bar{a}_1 \dots \bar{a}_p}(x_a, x_{\bar{a}})$ is not equivalent to a $(0, p)$ -form since it has wrong transformation behavior under the $U(1)$ part of the holonomy group, and the foregoing identifications, in particular the relation of $(-1)^p$ with chirality, do not hold.

Analogously, in orbifold string theory, as soon as we can define a complex basis (i.e., an $N = 2$ supersymmetry), we can introduce creation and annihilation operators using $\{\psi_{-n}^{\bar{a}}, \psi_n^b\} = \delta^{ab}$. In the standard (untwisted) Fock vacuum, we can take $\psi_0^{\bar{a}}, \psi_{-n}^{\bar{a}}$ and ψ_{-n}^a ($n > 0$) as creation and ψ_0^a, ψ_n^a and $\psi_n^{\bar{a}}$ as annihilation operators. A spinor field in the untwisted sector can be expanded as in (9), with additional higher mode contributions. In the twisted sectors, creation and annihilation operators have to be appropriately redefined. It is easier to employ the bosonic formulation here; a generic spinor on \mathcal{O} in the shifted sector $[m]$ ($m = 0, \dots, N - 1$) can then be expanded as (modulo derivatives)

$$|\Psi_{[m]}\rangle = \bigoplus_{\lambda} |\Psi_{[m]}^{(\lambda)}\rangle = \sum_{\lambda} c_{m,\lambda} : e^{i(\lambda+m\delta)\cdot H} : (\mathbf{0}) |\Omega\rangle. \quad (11)$$

Here, λ are vectors with arbitrary integer components, and the exponentiated shift vector δ represents the twist field in the fermionic sector of the theory.

We now compute $\text{ind}G_0^{+(int)}$ in the $2d$ dimensional internal sector (we denote internal and space-time sectors by (int) and (st) , respectively; $\vec{\delta}$ has non-zero components only in (int)). The trace in (4) runs only over those states on which $G^{+(int)}(z)$ has an integral mode expansion. Since $G^+(z) = G^{+(st)}(z) + G^{+(int)}(z)$ is well-defined on all states, the states are precisely those on which also $G^{+(st)}(z)$ has an integral mode expansion. Thus,

$$\text{ind}G_0^{+(int)} = \text{Tr}_{R^{(st)}} \left[(-1)^{F^{(int)}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right]. \quad (12)$$

$R^{(st)}$ indicates summing only over states whose space-time part is in the Ramond-sector. To evaluate the trace, we need to know the action of $(-1)^{F^{(int)}}$ on the states (11). In principle, there occurs an ambiguity in defining the action of $(-1)^{F^{(int)}}$ on the twisted orbifold ground

states (this corresponds to the ambiguity in defining different regularization procedures in the σ -model calculation below). For the index of the $N = 2$ supercharge, however, the natural choice is to identify $F^{(int)} = J_0^{(int)}$, and thus to determine the action of $(-1)^{F^{(int)}}$ on a ground state by its $U(1)$ charge^{*}. Denoting by $\Gamma_*^{(int)}$ the chirality operator on the fermionic excitations of the internal space, we have, therefore

$$(-1)^{F^{(int)}} |\Psi_{[m]}\rangle = (-1)^{m(V \cdot \delta)} \Gamma_*^{(int)} |\Psi_{[m]}\rangle . \quad (13)$$

Thus, for arbitrary first Chern class, $(-1)^{F^{(int)}}$ ceases to have eigenvalues ± 1 and to represent the chirality operator of the internal sector. This is the phenomenon of charge fractionalization. It follows

$$\text{ind} G_0^{+(int)} = \frac{1}{N} \sum_{m=0}^{N-1} (-1)^{m(V \cdot \delta)} \text{ind}_{[m]} G_0^{(int)} , \quad (14)$$

where $\text{ind}_{[m]} G_0^{(int)}$ is the index $\text{Tr} \Gamma_*^{(int)}$ of the Dirac-Ramond operator $G_0^{(int)} = G_0^{+(int)} + G_0^{-(int)}$ in the shift sector $[m]$. For $(V \cdot \delta) = 0 \pmod{2}$, i.e., $\frac{1}{2}c_1 = 0$, the above-mentioned ambiguity disappears and $\text{ind} G_0^{+(int)}$ becomes the same as the index of the Dirac-Ramond operator,

$$\text{ind} G_0^{+(int)} = \text{Tr}_{R^{(st)}} \left[\Gamma_*^{(st)} \bar{q}^{L_0 - \frac{c}{24}} q^{\bar{L}_0 - \frac{c}{24}} \right] \equiv \text{ind} G_0^{(int)} . \quad (15)$$

It is interesting to note that the expression (15) is precisely the chiral partition function [21] of the theory, as $\Gamma_*^{(int)} = \Gamma_*^{(st)}$ (see, e.g. [20]); it is algebraically defined even for conformal field theories not admitting a compactification interpretation. Of course, for theories admitting such a interpretation, $\text{ind} G_0$ is the loop space generalization of the index of the Dirac operator $\text{ind} \not{D}$ on \mathcal{M} [22].

In the same spirit we can regard $\text{ind} G_0^+$ as an elliptic generalization of the Dolbeault index $\text{ind} \bar{\partial}$, which can be defined for a differential operator on some Kähler manifold. It is obvious that (15) is a generalization of (10), which applies if $c_1^{(free)} = \frac{i}{2\pi} \text{Tr} R = 0$. However, (15) holds only for $\frac{1}{2}c_1^{(tot)} = 0$ (the condition for space-time supersymmetry) which is a stronger condition than the vanishing of the integral class $c_1^{(tot)}$ as defined in (7).

* If $c_1 = 0 \pmod{2}$, i.e., $\omega_2 = 0$, one can also choose $(-1)^{F^{(int)}}$ in such a way that it takes only eigenvalues ± 1 on all states, and thus can be identified with the chirality operator acting on spinors. With such a choice, the calculation would lead to the index of the Dirac-Ramond operator.

In order to see this in a different manner and to really show that $\text{ind}G_0^+$ generalizes $\text{ind}\bar{\partial}$, we evaluate $\text{ind}G_0^+ = \text{Tr}(-1)^F$ via the path integral for a $(2,0)$ σ -model on some $2d$ -dimensional Kähler target space \mathcal{M} :

$$S_E = \frac{1}{2\pi} \int d^2\sigma g_{\bar{a}b}(X) \partial X^{\bar{a}} \bar{\partial} X^b - \psi^{\bar{a}} g_{\bar{a}b}(X) \left(\delta_d^b \partial + \Gamma_{cd}^b(X) \partial X^c \right) \psi^d. \quad (16)$$

Separating out and rescaling the zero modes, the (in the small $Im\tau$ limit) leading part reads

$$S_E = \frac{1}{2\pi} \int d^2\sigma \partial X^{\bar{a}} (\delta_{\bar{a}b} \bar{\partial} + \mathcal{R}_{\bar{a}b}) X^b - \delta_{\bar{a}b} \psi^{\bar{a}} \partial \psi^b, \quad (17)$$

where X and ψ are now the quantum fluctuations around the classical zero modes $x_0^a, x_0^{\bar{a}}, \psi_0^a, \psi_0^{\bar{a}}$, and $\mathcal{R}_{\bar{a}b} = \frac{1}{2} R_{\bar{a}b\bar{c}d}(x_0^a, x_0^{\bar{a}}) \psi_0^{\bar{c}} \psi_0^d$ is the curvature two-form. The index of the $N=2$ supercharge $G_0^+ = \int d\sigma_1 \psi^a \left(-i \frac{D}{DX^a} + g_{\bar{a}b}(X) \frac{\partial X^b}{\partial \sigma_1} \right)$ is then obtained from

$$\text{ind}G_0^+ = \lim_{Im\tau \rightarrow 0} \int dX d\psi e^{-S_E} = \int_{\mathcal{M}} dx_0^{\bar{a}} dx_0^a d\psi_0^{\bar{a}} d\psi_0^a \frac{\det'(\partial \delta_{\bar{a}b})}{\det'(\partial \delta_{\bar{a}b}) \det'(\bar{\partial} \delta_{\bar{a}b} + \mathcal{R}_{\bar{a}b})}, \quad (18)$$

with periodic-periodic boundary conditions on the fermions. The computation of the remaining determinant is similar to the index calculation of the Dirac-Ramond operator [22], except that one has to regulate carefully using complex coordinates since \mathcal{R} takes now values in the holonomy group $U(d)$ instead of $SO(2d)$. Let us first consider

$$\begin{aligned} \log \left[\frac{\det'(\bar{\partial} + \nu)}{\det'(\bar{\partial})} \right] &= \sum'_{m,n} \log \left(1 + \frac{\nu}{m\tau + n} \right) \\ &= \lim_{s \rightarrow 0} \sum_{k=1}^{\infty} \sum'_{m,n} \frac{(-1)^{k+1}}{k} \frac{\nu^k}{(m\tau + n)^{k+s}} \\ &= \lim_{s \rightarrow 0} \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \nu^k \left(1 + e^{i\pi(k+s)} \right) G_{k+s}(\tau), \end{aligned} \quad (19)$$

where τ is the modular parameter of a skewed torus, and where

$$\begin{aligned} \frac{1}{2} G_s(\tau) &= \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} (m\tau + n)^{-s} + \sum_{n=1}^{\infty} n^{-s} \\ &= \frac{(-2\pi i)^s}{\Gamma(s)} \left(\frac{\Gamma(s)}{(-2\pi i)^s} \zeta(s) + \sum_{n=1}^{\infty} \sigma_{s-1}(n) q^n \right). \end{aligned}$$

Being careful about the contribution of the pole of the ζ -function for $k = 1$, (19) becomes

$$\begin{aligned} & i\frac{\pi}{2}\nu \lim_{s \rightarrow 0} (sG_{s+1}(\tau)) - \sum_{k=1}^{\infty} \frac{\nu^{2k}}{2k} G_{2k}(\tau) \\ & = i\pi\nu - \sum_{k=1}^{\infty} \frac{\nu^{2k}}{2k} G_{2k}(\tau) , \end{aligned} \quad (20)$$

where $G_{2k}(\tau)$ are the Eisenstein series (transforming with modular weight $2k$ for $k > 1$). Therefore, using the identities $\frac{\vartheta_1(\nu|\tau)}{\nu\vartheta_1'(0|\tau)} = \exp[-\sum \frac{\nu^{2k}}{2k} G_{2k}(\tau)]$, $\vartheta_1'(0|\tau) = 2\eta^3(\tau)$ and denoting the skew eigenvalues of $\frac{i}{2\pi}\mathcal{R}_{\bar{a}b}$ by ω_a , we get

$$\begin{aligned} \text{ind}G_0^+ &= \int_{\mathcal{M}} dx_0^{\bar{a}} dx_0^a d\psi_0^{\bar{a}} d\psi_0^a \prod_{a=1}^d \left[e^{\frac{1}{2}\omega_a} \frac{\frac{i\omega_a}{2\pi} \eta(\tau)}{\vartheta_1\left(\frac{i\omega_a}{2\pi}|\tau\right)} \right] \\ &= q^{-d/12} \int_{\mathcal{M}} dx_0^{\bar{a}} dx_0^a d\psi_0^{\bar{a}} d\psi_0^a \prod_{a=1}^d \left[e^{\frac{1}{2}\omega_a} \frac{\frac{1}{2}\omega_a}{\sinh(\frac{1}{2}\omega_a)} \right] \prod_{n=1}^{\infty} \det \left(1 - q^n e^{\frac{iR}{2\pi}} \right)^{-1} . \end{aligned} \quad (21)$$

Integrating over the fermionic zero modes results in substituting \mathcal{R} by $R = R_{\bar{c}d} dx^{\bar{c}} \wedge dx^d$, and rewriting the two products yields

$$\text{ind}G_0^+ = q^{-d/12} \int_{\mathcal{M}} td(R) Ch(q, R) \Big|_{top\ form} . \quad (22)$$

Here $td(R)$ is the Todd genus, which is related as follows to the Dirac genus $\hat{A}(R)$ [23]:

$$td(R) = \prod_{a=1}^d e^{\frac{1}{2}\omega_a} \cdot \left[\frac{\frac{1}{2}\omega_a}{\sinh(\frac{1}{2}\omega_a)} \right] = e^{\frac{1}{2}c_1} \cdot \hat{A}(R) , \quad (23)$$

with $c_1 \equiv c_1^{(free)}$. The series $Ch(q, R) = \sum_{n=0}^{\infty} q^n \text{Tr}_{[n]} e^{\frac{iR}{2\pi}}$ represents the Chern character for every string level. As the Todd genus is precisely the index density of the Dolbeault complex, the level $n = 0$ part of (22) is $q^{-d/12}$ times

$$\text{ind}\bar{\partial} = \int_{\mathcal{M}} td(R) \Big|_{top\ form} . \quad (24)$$

It follows that (22) is the string generalization of $\text{ind}\bar{\partial}$. For $c_1 = 0$, it becomes equal to the index of the Dirac-Ramond operator $\text{ind}G_0 = q^{-d/12} \int \hat{A}(R) Ch(q, R)$; this is the elliptic generalization of (10).

The factor $e^{\frac{1}{2}c_1}$ above arises from the ζ -function regularization (20) of the determinants, and is in a sense an artefact of this regularization. However, as the same divergence as in (20) arises already in the field theory computation of $\text{ind}\bar{\partial}$, we can appeal to the usual field theory reasoning [23] to resolve the regulator ambiguity as above.

The index of G_0 has well-defined transformation properties under the modular group, as long as the first Pontryagin class $p_1 = -\frac{1}{8\pi^2} \text{Tr} R^2 \in H^4(\mathcal{M}, R)$ vanishes [21][22] (this is because $\text{Tr} R^2$ is multiplied by the anomalous modular function $G_2(\tau)$; cf. (20)). Similarly, since there exists no compensating modular weight one function $G_1(\tau)$ to multiply $c_1 = \frac{i}{2\pi} \text{Tr} R$ with in (20)-(22), it follows that modular covariance of $\text{ind}G_0^+$ is spoiled if $c_1 \neq 0$. In other words, $c_1^{(free)} \neq 0$ and $p_1^{(free)} \neq 0$ are obstructions to generalizing the Dolbeault index to loop space. Both obstructions arise due to particular choices of regularization schemes that violate modular invariance: for the Dirac-Ramond operator, one chooses a regularization prescription which manifestly ensures holomorphicity; for the $N=2$ supercharge one chooses a prescription which (in addition) is adapted to complex coordinates.

This feature is also reflected in the formula (14) for $\text{ind}G_0^+$ on orbifolds. For arbitrary first Chern class, the phases in the sum over twisted sectors destroys modular covariance. The phases disappear only for $\frac{1}{2}c_1^{(tor)} = 0$. This is similar to the role played by the integral class $\lambda^{(tor)} = \frac{1}{2}p_1^{(tor)} \in H^4(\mathcal{O}, Z)$ for the Dirac-Ramond operator on orbifolds [19] [24-26]. The relevance of $\frac{1}{2}c_1^{(tor)}$ instead of $c_1^{(tor)}$ is already suggested by (23). More specifically, consider the character valued index associated with a group action g as in (5). According to the fixed point theorem, in the twisted sectors $\text{ind}\bar{\partial} = \int td(R)$ is replaced by the Lefschetz number of the Dolbeault complex [23]:

$$\begin{aligned} L_{Dol} &= \text{Tr} \left(gH_{\bar{\partial}}^{(0,even)} \right) - \text{Tr} \left(gH_{\bar{\partial}}^{(0,odd)} \right) \\ &= \sum_{\substack{\text{fixed} \\ \text{points}}} \prod_{a=1}^d \frac{1 - e^{-i\theta_a}}{2 - 2\cos(\theta_a)} = \sum_{\substack{\text{fixed} \\ \text{points}}} \prod_{a=1}^d e^{-\frac{i}{2}\theta_a} \left[\frac{i}{2\sin(\frac{1}{2}\theta_a)} \right], \end{aligned} \quad (25)$$

with $\theta_a \equiv \frac{2\pi}{N}k_a$. The expression in the brackets is the index 'density' for the twisted Dirac operator [23] [27], as it appears in the expression for the Dirac-Ramond index $\text{ind}G_0$ on orbifolds [28]. The phase corresponds to that in (14), and vanishes if $\sum k_a = 0 \pmod{2N}$, i.e., $\frac{1}{2}c_1^{(tor)} = 0$.

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