CALT-68-1551
CCNY-HEP-89/05
DOE RESEARCH AND
DEVELOPMENT REPORT

On Non-Renormalization Theorems for Four-Dimensional Superstrings

Olaf Lechtenfeld*

Physics Department, City College of New York
New York, NY 10031

and

Wolfgang Lerche†

California Institute of Technology, Pasadena, CA 91125

Abstract

For $d = 4$, $N = 1$ supersymmetric string theories we compute the quantum corrections to the superpotential to two-loop order. The vanishing of the amplitudes relies crucially on generalized Riemann identities. Beyond genus one, these map to non-trivial zeroes of theta functions implied by the Riemann vanishing theorem.

⋆ Work supported by NSF grant NSF-PHY-82-15364 and DOE grant DE-AC02-83ER40107.
† Work supported by DOE contract DE-AC03-81ER40050.

The very reason why one is interested in $N=1$ supersymmetric unified theories at all is the supposed absence of radiative corrections to the superpotential. Many indirect arguments have been given [1,2] why such non-renormalization theorems should hold also in string theory. However, *explicit* string computations of higher genus corrections to the superpotential have not been performed so far* . This is in contrast to higher-loop computations in $d=10$ (equivalent to $N=4$ in $d=4$), for which various non-renormalization results have been obtained [4-6]. Though lower-dimensional strings are more complex than those in ten dimensions, one can easily isolate the structure which is relevant for space-time supersymmetry in such theories. It turns out to be universal, *i.e.* the same for all possible supersymmetric string models. One expects that all features pertinent to supersymmetry, like non-renormalization theorems in particular, can be understood solely in terms of this structure. We wish to demonstrate this by directly computing the one- and two-loop corrections to the superpotential (without relying on contour arguments).

We shall adopt a meromorphic generalization of the unitary gauge in supermoduli space that has proven to be useful in ten dimensions. As is well-known, there are generic difficulties with higher-loop computations in superstring theory [2] [7]. These have to do with the ambiguous supermoduli integration, with the placing of picture changing operators and so on. It is clear that these subtleties are also universal, *i.e.* independent of the particular model and space-time dimension. Our aim here is not to resolve these problems. Rather, we will focus on how supersymmetry cancellations work, which is in a sense orthogonal to these issues. Thus, our results will be rigorous to the extent higher-loop calculations are reliable in ten dimensions.

One crucial ingredient is that all information about space-time supersymmetry in string theory is encoded in the representation theory of the exceptional groups [8]. In particular, $N=1$, $N=2$ and $N=4$ supersymmetry in four dimensions are characterized by E_6, E_7 and E_8 (similarly for other dimensions). Consider for example the $N=1$ case. It can be shown [9][8] [10] that part of the internal sector of

---

* For a related one-loop calculation, see however [3].

any $N=1$ theory can always be represented by a free boson whose momenta span a certain one-dimensional lattice $\mathcal{U}$. Similarly, upon bosonizing the NSR fermions, the light-cone space-time sector (described by the transverse Lorentz group $SO(2)$) can be described by a lattice D_1. Thus the theory has the form *(internal sector)*$\times\mathcal{U}D_1$ in the right-moving sector. The point is that in a supersymmetric theory the possible quantum numbers of $\mathcal{U}\times SO(2)$ are correlated in a specific way that is encoded in the weight lattice of E_6. This correlation is best characterized by writing $\mathcal{U}\times SO(2)$ as the maximal torus of $E_6/SO(8)$.

More generally, any supersymmetric theory has (partly) the structure of a coset conformal field theory $E_n/SO(8)$; it is this feature which is responsible for all aspects of supersymmetry‡ . It implies that the partition function contains as a piece the corresponding coset branching function, defined by

$$\mathrm{Ch}_\Lambda^{E_n}(\tau) \;=\; \bigoplus_{\lambda=0,v,s,c} \mathrm{Ch}_{\Lambda,\lambda}^{E_n/SO(8)}(\tau)\cdot \mathrm{Ch}_\lambda^{SO(8)}(\tau)\ .$$

Here, Λ and λ label the level one highest weight representations of $\hat{E}_n$ and $S\hat{O}(8)$, respectively; for E_6, Λ labels the conjugacy classes $0,1,\bar{1}$, corresponding to gauge and (anti-)matter sectors. For the sake of clarity we momentarily write our formulae for the torus; however, they trivially generalize to higher genera by assigning one representation to each loop. The partition function has the form

$$Z(\overline{\tau},\tau) \;=\; \sum_\Lambda P_\Lambda^{int}(\overline{\tau},\tau)\left\{\sum_\lambda \mathrm{Ch}_{\Lambda,\lambda}^{E_n/SO(8)}(\tau)\,Y_\lambda(\tau)\right\}\ ,$$

where Y_λ denotes the contributions of the superghosts and longitudinal NSR fermions, and P_Λ^{int} the (arbitrarily complicated) partition function of the rest of the theory. All building blocks have simple modular properties corresponding to the

‡ It also implies the presence of extended world-sheet supersymmetries [9]. However, it is doubtful whether this fact has any particular significance for space-time supersymmetry.

level one Kac-Moody labels they carry. Similarly, any correlation function involving some operator(s) $\mathcal{O} = \mathcal{O}^{int}\mathcal{O}^{E_n/SO(8)}\mathcal{O}^{Y}$ factorizes as

$$\left\langle\, \mathcal{O}\, \right\rangle = \sum_{\Lambda} \left\langle \mathcal{O}^{int} \right\rangle_{\Lambda} \left\{ \sum_{\lambda} \left\langle \mathcal{O}^{E_n/SO(8)} \right\rangle_{\Lambda,\lambda} \left\langle \mathcal{O}^{Y} \right\rangle_{\lambda} \right\} . \tag{1}$$

It is clear that the precise form of the internal sector is irrelevant for our purposes. Rather, all non-renormalization features are due to the vanishing of the braces above, for any Λ. This happens because the branching functions obey certain identities [11][10] which are a reflection of supersymmetry. These identities can be derived from the fact that the triality outer automorphism of $SO(8)$ becomes an inner automorphism when embedded into E_n. They can be summarized by

$$\mathrm{Ch}_{\Lambda,\lambda}^{E_n/SO(8)}(\nu\,|\,\tau) \;=\; \mathrm{Ch}_{\Lambda,\sigma(\lambda)}^{E_n/SO(8)}(T_\sigma \nu\,|\,\tau) \;, \tag{2}$$

where $\sigma(\lambda)$ denotes a triality automorphism acting on the $SO(8)$ labels, and T_σ is the representation matrix of σ acting on the Cartan subalgebra (note that (2) applies to all genera). Choosing the particular automorphism $\sigma^* :\; v \to s, s \to v, c \to c$ one can write equation (2), at one loop, as $\mathrm{Ch}_v - \mathrm{Ch}_c = \mathrm{Ch}_s - \mathrm{Ch}_c$. Transforming from conjugacy class to spin structure basis, $\mathrm{Ch}_v = \frac{1}{2}(\mathrm{Ch}[{0 \atop 0}] - \mathrm{Ch}[{0 \atop 1}])$, $\mathrm{Ch}_s = \frac{1}{2}(\mathrm{Ch}[{1 \atop 0}] + \mathrm{Ch}[{1 \atop 1}])$ and so on, this takes a more familiar form:

$$\sum_{a,b\,\in\,\{0,1\}} e^{i\pi(a-b)}\,\mathrm{Ch}_{\Lambda}^{E_n/SO(8)}\begin{bmatrix} a \\ b \end{bmatrix}(\nu\,|\,\tau) \;=\; 2\,\mathrm{Ch}_{\Lambda}^{E_n/SO(8)}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(T_{\sigma^*}\nu\,|\,\tau) \,. \tag{3}$$

In our application, $[{a \atop b}]$ labels the spin structures of the NSR fermions with space-time indices. Explicitly, for E_6, one has $\nu = (\nu_{\mathcal{U}}, \nu_1)$,

$$T_{\sigma^*} \;=\; \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix} \,,$$

and

$$\mathrm{Ch}_{\Lambda}^{E_6/SO(8)}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(T_{\sigma^*}\nu\,|\,\tau) \;=\; \vartheta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tfrac{1}{2}\nu_1 + \tfrac{1}{2}\sqrt{3}\nu_{\mathcal{U}}|\tau)\cdot \mathcal{F}_{\Lambda}(3\nu_1 - \sqrt{3}\nu_{\mathcal{U}}|\tau) \,, \tag{4}$$

with

$$\mathcal{F}_0(\nu|\tau) = \Big\{\Theta_{3,6} - \Theta_{-3,6}\Big\}(\nu|\tau)$$
$$\mathcal{F}_1(\nu|\tau) = \Big\{\Theta_{-1,6} - \Theta_{5,6}\Big\}(\nu|\tau)$$
$$\mathcal{F}_{\bar{1}}(\nu|\tau) = \Big\{\Theta_{-5,6} - \Theta_{1,6}\Big\}(\nu|\tau)\ ,$$

where

$$\Theta_{n,m}(\nu|\tau) = \sum_{l\in Z+\frac{n}{2m}} e^{2\pi i\tau m l^2+2\pi i l\nu}$$

are theta functions of level m [11]. Choosing a different automorphism, *e.g.* the one exchanging v and c instead, leads to a different identity with different $T_{\sigma^\bullet}$.

More generally, for arbitrary genus g, one can derive

$$\sum_{\vec{a},\vec{b}\in Z^g/2Z^g} e^{i\pi(\vec{\alpha}_1\vec{b}-\vec{\alpha}_2\vec{a})}\mathrm{Ch}_{\vec{\Lambda}}^{E_n/SO(8)}\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{\nu}\,|\,\Omega) = 2^g\,\mathrm{Ch}_{\vec{\Lambda}}^{E_n/SO(8)}\begin{bmatrix}\vec{\alpha}_1\\ \vec{\alpha}_2\end{bmatrix}(T_{\sigma^\bullet}\vec{\nu}\,|\,\Omega)\ , \quad (5)$$

where $\vec{x} = (x_1, ..., x_g)$ and Ω denotes the period matrix. The ambiguity in choosing an arbitrary (not necessarily odd) reference spin structure $\alpha = [{\vec{\alpha}_1 \atop \vec{\alpha}_2}]$ comes from an ambiguity in taking linear combinations when changing from conjugacy class to spin structure basis. For E_8 ($N=4$ supersymmetry in $d=4$), $\mathrm{Ch}[{\vec{a} \atop \vec{b}}] = \vartheta[{\vec{a} \atop \vec{b}}]^4$ and (5) is identical to the well-known Riemann identity. For E_6 ($N=1$), the r.h.s. of (5) looks complicated, but all what matters is the form

$$\mathrm{Ch}_{\vec{\Lambda}}^{E_6/SO(8)}\begin{bmatrix}\vec{\alpha}_1\\ \vec{\alpha}_2\end{bmatrix}(T_{\sigma^\bullet}\vec{\nu}\,|\,\Omega) = \vartheta\begin{bmatrix}\vec{\alpha}_1\\ \vec{\alpha}_2\end{bmatrix}(\tfrac{1}{2}\vec{\nu}_1 + \tfrac{1}{2}\sqrt{3}\vec{\nu}_{\mathcal{U}}|\Omega)\cdot\mathcal{F}_{\vec{\Lambda}}\begin{bmatrix}\vec{\alpha}_1\\ \vec{\alpha}_2\end{bmatrix}(3\vec{\nu}_1 - \sqrt{3}\vec{\nu}_{\mathcal{U}}|\Omega) \quad (6)$$

for some $\mathcal{F}_{\vec{\Lambda}}$ (composed of higher genus level six theta functions).

The claim is that for $N = 1$ supersymmetry, F-terms are not renormalized beyond tree level. They have the form

$$\int d^2\theta\, W(\Phi) = W_i'(\varphi)F_i - \tfrac{1}{2}W_{ij}''(\varphi)\psi_i\psi_j\ , \quad (7)$$

where $\Phi = \varphi + \theta\psi + \theta\theta F$ is a chiral superfield. The vertex operators for a given supermultiplet are easily inferred from the exceptional group structure, *i.e.* from the weights of $\underline{27}$ of E_6:

$$\begin{aligned} V^{\varphi}_{-1} &= V^{(\frac{1}{3}\sqrt{3},0,0,\,|\,-1)}V^{int}e^{ikX} \\ V^{\psi(\pm)}_{-\frac{1}{2}} &= V^{(-\frac{1}{6}\sqrt{3},\pm\frac{1}{2},\mp\frac{1}{2},\,|\,-\frac{1}{2})}V^{int}e^{ikX} \\ V^{F}_{0} &= V^{(-\frac{2}{3}\sqrt{3},0,0,\,|\,0)}V^{int}e^{ikX}\ . \end{aligned} \tag{8}$$

We use a bosonic lattice notation here: the first entry in V indicates the $\mathcal{U}$-charge, the following ones an $SO(4)$ weight and the last entry the ϕ-ghost charge. As the precise form of V^{int} is not relevant for us here, we will suppress it in the following. Furthermore, we will need the picture-changing operator

$$\begin{aligned} \mathcal{P} \;\equiv\; \{Q_{BRST},\xi\} &= \mathcal{P}_{matter} + \mathcal{P}_{ghost}\,, \qquad \text{where} \\ \mathcal{P}_{matter} &= e^{i\phi}\{T_F^{space-time} + T_F^{int}\} = \mathcal{P}^0_{+1} + \mathcal{P}^+_{+1} + \mathcal{P}^-_{+1} \\ \mathcal{P}^0_{+1} &= V^{(0,\{0,\pm1\}|+1)}\cdot\partial X \\ \mathcal{P}^{\pm}_{+1} &= V^{(\pm\frac{1}{3}\sqrt{3},0,0,\,|+1)}\mathcal{P}^{\pm}_{int}\,, \qquad \text{and} \\ \mathcal{P}_{ghost} &= \mathcal{P}^0_0 + \mathcal{P}^0_{+2} \\ \mathcal{P}^0_0 &= c\partial\xi \\ \mathcal{P}^0_{+2} &= [2\partial\eta b + \eta\partial b + 2\eta\partial\phi b]\,V^{(0,0,0|+2)} \end{aligned} \tag{9}$$

(in obvious notation). The subscript indicates the ϕ-ghost charge, and the superscript the $\mathcal{U}$-charge. When $\mathcal{P}$ acts on a vertex V, it simply changes its picture. Note that $\mathcal{P}^0_0 V$ must be omitted since it compensates for the difference between $\mathcal{P}V$ and $\{Q_{BRST},\xi V\}$ prior to integration.

As a warm-up we start with the one-loop calculation. In this paper we will consider only the second term in (7) and hence amplitudes of the form $<\psi\psi\varphi\ldots\varphi>$ for vanishing momenta and $\alpha'\to 0$ (the computation of the first term is completely analogous). We furthermore restrict ourselves to massless external fields. It is sufficient to consider only the spin-structure dependent, that is the zero-mode part of the braces in (1) (we shall discuss poles later). It is given by

$$\Big\langle V^{(\lambda_{\mathcal{U}},\lambda_1,\lambda_2|\lambda_\phi)}(z)V^{(\mu_{\mathcal{U}},\mu_1,\mu_2|\mu_\phi)}(w)\ldots\Big\rangle^{\Lambda}_{zero\ modes}$$

$$= \sum_{a,b} \mathrm{Ch}_{\Lambda}^{E_6/SO(8)}\begin{bmatrix}a\\b\end{bmatrix}(\lambda_{\mathcal{U}}z+\mu_{\mathcal{U}}w+\ldots,\lambda_1 z+\mu_1 w+\ldots\,|\,\tau) \qquad (10)$$

$$\times\, Y\begin{bmatrix}a\\b\end{bmatrix}(\lambda_2 z+\mu_2 w+\ldots,\lambda_\phi z+\mu_\phi w+\ldots-2\Delta|\tau)\ ,$$

where

$$Y\begin{bmatrix}a\\b\end{bmatrix}(z_2,z_\phi-2\Delta|\tau) \;=\; \frac{\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z_2|\tau)}{\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z_\phi-2\Delta|\tau)}\ , \qquad \Delta=\tfrac{1}{2}(1-\tau)$$

is the usual ratio of determinants.

Consider first the fermion two-point function, with $\mathcal{P}$ attached to one of the vertex operators. By $\mathcal{U}$-charge conservation only $\mathcal{P}^+_{+1}$ can contribute, and thus

$$\Big\langle \psi\psi\Big\rangle^{\Lambda} \sim \Big\langle \Big(\mathcal{P}^+_{+1}V^{\psi(-)}_{-\frac12}\Big)(z)\,V^{\psi(+)}_{-\frac12}(0)\Big\rangle^{\Lambda}_{zero\ modes}$$

$$= \sum_{a,b} \mathrm{Ch}_{\Lambda}^{E_6/SO(8)}\begin{bmatrix}a\\b\end{bmatrix}(\tfrac16\sqrt3 z,-\tfrac12 z\,|\,\tau)\cdot Y\begin{bmatrix}a\\b\end{bmatrix}(\tfrac12 z,\tfrac12 z-2\Delta|\tau)\ . \qquad (11)$$

From

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z-2\Delta|\tau) \;=\; e^{i\pi(a-b)-i\pi\tau-2\pi i z}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau)$$

follows $Y[{a \atop b}](z,z-2\Delta) \sim e^{i\pi(a-b)}$. Thus, we can use the identity (3) and the quantum numbers just happen to be such that (4) gives*

$$\Big\langle \psi\psi\Big\rangle^{\Lambda} \sim \vartheta\begin{bmatrix}1\\1\end{bmatrix}(0|\tau) = 0 \qquad \text{for}\quad \Lambda=0,1,\bar{1}\ .$$

One might worry about what happens if the two vertex operators collide, that is about derivative terms appearing in $\psi\cdot\psi$ for $z\to0$. It is trivial to see that there is no such contribution, due to the fact that $(\frac{\sqrt3}{6}\frac{\partial}{\nu_{\mathcal{U}}}-\frac12\frac{\partial}{\nu_1})$ acting on (4) vanishes.

---

* Had we chosen different spinor indices, the character arguments in (11) would have been changed. Cancellation still occurs, as then another Riemann identity with a different T_{σ^*} is relevant. We will always choose only such configurations of Lorentz quantum numbers that allow for an application of (3).

Consider now M additional external bosons. Only picture changed bosonic vertices, $V_0^\phi=\{Q_{BRST},\xi V_{-1}^\phi\}$, occur. Counting conformal dimensions, it follows that in the picture changing operation $\mathcal{P}_{+1}^{+}$ and $\mathcal{P}_{+2}^{0}$ cannot contribute, so that

$$V_0^\phi(w) \;=\; V^{(0,0,0|0)}\cdot \lim_{z\to w}(z-w)^{-\frac{1}{3}}\mathcal{P}_{int}^{-}(z)V^{int}(w) + k\cdot V^{(\frac{1}{3}\sqrt{3},\{\pm1,0\}|0)}V^{int}(w)\,. \quad (12)$$

Let us focus on the first term. Then, because of the complete neutrality in the $E_6/SO(8)$ sector any number of insertions does not spoil the structure of (11) and thus this contribution vanishes. It is also clear that poles coming from colliding operators do not matter. What remains are possible contributions from the momentum-dependent terms. Although we are considering processes with $k\to 0$, we must be careful [3] [12] about possible contact terms proportional to $\frac{k\cdot k}{k^2}$. There are two such correlators proportional to k^2. One is

$$\begin{aligned}&\Big\langle \Big(\mathcal{P}_{+1}^{0}V_{-\frac{1}{2}}^{\psi(-)}\Big)(z)\,V_{-\frac{1}{2}}^{\psi(+)}(0)\,(\mathcal{P}_{+1}^{0}V_{-1}^{\varphi})(y)\prod_{i=1}^{M-1}(\mathcal{P}_{+1}^{-}V_{-1}^{\varphi})(x_i)\Big\rangle_{zero\ modes}^{\Lambda}\\ &\sim \sum_{a,b}\mathrm{Ch}_{\Lambda}^{E_6/SO(8)}\begin{bmatrix}a\\b\end{bmatrix}(-\tfrac{1}{6}\sqrt{3}z+\tfrac{1}{3}\sqrt{3}y,\tfrac{1}{2}z-y\,|\,\tau)\cdot Y\begin{bmatrix}a\\b\end{bmatrix}(\tfrac{1}{2}z,\tfrac{1}{2}z-2\Delta|\tau)\,,\end{aligned} \quad (13)$$

which vanishes identically by the same reason as (11) does. The other term is

$$\begin{aligned}&\Big\langle \Big(\mathcal{P}_{+1}^{-}V_{-\frac{1}{2}}^{\psi(-)}\Big)(z)\,V_{-\frac{1}{2}}^{\psi(+)}(0)\prod_{j=1}^{2}(\mathcal{P}_{+1}^{0}V_{-1}^{\varphi})(y_j)\prod_{i=1}^{M-2}(\mathcal{P}_{+1}^{-}V_{-1}^{\varphi})(x_i)\Big\rangle_{zero\ modes}^{\Lambda}\\ &\qquad\sim\ \vartheta\begin{bmatrix}1\\1\end{bmatrix}(y_1-z|\tau)\,\mathcal{F}_\Lambda(2y_1-4y_2|\tau)\,,\end{aligned} \quad (14)$$

which does not vanish identically. However, for a possible contribution there must occur a $1/k^2$ pole. Such a pole only obtains if vertex operators collide to produce a massless intermediate state. This can be treated by analytic continuation [12] unless all but one operators collide. In (14) non-zero contributions only occur for $x_i, z, y_2\to 0$ or $x_i, y_1, y_2 \to 0$. Consider the first case. This configuration corresponds to the situation $k\cdot <\psi\psi\varphi\ldots\varphi>_{tree\ level} \frac{1}{k^2}k\cdot <\varphi^*\varphi>_{1-loop}$, and represents wavefunction renormalization† of an external leg. Hence, this term is not in conflict with the non-renormalization theorem. The same is true for the other configuration. This ends the discussion at one loop.

We now would like to consider higher genera. As long as the auxiliary η,ξ system does not appear explicitly, the correlators are readily generalized. In particular, (10) takes the form

$$\Big\langle \prod_k V^{\lambda^k}(z_k)\Big\rangle^{\vec{\Lambda}}_{zero\ modes} = \sum_{\vec{a},\vec{b}\in Z^g/2Z^g} \mathrm{Ch}^{E_6/SO(8)}_{\vec{\Lambda}}\left[\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right]\Big(\sum_k \lambda^k_{\mathcal{U}}\int_P^{z_k}\vec{\omega}, \sum_k \lambda^k_1\int_P^{z_k}\vec{\omega}\,|\,\Omega\Big)$$
$$\times\, Y\left[\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right]\Big(\sum_k \lambda^k_2\int_P^{z_k}\vec{\omega}, \sum_k \lambda^k_\phi\int_P^{z_k}\vec{\omega} - 2\vec{\Delta}|\Omega\Big)\ , \tag{15}$$

where $\vec{\omega}$ are the abelian differentials and P is an arbitrary reference point. For simplicity we shall denote by z_k its image $\int_P^{z_k}\vec{\omega}$ under the Jacobi map in the following.

At genus g, one needs $2g-2$ extra picture changing insertions, say at locations $p_1,\ldots,p_{2g-2}$. However, as for $g\geq 3$ certain ghost correlations cannot yet be simplified sufficiently as to allow for an explicit evaluation [13][5], we restrict ourselves to $g=2$ (apart from this, our computation generalizes immediately to arbitrary genera). Let us again consider first the fermionic two-point function. It is convenient to point-split $\mathcal{P}V^{\psi(-)}_{-\frac{1}{2}}$ so that the picture-changing operators can be treated on the same footing. Ghost and $\mathcal{U}$-charge conservation then leaves the following list of terms:

$$\Big\langle \psi\psi\Big\rangle^{\vec{\Lambda}} \sim \Big\langle V^{\psi(-)}_{-\frac{1}{2}}(z)\,V^{\psi(+)}_{-\frac{1}{2}}(w)\prod_{a=1}^{3}\mathcal{P}(p_a)\Big\rangle^{\vec{\Lambda}}_{zero\ modes} = A+B+C\ , \tag{16}$$

† We assume the D-term [3] that contributes to $<\phi^*\phi>$ to vanish.

$$
\begin{aligned}
A &= \left\langle V^{\psi(-)}_{-\frac{1}{2}}(z)\, V^{\psi(+)}_{-\frac{1}{2}}(w)\; \mathcal{P}^{+}_{+1}(p_1)\, \mathcal{P}^{+}_{+1}(p_2)\, \mathcal{P}^{-}_{+1}(p_3) \right\rangle^{\vec{\Lambda}}_{zero\ modes} \\
B &= \left\langle V^{\psi(-)}_{-\frac{1}{2}}(z)\, V^{\psi(+)}_{-\frac{1}{2}}(w)\; \mathcal{P}^{+}_{+1}(p_1)\, \mathcal{P}^{0}_{+1}(p_2)\, \mathcal{P}^{0}_{+1}(p_3) \right\rangle^{\vec{\Lambda}}_{zero\ modes} \\
C &= \left\langle V^{\psi(-)}_{-\frac{1}{2}}(z)\, V^{\psi(+)}_{-\frac{1}{2}}(w)\; \mathcal{P}^{+}_{+1}(p_1)\, \mathcal{P}^{0}_{0}(p_2)\, \mathcal{P}^{0}_{+2}(p_3) \right\rangle^{\vec{\Lambda}}_{zero\ modes} .
\end{aligned}
$$

The situation is not as simple as for the one-loop case as the picture-changing insertions spoil the simple cancellation mechanism. We will adopt the strategy [4][5] to relate the above terms not to the trivial zero of $\vartheta[\begin{smallmatrix}\vec{\alpha}_1\\ \vec{\alpha}_2\end{smallmatrix}]$ but to non-trivial zeroes implied by the Riemann vanishing theorem.

To this end we have to make a clever choice of locations p_a so that Y reduces to a phase and the sum over spin structures can be performed. The most natural and standard selection is to take as insertion points p_a the two zeroes of some holomorphic one-form. This implies the restriction

$$
\tfrac{1}{2}(p_1 + p_2) \;=\; \Delta_\alpha \tag{17}
$$

where

$$
\Delta_\alpha \;=\; \Delta - \tfrac{1}{2}(\Omega\vec{\alpha}_1 + \vec{\alpha}_2)
$$

corresponds to the divisor class of the spin-$\frac{1}{2}$ bundle S_α, α being an arbitrary, dummy reference spin structure. This so-called unitary gauge has been used successfully in ten-dimensional string calculations [4][5]. Since there does not exist a modular invariant one-form on the surface, it may seem that unitary gauges break modular invariance. However, the appearance of a spin structure α is a technical artefact of our computational scheme and merely accounts for the change from the single, invariant one-loop Riemann identity to the modular *co*variant *set* of higher-loop identities. It only reflects an irrelevant choice of integration path in the definition of the Jacobi images on the l.h.s. of (17) and drops out of all final results [5], thus restoring modular invariance.

In order to treat the three picture-changing insertions in (16) symmetrically, it is convenient to generalize the unitary gauge to

$$\tfrac{1}{2}\Big(\sum_{a=1}^{3} p_a - z\Big) \;=\; \Delta_\alpha \quad , \tag{18}$$

which indicates a *meromorphic* gauge slice. In fact, this can be regarded as putting the picture-changing operators at the three zeroes of some meromorphic one-form with a single pole at z. In the limit $p_a \to z$ for one p_a, we recover the unitary gauge. Using Lorentz invariance we can always arrange terms A and B to produce

$$Y\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\tfrac{1}{2}[z-w]\,, -\tfrac{1}{2}[z+w] + \sum p_a - 2\Delta|\Omega)$$

for the longitudinal and spinor ghost contributions. Employing our gauge (18) and

$$\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(z - \Omega\vec{\alpha}_1 - \vec{\alpha}_2|\Omega) \;=\; e^{i\pi(\vec{\alpha}_1\vec{b} - \vec{\alpha}_2\vec{a}) - i\pi\vec{\alpha}_1(\Omega\vec{\alpha}_1 - 2z)}\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(z|\Omega)$$

this simplifies $Y[{\vec{a} \atop \vec{b}}] \to e^{i\pi(\vec{\alpha}_1\vec{b} - \vec{\alpha}_2\vec{a})}e^{i\pi\vec{\alpha}_1(\Omega\vec{\alpha}_1 + w - z)}$ (therefore "unitary gauge"). Thus, we can apply the Riemann identity (5), with coset branching function arguments $(\nu_{\mathcal{U}}, \nu_1)$

$$\begin{aligned}
&\Big(\tfrac{1}{\sqrt{3}}(-\tfrac{1}{2}[z+w] + p_1 + p_2 - p_3)\,, -\tfrac{1}{2}[z-w]\Big) &&\text{for } A \text{ and}\\
&\Big(\tfrac{1}{\sqrt{3}}(-\tfrac{1}{2}[z+w] + p_1)\,, -\tfrac{1}{2}[z-w] \pm p_2 \mp p_3\Big) &&\text{for } B\,.
\end{aligned}$$

By (6) we see that both results are proportional to

$$\vartheta\begin{bmatrix}\vec{\alpha}_1\\ \vec{\alpha}_2\end{bmatrix}(\tfrac{1}{2}[-z + p_1 \pm p_2 \mp p_3]|\Omega) \;=\; \vartheta\begin{bmatrix}\vec{\alpha}_1\\ \vec{\alpha}_2\end{bmatrix}(\Delta_\alpha - p_a|\Omega) \quad , \tag{19}$$

with $a = 3$ or 2 corresponding to the upper and lower signs, respectively.

To proceed we need not take dangerous limits $p_a \to p_b$; rather, we can employ the Riemann vanishing theorem. It describes the zeroes of theta functions on the Jacobian variety. The statement is that $\vartheta[{\vec{\alpha}_1 \atop \vec{\alpha}_2}](\vec{z}|\Omega)$ vanishes on a submanifold of complex codimension one (the theta divisor) which is generated by the Jacobi images

$$\vartheta\begin{bmatrix}\vec{\alpha}_1\\ \vec{\alpha}_2\end{bmatrix}(\vec{e}|\Omega) = 0 \quad \Longleftrightarrow \quad \vec{e} = \vec{\Delta}_\alpha + \sum_{k=1}^{g-1} \int_P^{p_k} \vec{\omega} \tag{20}$$

of a set of $g-1$ points p_k on the surface. Each choice for $\{p_k\}$ yields a zero of the theta function. Note that *any* point in the Jacobian can be written like the r.h.s. of (20) if we increase the summation index to g. Taking $g = 2$ it is obvious that (19) describes non-trivial zeroes of the two-loop theta function, and terms A and B both vanish for any α.

For the "ghost term", C in (16), one has to work a little harder. Fortunately, the results of [4][5] apply here. We find

$$Y_C = \frac{\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\frac{1}{2}[z-w]|\Omega)\ \{f_1 + f_2\,\vec{\omega}(p_3)\cdot\vec{\partial}\}\,\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(-\frac{1}{2}[z+w] + p_1 + 2p_2 - 2\Delta|\Omega)}{\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(-\frac{1}{2}[z+w] + \sum p_a - 2\Delta|\Omega)^2}\ ,$$

which in our gauge (18) becomes

$$e^{i\pi(\vec{\alpha}_1\vec{b}-\vec{\alpha}_2\vec{a})}\, e^{i\pi\vec{\alpha}_1(\Omega\vec{\alpha}_1+w-z+2p_2-2p_3)}\ \{f_1 + f_3\,\partial_{p_3}\}\frac{\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\frac{1}{2}[z-w] + p_2 - p_3|\Omega)}{\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\frac{1}{2}[z-w]|\Omega)} \tag{21}$$

where f_i are well-defined expressions independent of spin structure. Appending the other degrees of freedom and employing again Lorentz invariance (we can swap

$\lambda_1 \leftrightarrow -\lambda_2$ in (15)), one gets

$$C \sim \sum_{\vec{a},\vec{b}\in Z^g/2Z^g} e^{i\pi(\vec{\alpha}_1\vec{b}-\vec{\alpha}_2\vec{a})}\,\mathrm{Ch}_{\vec{\Lambda}}^{E_6/SO(8)}\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\tfrac{1}{\sqrt{3}}(p_1-\tfrac{1}{2}[z+w]),-\tfrac{1}{2}[z-w]\,|\,\Omega)$$

$$\times\; Y\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\tfrac{1}{2}[z-w]+p_2-p_3,\tfrac{1}{2}[z-w]|\Omega)$$

$$= \sum_{\vec{a},\vec{b}\in Z^g/2Z^g} e^{i\pi(\vec{\alpha}_1\vec{b}-\vec{\alpha}_2\vec{a})}\,\mathrm{Ch}_{\vec{\Lambda}}^{E_6/SO(8)}\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\tfrac{1}{\sqrt{3}}(p_1-\tfrac{1}{2}[z+w]),p_3-p_2-\tfrac{1}{2}[z-w]\,|\,\Omega)$$

where the derivative w.r.t. p_3 has been pulled out. This expression coincides with that for case B. Hence, the two-loop two-fermion amplitude has been shown to vanish.

To complete the argument we have to include arbitrarily many bosonic vertex operators. Ghost charge conservation requires the zero picture which, according to (12), consists of two terms. Like at $g=1$, insertions of the first, neutral term do not effect the above argument; the spin structure dependence of

$$\Big\langle V_{-\frac{1}{2}}^{\psi(-)}(z)\,V_{-\frac{1}{2}}^{\psi(+)}(w)\prod_{a=1}^{3}\mathcal{P}(p_a)\prod_{i=1}^{M}(\mathcal{P}_{+1}^{-}V_{-1}^{\varphi})(y_i)\Big\rangle_{zero\ modes}^{\vec{\Lambda}} \tag{22}$$

is not altered compared to that of $\Big\langle\,\psi\psi\,\Big\rangle^{\vec{\Lambda}}$. Replace now somewhere in (22) the first by the second term of (12), say at location y_1. $\mathcal{U}$-charge conservation then demands either to replace some $\mathcal{P}_{+1}^{+}$ by a $\mathcal{P}_{+1}^{0}$, or some $\mathcal{P}_{+1}^{0}$ by a $\mathcal{P}_{+1}^{-}$, at some p_a. This will change the branching function arguments like

$$(\nu_{\mathcal{U}},\nu_1)\quad\longrightarrow\quad(\nu_{\mathcal{U}}+\tfrac{1}{\sqrt{3}}(y_1-p_a),\nu_1-y_1+p_a)\quad .$$

A look at (5) and (6) reveals that this shift is cancelled in the final theta function argument, eq. (19). However, this procedure does not iterate and, like in the one-loop case, there occur contributions $\sim k^2/k^2$ from configurations where $M+1$ vertices collide. But the one-loop reasoning applies here, too, and also by general arguments [12] we conclude these must describe only wavefunction renormalization.

We hasten to add that our results for $g=2$ are only true up to possible boundary terms in moduli space, due to the integration ambiguity [7] (which is a generic difficulty for any kind of higher loop computation). The appearance of such terms depends on the compatibility of the unitary gauge with BRST invariance, which dictates the behavior of the gauge slice at the boundary of moduli space. In particular, one has to be very careful to correctly describe the limit in which the genus two surface degenerates into two tori. In fact, it is known [14] that in this limit the vacuum amplitude (which is obtained from the above by dropping z, w and p_1) can get a non-zero contribution due to D-term exchange if $Tr\, Q \neq 0$ for some $U(1)$ factor. We expect something similar here, too, that is a non-zero result from the pinching limit whenever $Tr\, Q \neq 0$ (then supersymmetry is spontaneously broken). To settle this question one would have to investigate the amplitudes for degenerating surfaces and look at their factorization properties, which is beyond the scope of the present work.

ACKNOWLEDGEMENTS

One of us (O.L.) is grateful to the Physics Department of Caltech for hospitality extended to him at the beginning of this project. We thank Osamu Yasuda and Michael Dine for discussions and John Schwarz for reading the manuscript.

REFERENCES

[1]. E. Martinec, Phys. Lett. 171B (1986) 189; M. Dine and N. Seiberg, Phys. Rev. Lett. 57 (1986) 2625.

[2]. E. Verlinde and H. Verlinde, Phys. Lett. 192B (1987) 95.

[3]. M. Dine, I. Ichinose and N. Seiberg, Nucl. Phys. B293 (1987) 253; J. Atick, L. Dixon and A. Sen, Nucl. Phys. B292 (1987) 109.

[4]. A. Morozov and A. Perelomov, preprint ITEP-88-80 (1988); A. Morozov, preprint ITEP-88-125 (1988); O. Lechtenfeld, preprint CCNY-HEP-88/13 (1988), Nucl.Phys.B in print; A. Parkes, Phys. Lett. 217B (1988) 458.

[5]. O. Lechtenfeld and A. Parkes, *On Covariant Multi-Loop Superstring Amplitudes*, preprint CCNY-HEP-89/04 UCD-89-8 (1989).

[6]. see *e.g.*, E. Gava, R. Iengo and G. Sotkov, Phys. Lett. 207*B* (1988) 283; R. Iengo, and C. Zhu, Phys. Lett. 212*B* (1988) 309,313; M. Bershadskii, Nucl. Phys. *B*310 (1988) 79; O. Yasuda, *Nonrenormalization theorem for the Green-Schwarz counterterm and the low-energy effective action*, preprint CALT-68-1522; J. Ellis and L. Mizrachi, *Multiloop no renormalization theorem for a class of four-dimensional superstrings*, preprint CERN-TH.5264/88.

[7]. see *e.g.* J. Atick, J. Rabin and A. Sen, Nucl. Phys. *B*299 (1988) 279; J. Atick, G. Moore and A. Sen, Nucl. Phys. *B*307 (1988) 221; Nucl. Phys. *B*308 (1988) 1 and references in these papers.

[8]. W. Lerche, D. Lüst and A.N. Schellekens, Nucl. Phys. *B*287 (1987) 477; J. Lauer, D. Lüst and S. Theisen, Nucl. Phys. *B*309 (1988) 771; for a review, see W. Lerche, A.N. Schellekens and N.P. Warner, Physics Reports, to appear.

[9]. D. Friedan, A. Kent, S. Shenker and E. Witten, unpublished; W. Boucher, D. Friedan and A. Kent, Phys. Lett. 172*B* (1986) 316; T. Banks, L.J. Dixon, D. Friedan and E. Martinec, Nucl. Phys. *B*299 (1988) 613.

[10]. W. Lerche, A.N. Schellekens and N.P. Warner, Phys. Lett. 214*B* (1988) 41.

[11]. W. Lerche, Nucl. Phys. *B*308 (1988) 102; D. Lüst and S. Theisen, preprint MPI-PAE/PTh-41/88, *Superstring partition functions and the characters of exceptional groups*, preprint MPI-PAE/PTh-41/88.

[12]. M. Green and N. Seiberg,Nucl. Phys. *B*299 (1988) 559.

[13]. O. Lechtenfeld, Nucl. Phys. *B*309 (1988) 361.

[14]. J. Atick and A. Sen, Nucl. Phys. *B*296 (1988) 157.