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# Chiral Four-dimensional Heterotic Strings from Self-dual Lattices

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**Abstract**

It is shown how our previous work on lattice constructions of ten-dimensional heterotic strings can be applied to four dimensions. The construction is based on an extension of Narain's lattices by including the bosonized world-sheet fermions and ghosts, and uses conformal field theory as its starting point. A natural embedding of all these theories in the bosonic string is automatically provided. Large numbers of chiral string theories with and without $N=1$ supersymmetry can be constructed. Many features of their spectra have a simple interpretation in terms of properties of even self-dual lattices. In particular we find an intriguing relation between extended supersymmetry and exceptional groups.

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## 1. Introduction

In the early days of string theory it was considered a major embarrassment when it turned out that string theories could only be formulated consistently in 26 or 10 dimensions. The revival of interest in the subject was for a small, but not unimportant part due to a change in attitude towards extra dimensions, namely the acceptance of the idea that they can be compactified. This idea dates back to the first half of this century, but received serious attention only during the last ten years, after the end of the first string era. When strings were reconsidered one initially attempted to compactify their field theory limits, with the help of the technology developed during the past decade. More recently the attention has slowly shifted towards more "stringy" compactifications.

Actually the concept of a critical dimension for string theory is a misconception. This is clear once one realizes what this concept is based upon. All string theories which might be relevant for four-dimensional physics (and this certainly includes the ones considered in this paper) are based on just two conformally invariant two-dimensional field theories. In conventional terms, one consists of 26 bosons whose conformal anomaly cancels that of the reparametrization ghost; the other is a supersymmetric model with ten bosons and ten fermions, cancelling the anomaly of a reparametrization ghost and a superconformal ghost. One may use either one as the left-moving or right-moving sector of a closed string theory. The space-time dimension is not simply the number of bosons, but is determined by how many of the bosons play the role of space-time coordinates. Of course this is not just a matter of choice, and one has to respect the consistency conditions of the interacting theory.

The compactified bosonic string may be regarded as a string theory formulated in less than 26 dimensions, or as a string theory in 26 dimensions, of which some have been compactified. The latter point of view is already less appropriate for the bosonic sector of the heterotic string [1] or a Narain-type [2] "compactification" of the bosonic string. The latter differs from ordinary compactification of n dimensions because left-moving and right-moving coordinates are treated as independent. It is difficult to see such theories as 26-dimensional strings in compactified space-time, since one cannot compactify n space-time dimensions on a (n,n) Lorentzian lattice without the notion of strings.

The theories we will consider are a further step away from compactification. Although there may be some way to regard them as compactifications of ten-dimensional theories, they are far more easily described as four-dimensional string theories. The starting point of the construction is a formulation of two-dimensional field theories in which all fields are bosonic. Bosonization of the Neveu-Schwarz-Ramond fermions has been studied recently for two entirely different purposes. The authors of [3] [4] were interested in the resulting simplifications of superconformal field theory. The authors of [5] [6] used bosonization as a first step towards a construction of fermionic strings from the bosonic string. These two ideas play a crucial role in our construction, and all theories we will obtain can be regarded from both points of view.

Cohn et. al. [3] observed that the bosonized fermions plus ghosts of the superstring can be described by an odd self-dual Lorentzian lattice $\Gamma_{5,1}$ with metric $(+++++-)$. This was extended in [7] to heterotic strings, where also the relation of the self-duality of the lattice and modular invariance was understood. It was shown that the "old" $E_8 \times E_8$ and SO(32) strings, as well as some of the new ones, such as the $O(16) \times O(16)$ theory can be described by odd self-dual Lorentzian lattices $\Gamma_{16;5,1}$ with metric $(-)^{16}(+)^5(-)$ (the semicolon separates left and right-movers). Such Lorentzian lattices are unique up to Lorentz transformations, but not every Lorentz transformation gives a sensible theory. The lattices that do give sensible 10-dimensional theories were classified in [8] by mapping the odd self-dual lattices to even self-dual Lorentzian lattices $\Gamma_{16;8}$, which in turn could be mapped to the Niemeier lattices $\Gamma_{24}$. We will give a brief summary of these ideas in section 2.

It was also suggested in [7] and [8] how this construction could be applied in $10-d$ dimensions by extending the lattices to incorporate d "compactified" bosons. In that way one is led to consider odd Lorentzian self-dual lattices $\Gamma_{16+d;5+d,1}$ which can be mapped onto even Lorentzian self-dual ones $\Gamma_{16+d;8+d}$. An interesting fact is that a priori there does not seem to be a reason why one could not get chiral theories. In Narain's construction the world-sheet fermions are compactified trivially on a torus, and this prevents the appearance of chiral fermions. In the theories we consider the world-sheet fermions are entangled non-trivially with the bosons, and the argument is no longer valid. The purpose of this paper is to explore these lower-dimensional theories.

Similar ideas have been discussed by Kawai et. al. [9]. Their starting point is exactly opposite to ours, and is based on a fermionization of all fields except the space-time coordinates. These authors have already demonstrated the existence of four-dimensional chiral theories in [9]. In a recent paper [10] they have bosonized their construction and have also obtained a lattice formulation. Their lattices are different from ours, and we believe that our lattice formulation (and in particular the even self-dual one) has considerable advantages. The difference is however not a fundamental one: In section 2.3 we will show that the two approaches lead to the same theories.

The only new ingredient (in comparison with [7] and [8] ) needed in the construction of theories below ten dimensions is a mechanism to maintain world-sheet supersymmetry of the right-moving sector. In a completely bosonic formulation one needs a bosonic realization of this supersymmetry, and one cannot expect this to emerge automatically. Indeed, in [9] it was found that an additional constraint is needed, and an elegant formulation of a sufficient condition (the "triplet constraint") was given. In section 3 we will show how one can arrive at the same condition by requiring that only massless chiral states appear in the spectrum, and no massive ones. Thus all massive excitations of these chiral ground states must cancel or pair off, no small feat in a theory with an infinite number of excitations.

A consequence of this is that the ***chiral*** partition function is a holomorphic modular function, which is subject to the same classification theorems which apply in ten dimensions. In general, partitions functions of strings below ten dimensions are not holomorphic, and such functions cannot be classified.

In section 4 we discuss the spectrum of the theories we can obtain. Here the use of even Lorentzian lattices turns out to be extremely helpful, since many aspects of the spectrum can be directly related to length two vectors (roots) of the lattice. For example, the presence of $N=1,2$ or 4 supersymmetry in four dimensions can be related to the presence of $E_6$, $E_7$ or $E_8$ sublattices, in which the space-time lattice is embedded.

In section 5 we show how large numbers of chiral four-dimensional (and other) theories can be constructed from the Niemeier lattices. It is also made clear that, unlike the ten-dimensional case we obtain only a very small fraction of a finite, but presumably extremely large set of theories. Some examples are presented in section 6. In section 7 we formulate some conclusions.

## 2. Bosonic constructions

### *2.1 The covariant lattice approach*

To discuss string spectra, the covariant lattice approach [3] [7] [11] is particularly useful. By "covariant" we mean that the charges of the bosonized fermions of the NSR-Model are combined with those of the bosonized superconformal ghost system into a (self-dual) lattice. In order to review this approach, we focus on the case of the ten dimensional spinning string and superstring.

Physical states are generated by covariant vertex operators $V(\bar{z})$ at zero momentum, acting on the SL(2,C) non-invariant vacuum $|0\rangle = c(0)|0\rangle_{SL(2,C)}$, c representing the reparametrization ghost. These operators have the form ($c_w$ denotes a cocycle-generating Klein factor):

$$V_w(\bar{z}) = (\text{derivatives}) \cdot e^{\lambda\cdot H} e^{q\phi}(\bar{z}) c_w\,, \qquad w = (\lambda,q). \tag{2.1}$$

Here, $\phi$ is part of the superconformal ghost system $\beta$, $\gamma$ with $\beta\gamma = \partial\phi$, and H denotes a five-vector of bosonized fermion coordinates, $i\psi^{2j}\psi^{2j-1} = \partial H^j$. In (2.1), $\lambda$ denotes a vector of the O(10) weight lattice $D_5$. Since the vacuum $|0\rangle$ has level $-1$ while $\exp\lambda\cdot H$ and $\exp q\phi$ have conformal weights $1/2\lambda^2$ and $-1/2q(q+2)$[1], respectively, we have for the mass of a state generated by $V_w$

$$1/8m^2 = 1/2\lambda^2 - 1/2q^2 - q + N - 1, \tag{2.2}$$

where N denotes the total oscillator number (number of derivatives in (2.1)). In the following discussion, we ignore the effect of oscillators.

Let us now discuss the spectrum of the covariant spinning string [11]. The purpose of the factor $\exp q\phi$ is to provide the correct 'vacuum' state for $\exp\lambda\cdot H$ to act on; it is well-known [4] that the vacuum states of the spinning string are characterized by non-zero Bose ghost sea charges $|q\rangle = \exp q\phi|0\rangle$. For instance, the tachyonic ground state of the spinning string is given by $|-1\rangle = \exp(-\phi)|0\rangle$, while the massless vector state is given by $|\mu,-1\rangle = \exp(\mu\cdot H)\exp(-\phi)|0\rangle$, where $\mu$ is an O(10) vector weight. More generally, all states in the Neveu-Schwarz-sector have ghost charge $q=-1$, and are built on the vacuum $|-1\rangle$ with mass

[1]The linear term in q arises due to the holomorphic anomaly in the ghost number current [4].

$-1/2q(q+2)-1=-1/2$. On the other hand, in the Ramond-sector the states are build on $\exp(-\phi/2)|0>$ with level $3/8-1 = -5/8$. This is precisely the value needed for the physical ground state $|\alpha,-1/2> = \exp(\alpha\cdot H)\exp(-\phi/2)|0>$ of the Ramond-sector to be massless, as the O(10) spinor weight obeys $\alpha^2/2 = 5/8$.

In general, all states in the Neveu-Schwarz- or Ramond-sectors obey $|\lambda,q> = |\text{tensor},-1>$ or $|\text{spinor},-1/2>$, respectively [4]. However, $q=-1$ or $q=-1/2$ are not the only sectors in the theory. Since in operator products ($<w,w'> = \lambda\cdot\lambda'-qq'$)[1]

$$V_v(z)\cdot V_w(0) \sim z^{<v,w>} V_{v+w}\epsilon(v,w) \tag{2.3}$$

the vectors w and v add, intermediate states factorize in general on different than the canonical ghost sea levels q. Thus the Hilbert space must be extended to include all ghost charge sectors. However, as discussed in [4], sectors differing by integer units of q are physically equivalent and related by the 'picture changing' operation. In other words, any physical state has a representative in each sector q mod 1, so that Neveu-Schwarz- and Ramond-states are characterized by $q\in Z$ and $q\in Z+1/2$, respectively.

Allowing thus for arbitrary ghost charges q, we find that the vector $w=(\lambda,q)$ belongs to a lattice extending $D_5$, on which all vector additions corresponding to operator products are allowed. From (2.3) it is clear that the natural inner product on this lattice is characterized by a Lorentzian metric $(+++++-)$; therefore, let us denote it by $\overline{\Gamma}_{5,1}$.

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[1]The minus sign arises due to the opposite statistics of the ghost; $\epsilon(v,w)$ below is a cocycle phase.

Since all components of $w \epsilon \bar{\Gamma}_{5,1}$ are either integer or half-integer, we can define conjugacy classes (0),(v),(s) and (c) for $\bar{\Gamma}_{5,1}$ [8] [11], analogous to the O(2n) weight lattices $D_n$. In particular, (0) contains the massless vector $|\mu,-1>$ of the Neveu-Schwarz-sector, (v) contains the scalar tachyon $|-1>$ as well as state $|\mu,0> = \psi^\mu|0>$ while (s) contains the massless spinor $|\alpha,-1/2>$ of the Ramond-sector; (c) contains spinors of opposite chirality. In general (0) with odd ghost charge and (v) with even ghost charge contain states of the Neveu-Schwarz sector with the usual GSO projection, while (0) with even and (v) with odd ghost charge contain the opposite GSO projection. The sectors (s) and (c) yield Ramond states with opposite chirality if their ghost charges are the same.

In operator products (2.3), scalar products $<\ ,\ >$ between different conjugacy classes occur. For $\bar{\Gamma}_{5,1}$ one has the multiplication rules [11] ((i) = (0),(v),(s) or (c)):

$$\begin{aligned} &<(i),(i)> \ \epsilon\ Z \\ &<(0),(i)> \ \epsilon\ Z \\ &<(v),(s)> \ \epsilon\ Z+1/2 \\ &<(v),(c)> \ \epsilon\ Z+1/2 \\ &<(s),(c)> \ \epsilon\ Z+1/2 \end{aligned} \tag{2.4}$$

This means the spinning string theory based on $\bar{\Gamma}_{5,1}$ is non-local [4]. The GSO-projection, which generates the (local) superstring, reduces $\bar{\Gamma}_{5,1}$ to an integral sublattice $\Gamma_{5,1}$, which contains only the classes (0) and (s) of $\bar{\Gamma}_{5,1}$. In particular, all even rank tensor fields belonging to (v) like the tachyon are projected out.

The covariant superstring lattice $\Gamma_{5,1}$ is not only integral, but odd self-dual, as was first noted in [3]. Actually one-loop modular invariance of the partition function requires invariance (up to phases and weight factors) of the lattice function $(e=(000001))$[1]

$$\theta_{5,1}(e|\bar{\tau}) = \sum_{w\in\Gamma_{5,1}} e^{-i\pi\bar{\tau}<w+e,w+e>} e^{2\pi i<w,e>} \tag{2.5}$$

under $\bar{\tau}\rightarrow -1/\bar{\tau}$. This implies the lattice to be self-dual [7]. Invariance under $\bar{\tau}\rightarrow\bar{\tau}+1$ requires the lattice to be odd,

$$1/2<w,w> \;=\; <w,e> \bmod 1 \;=\; q \bmod 1, \tag{2.6}$$

so that Neveu-Schwarz-(Ramond-)states belonging to (0) ((s)) are associated with even (odd) points on the lattice. This in turn ensures also proper (anti-)commutation rules of the vertex operators.

In the canonical ghost sectors, all states have $q=-1$ or $q=-1/2$. Physical light cone states are characterized by fixed two last entries, $x_0=(0,-1)$ in the Neveu-Schwarz- and $x_0=(-1/2,-1/2)$ in the Ramond-sector. These values provide the correct normal ordering constants for both sectors: if $u\in D_4$ denotes the first four entries, i.e., the light cone part of $w=(u,x_0)$, then the mass of a light cone state can be written as

$$1/2m^2 \;=\; 1/2u^2 \;-\; 1/2. \tag{2.7}$$

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[1]Whenever both string sectors are discussed simultaneously we will use $\tau$ to refer to leftmovers and $\bar{\tau}$ for rightmovers.

Regarding the fixed two last entries $x_0$ of w as belonging to a sublattice $\Gamma_{1,1}$, we can identify the physical states of the superstring by decomposing $\Gamma_{5,1}$ to $D_4 \times \Gamma_{1,1}$:

$$(0) \rightarrow ((v),(v))$$

$$(s) \rightarrow ((s),(s)) \tag{2.8}$$

and fixing the $\Gamma_{1,1}$ components to the entries $x_0$ defined above.

From this the advantage of our approach compared to the conventional light cone treatment considering only $D_4$ becomes evident: the light cone states $u \in D_4$ themselves do not form a lattice, since only the classes (v) and (s) of $D_4$ occur but not the neutral class (0). Only when u is non-trivially combined with vectors $x_0 \in \Gamma_{1,1}$ (or equivalently $D_4$, as will be explained below), does one obtain a sensible lattice. This is quite convenient since, for example, investigation of modular invariance requires only a discussion of the general structure of the lattice, and not the analysis of complicated expressions like sums over spin structures.

The above concept generalizes easily to heterotic theories in any dimension $D \leq 10$ [7]. Heterotic string states are generated by vertex operators

$$V_w(\bar{z},z) = e^{w_L \cdot F}(z) \cdot e^{\lambda_R \cdot H} e^{q\phi}(\bar{z}), \quad w = (w_L; w_R = (\lambda_R, q)) \tag{2.9}$$

The vectors w generate a lattice $\Gamma_{26-D;15-D,1}$ with natural metric $(-)^{(26-D)}(+)^{(15-D)}(-)$. The condition for modular invariance becomes that this lattice has again to be odd self-dual [7], obeying (2.6). In general all four conjugacy classes of the right part of the lattice may now appear, but the physical state selection rule (i.e that the last two entries should be $(0,-1)$ or $(-1/2,-1/2)$) remains the same.

As is well known, all self-dual Lorentzian lattices of given rank are isomorphic to each other in the sense that they are related by Lorentz-type rotations. However, for a lattice to represent a physically sensible string theory, certain physical constraints like correct statistical behavior and Lorentz transformation properties, as well as superconformal invariance have to be imposed.

For the ten-dimensional case, all lattices $\Gamma_{16;5,1}$ satisfying the additional physical constraints have been classified in [8]. They correspond precisely to the theories found in [12] [13], constructed by different methods. To these belong of course the well-known supersymmetric SO(32) and $E_8 \times E_8$ theories, which are based on the direct product lattices $\Gamma_{16;5,1} = (\Gamma_{16})_L \times (\Gamma_{5,1})_R$.

In the following, we will consider theories in less than ten dimensions: $D = 10 - 2n$. To these belong the Narain-type [2] compactifications of the supersymmetric theories associated with odd self-dual lattices

$$\Gamma_{16+2n;5+2n,1} = (\Gamma_{16} \times \Gamma_{2n;2n}) \times (\Gamma_n \times \Gamma_{5-n,1}). \tag{2.10}$$

Here, $\Gamma_{2n;2n}$ corresponds to the compactified left- and right-moving space-time bosons, $\Gamma_n = D_n$ to the compactified fermions and $\Gamma_{5-n,1} = D_{4-n} \times \Gamma_{1,1}$ to the covariant space-time fermions plus ghosts. Lorentz-type rotations acting only inside of the first bracket in (2.10) are not restricted at all, and produce a continuous infinity of compactified theories [2]. However, these are all vectorlike, since the second bracket corresponds to a trivial torus compactification of the superstring.

On the other hand, Lorentz-type rotations acting on the whole lattice do not correspond, in general, to torus compactifications[1] of ten-dimensional theories. Since

[1]Some might be related to orbifold-type [14] compactifications.

in general for these theories the non-trivial relation between conjugacy classes of $\Gamma_n$ and $\Gamma_{5-n,1}$ in (2.10) is destroyed, there arise strong constraints due to superconformal invariance. This will be discussed in section 3, where it is shown that if one demands Lorentz invariance for chiral string theories only a finite, discrete set of lattices remains physical. In the next section we show how the lattice $\Gamma_{1,1}$ appended to the light cone "lattice" $D_{4-n}$ can naturally be re-interpreted in the context of embedding NSR-strings into the bosonic string.

### *2.2 The bosonic string approach*

The idea that fermionic strings can be obtained from the bosonic string was first suggested by [15]. A more concrete realization was proposed in [5] and was worked out in detail, and presented in the most suitable form for our purpose in [6]. We use only the weak version of this idea, namely the fact that the light cone formulation of fermionic strings can be embedded in the bosonic string, provided that one makes certain truncations. This can be demonstrated explicitly. The more interesting strong version, namely that fermionic strings are actually some sort of "ground state" of the bosonic string is of course much harder to prove (see [16] for a discussion of some of the issues involved). The following will only be a brief summary of the construction, with less emphasis on 10 dimensions than was previously the case.

One starts with the 26-dimensional bosonic string compactified on an even Lorentzian self-dual lattice $\Gamma_{16+2n,16+2n}$ to $10-2n$ dimensions. This lattice has a Frenkel-Kac symmetry $G_L \times G_R$. To obtain heterotic strings one considers an

$SO(8-2n)$ regular subgroup of $G_R$, and adds its generators to those of the transverse Lorentz group. In this way $SO(8-2n)$ gauge symmetries get a new interpretation as internal symmetries, and in particular if $SO(8-2n)$ spinor representations are present one will get states which transform as space-time fermions. To describe the newly generated (bosonized) world-sheet fermions covariantly one needs to introduce new ghosts, as well as two additional fermionic components to extend $SO(8-2n)$ to $SO(10-2n)$. Altogether these new fields give a contribution of 12 to the conformal anomaly (in units where a boson contributes 1), and conformal invariance can only be maintained if 12 world-sheet bosons are somehow converted to these new fields. This means that their momenta and excitations cannot be present in the light cone spectrum. The fermionic string light cone spectrum can thus only be obtained from the compactified bosonic string spectrum by truncating 12 bosonic string degrees of freedom.

This truncation should respect modular invariance. Before truncation, the lattice partition function of the theory can be written as follows

$$P_\Gamma(\tau,\bar{\tau}) = \Sigma\, \theta_i^{4-n}(0|\tau)\, P_i(\tau,\bar{\tau}) \qquad (2.11)$$

Here we have made use of the fact that the lattice should at least contain a factor $D_{4-n}$, which is represented by the theta function. The sum is over all conjugacy classes of $D_{4-n}$ (or more precisely, it is over the sums and differences of conjugacy classes; for example $\theta_3$ represents the sum of the root lattice and the vector weight lattice). The function $P_i$ represents all remaining components of the lattice vectors. The full partition function is given by (2.11) multiplied by $\eta(\tau)^{-24}\eta(\bar{\tau})^{-24}$. To perform the truncation we have to remove 12 factors $\eta(\bar{\tau})^{-1}$ and a 12-dimensional part of the right lattice. In [16] it was shown that one can do this in a modular invariant way by defining

$$\hat{P}_i(\tau,\bar{\tau}) = P_i(\tau,\bar{\tau})/S_i(\bar{\tau}) \tag{2.12}$$

and replacing $P_i$ by $\hat{P}_i$ in (2.11). The functions $S_i$ must transform into each other in the same way as the $\theta$-functions but without phase or weight factors. They must thus transform as follows

$$\begin{array}{lll} S_3(\tau+1) = S_4(\tau); & S_4(\tau+1) = S_3(\tau); & S_2(\tau+1) = S_2(\tau) \\ S_4(-1/\tau) = S_2(\tau); & S_2(-1/\tau) = S_4(\tau); & S_3(-1/\tau) = S_3(\tau) \end{array} \tag{2.13}$$

If we assume that $S_i$ is constructed out of factors $\theta/\eta$ (i.e that the right lattice has a decomposition in terms of $D_n$-factors), then a nontrivial solution requires at least 12 such factors, which is interestingly precisely the number required by conformal invariance. One of the solutions is

$$S_i(\tau) = \eta(\tau)^{-12}\theta_i^{\,4}(0|\tau)\Sigma\theta_j^{\,8}(0|\tau). \tag{2.14}$$

Dividing by it must have the effect of removing the contribution of 12 bosons. The numerator of (2.14) tells us then that the original lattice must have had a factor $D_{8-n}\times E_8$, where $D_{4-n}$ is embedded in the first factor. In other words, the even Lorentzian self-dual lattice on which the bosonic string is compactified has to have the direct product form

$$\Gamma_{16+2n;16+2n} = \Gamma_{16+2n;8+2n}\times E_8 \tag{2.15}$$

There is a second independent solution corresponding to an extension of $D_{4-n}$ to $D_{16-n}$, but then division by the $S_i$-factors cannot be interpreted as a truncation of the bosonic string spectrum [16].

Division by the functions (2.14) (with argument $\bar{\tau}$) is easily seen to correspond to the following truncation of the spectrum:

(i) Removal of all states with non-vanishing $E_8$ lattice momenta and excitations.

(ii) Removal of all states with $D_4$ oscillator excitations, and all $D_4$ lattice momenta except one fixed vector weight and one fixed spinor weight, both of length 1: for example, $x_e = (1,0,0,0)$ and $x_e = (1/2,1/2,1/2,1/2)$.

(iii) Adding a $-$ sign to the partition function for those states with a $D_4$ spinor weight, to compensate for the one in (2.14).

The removal of a $D_4$ factor from $D_{8-n}$ reduces it to the transverse Lorentz algebra. The sign in (iii) produces the correct spin-statistics relation for the fermions.

These rules where first written like this as a rather ad hoc way of obtaining the light cone spectrum of the superstring from $E_8 \times E_8$ [6]. As we have shown, they follow also from the requirement of modular invariance in the truncation, and have clearly a more general validity.

A third way of getting them is to use the conjugacy class map introduced in [8] on the "covariant" lattices $\Gamma_{5-n,1} = D_{4-n} \times \Gamma_{1,1}$ discussed in section 2.1. One simply maps these odd lattices to even lattices $\Gamma_{8-n} = D_{4-n} \times D_4$ by mapping[1]

$\Gamma_{1,1}$ to $D_4$, i.e., the four conjugacy classes of $\Gamma_{1,1}$ on the same classes of $D_4$. It is easy to see that all dot-products change in precisely the right way to turn odd self-dual lattices $\Gamma_{16+2n;5+2n,1}$ into even self-dual lattices $\Gamma_{16+2n;8+2n}$. These are precisely the lattices indicated in (2.15). The physical state selection rule for $\Gamma_{1,1}$ described in sect. 2.1 translates exactly to the truncation rules for $D_4$ given above, by $x_o \to x_e$.

We find thus that from the bosonic string point of view all theories are on the same footing, and that the truncation is the same in all cases.

## *2.3 The light cone approach*

In this section, we explain the relation of our approach to the one of Kawai et al.[2] [10]. For simplicity, we focus on the ten-dimensional superstring case; extension to the heterotic case and to lower dimensions is trivial.

As explained in section 2.1, the bosonized light cone states correspond to the conjugacy classes (v) and (s) of $D_4$; therefore, the vectors do not form a lattice. An odd (even) self-dual lattice is achieved by appending a covariantizing factor $\Gamma_{1,1}$ ($D_4$) to $D_4$. The observation made in [10] is that one can also obtain a lattice by shifting the charge vectors $u \in D_4$ of the light cone states by a constant vector $S=(1000)$. This maps the conjugacy classes (v) and (s) to (0) and (c), respectively, which form an odd self-dual lattice. Modular invariance furthermore requires that

[1] We use this mapping here only as a formal operation which establishes the relation between the two approaches, although it clearly suggests a more fundamental connection, which remains to be understood properly.

[2] Meant here is the bosonic construction, not the fermionic one of [9].

$(w = u + S)$

$$1/2w^2 = S \cdot w \bmod 1, \tag{2.16}$$

so that tensors (spinors) correspond to even (odd) points on the lattice. In fact, (2.16) is equivalent to our equation (2.6). S plays the same role as the vector e used in eq. (2.5), which also represents a sum over a shifted lattice. In the covariant approach, shift by e is equivalent to shift by S, because the vectors differ only by a "root" vector. The only difference is thus the lattice factor $\Gamma_{1,1}$; this is irrelevant, however, since, as was shown in [7], all additional states due to $\Gamma_{1,1}$ factor out in the partition function. Hence the two approaches are equivalent, although the odd self-dual lattices which are used are different.

However, we think that our approach has several advantages. First, it is physically motivated by conformal field theory: the shift vector e originates in the ghost number current anomaly. Second, we are able to relate our odd self-dual lattices to even ones by mapping $\Gamma_{1,1} \rightarrow D_4$. This allows for an elegant and direct method of constructing such lattices, as will be detailed further below.

## 3. Chirality and Lorentz Invariance

In this section, we investigate properties of lattices associated with chiral theories. The most convenient approach is the one of sect. 2.2: Instead of odd self-dual ones, we consider equivalent even self-dual lattices $\Gamma_{16+2n;8+2n}$. In general, there exists a continuous variety of such lattices. We show below that if one demands chiral theories, this variety is reduced to a discrete, finite one.

The presence of chiral fermions in the spectrum is easily seen to impose a rather strong condition on a string theory. Because of Lorentz invariance, chiral states can only be massless. However, in string theory there will be an infinite number of massive excitations of such a chiral state. Somehow these will all have to be non-chiral.

It is well-known how this works in ten dimensions. The chiral space-time states are due to world-sheet fermions with periodic boundary conditions (PP) along both non-contractable loops on the world-sheet torus. Their one-loop partition function is thus given by

$$P_1(\tau) = \eta^{-4}(\tau)\theta_1{}^4(0|\tau) = 0. \tag{3.1}$$

This vanishes, precisely because this sector of the theory has a chiral ground state. The two chiralities contribute with opposite sign to the partition function, and therefore cancel each other.

What we are really interested in is not (3.1), but the factor of (3.1) that describes the excitations of the chiral ground state. This is easily obtained by factoring out the zero mode that causes (3.1) to vanish. This can be achieved by twisting the boundary conditions, or coupling the theory to a weak gravitational background field, which does see the difference between the two chiralities (see [17] for a more detailed discussion). After factorizing the zero mode we get

$$P_1{}'(\tau) = \eta^{-4}(\tau)\theta_1{}'^4(0|\tau) \neq 0, \tag{3.2}$$

where the prime on the $\theta$-function indicates differentiation with respect to the first argument.

In addition to fermionic oscillator excitations, the chiral vacuum is excited by bosonic oscillators $\alpha_{-n}$. They contribute an additional factor $\eta(\tau)^{-8}$ to (3.2). By using the identity

$$\theta_1'(0|\tau) = 2\eta^3(\tau) \tag{3.3}$$

we find then that the result is a constant. This means that with the exception of the ground state there is an equal number of left-handed and right-handed states at every level, so that they can combine into massive states. This cancellation is a consequence of world-sheet supersymmetry which ensures an exact matching between the bosonic excitations and those of periodic fermions. It is important that not only the multiplicities of chiral pairs are the same, but that they also belong to the same representations of any relevant symmetry, i.e. any symmetry which is gauged. In ten dimensions this gauge symmetry is gravity, and the requirement is that not only $P_1'(\tau)\eta^{-8}(\tau)$ is a constant, but that also its character-valued generalization is a constant. The character-valued partition function can be thought of as the integrand of the one-loop string diagram in a background field [18]. It can be shown that it is indeed a constant [17].

Below ten dimensions a more elaborate mechanism will be needed. The cancellation between world-sheet fermions and bosons with space-time indices still occurs in exactly the same way. But in addition we have now the oscillator excitations of the remaining bosons, and the contribution from their soliton sector. If we consider heterotic strings in $D=10-2n$ dimensions, there are $(2+1)n$ additional bosons (cf. eq. (2.10)). The fermionic part of the related even self-dual lattice $\Gamma_{16+2n;8+2n} = \Gamma_{16+2n;3n} \times D_{8-n}$ has the following structure

$$(\Delta^s{}_L;\Delta^s{}_R,(s)) \ + \ (\Delta^c{}_L;\Delta^c{}_R,(c)), \tag{3.4}$$

where $\Delta_L$ is a set of $16+2n$ dimensional vectors, $\Delta_R$ a set of $3n$ dimensional vectors and (s),(c) denote conjugacy classes of the space-time lattice $D_{8-n}$. The symmetries associated with the left lattice are gauged, so that a chiral pair must have the same lattice vector $\Delta_L$ in order to become massive. Consider one fixed such vector, $\vec{w}_L$. We are interested in all pairs of states of the form

$$(\vec{w}_L;\Delta^s{}_R,(s)) \ + \ (\vec{w}_L;\Delta^c{}_R,(c)), \tag{3.5}$$

After cancelling the space-time factor using (3.3), we are left with the following factor from the bosonic excitations and the right lattice $\Gamma_{3n}$:

$$\eta(\bar{\tau})^{-9}[\Gamma_R{}^s(\bar{q}) \ - \ \Gamma_R{}^c(\bar{q})] \tag{3.6}$$

where $\Gamma_R{}^s$ and $\Gamma_R{}^c$ are the partition functions of $\Delta^s{}_R$ and $\Delta^c{}_R$, respectively. According to our previous arguments, (3.6) should be a constant, which should be non-zero, since otherwise we would not obtain a massless chiral fermion. For arbitrary lattices (3.6) may be difficult to solve in general, but it becomes easy if we assume that the right lattice consists of products of $D_m$ lattices[1]. Then we know that the partition function must be a sum of products of 3n $\theta$-functions, and we are looking for a combination of 3n factors $\theta/\eta$ which is equal to one. In particular this

[1]The conjugacy classes of these $D_m$'s may be correlated with each other and with the left lattice.

combination should be modular invariant with weight zero. In [17] all such modular invariant combinations have been constructed, and there turns out to be just one way to get a modular invariant function out of 3 factors $\theta/\eta$. Fortunately this combination is indeed equal to a constant:

$$\eta(\tau)^{-3}\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau) = 2. \tag{3.7}$$

It is this fact that allows us to construct chiral theories. To give a lattice interpretation to (3.7) we write it in terms of $D_1$ conjugacy classes[1], of which the $\theta$-functions are the partition functions. Define

$$\begin{aligned} \theta_0(0|\tau) &= 1/2[\theta_3(0|\tau) + \theta_4(0|\tau)] \\ \theta_V(0|\tau) &= 1/2[\theta_3(0|\tau) - \theta_4(0|\tau)] \\ \theta_S(0|\tau) &= \theta_C(0|\tau) = 1/2\theta_2(0|\tau) \end{aligned} \tag{3.8}$$

Then the identity given above may also be written as

$$\eta^{-3}(\tau)\theta_S(0|\tau)[\theta_0{}^2(0|\tau) - \theta_V{}^2(0|\tau)] = 1. \tag{3.9}$$

Here $\theta_0$ is the partition function of the root lattice of $D_1$, $\theta_V$ the partition function of the vector weight lattice, etc. In $D=10-2n$ dimensions we need n factors of the form (3.9). We can now give this result a lattice interpretation by associating the positive terms with $\Delta^S{}_R$ and the negative ones with $\Delta^C{}_R$.

---

The root lattice (0) of $D_1$ is the set of even integers; the (v), (s) and (c) conjugacy classes are the even integers shifted by 1, 1/2 and $-1/2$, respectively.

The identity (3.7) does not hold if the $\theta$-functions have a non-zero argument. This means there should be no gauge symmetries associated with the right lattice, because this would lead to a character valued partition function which is not a constant.

We specialize now to n = 3, i.e., to 4 dimensions, corresponding to the even lattice $\Gamma_{22;14} = \Gamma_{22;9} \times D_5$ (other even dimensions are not essentially different). The right lattice can be decomposed to $(D_1)^9 \times D_5$, where $D_5 = D_1 \times D_4$ is the space-time lattice, consisting of the light-cone factor $D_1$ and the appended factor $D_4$. It may be possible to combine the $(D_1)^9$ factor into larger $D_n$'s, but this is irrelevant. Taking the third power of (3.9), we find that the lattice must have the following conjugacy classes of $D_5 \times (D_1)^9$:

$$\begin{array}{c} ((s),(F)^3(0)^6) \\ 3((c),(F)^3(v)^2(0)^4) \\ 3((s),(F)^3(v)^4(0)^2) \\ ((c),(F)^3(v)^6) \end{array} \tag{3.10}$$

Here "F" stands for c or s. All these conjugacy classes should be associated with the same left vector $\vec{w}_L$. Because the $D_1$'s are not gauged, only the total number of states one gets from $(D_1)^9$ is relevant. It is therefore not a priori clear how to distribute the conjugacy classes over the nine $D_1$'s.

If we subtract the first line in (3.10) from the seven others we get a set of lattice vectors with vanishing left components. These vectors must have even length, and generate a certain sublattice. If we write the first $D_5 \times (D_1)^9$ conjugacy class as

$$(\ (s),(s,s,s,0,0,0,0,0,0)\ ), \tag{3.11}$$

then one can show that to reproduce (3.10) the sublattice must contain a set of vectors of the form

$$
\begin{aligned}
&(\ (v),(x,v,v,0,0,0,0)\ )\\
&(\ (v),(y,0,0,v,v,0,0)\ )\\
&(\ (v),(z,0,0,0,0,v,v)\ ),
\end{aligned}
\tag{3.12}
$$

where x, y and z are 3-vectors of (v)'s and (0)'s with odd length. Without loss of generality one may assume that the $(F)^3$ terms in (3.10) belong always to the same three $D_1$ factors. If one chooses them differently, one either generates a sublattice with half-integer length vectors, or a sublattice bigger than (3.12), which contains a set of vectors of the form (3.12). The shortest vector in the conjugacy class $((s),(s^3,0^6))$ has length 2, and will lead to chiral fermions if $\vec{w}_L{}^2=2$ and if there is no vector $((c),(F^3,0^6))$ associated with $\vec{w}_L$. The existence of a set of vectors (3.12) is then a necessary condition for the absence of massive chiral excitations of the chiral ground state. However, there are many other spinors on the lattice, for which either another set (3.12) must exist, or which must come in chiral pairs. The existence of other space-time fermions is restricted by (3.12) and the self-duality of the lattice, because they must have integer dot products with (3.12).

Modulo permutations, there are seven ways to choose the vectors x, y and z. One of these choices has the property that any fermion allowed by (3.12) automatically has a chiral partner if it is massive. This choice is

$$
\begin{aligned}
&(\ (v),(v,0,0,v,v,0,0,0,0)\ )\\
&(\ (v),(0,v,0,0,0,v,v,0,0)\ )\\
&(\ (v),(0,0,v,0,0,0,0,v,v)\ ).
\end{aligned}
\tag{3.13}
$$

For example, a chiral partner for the vector ((s),(s,s,s,s,s,0,0,0,0)) associated with any left vector, is obtained by adding the first vector of (3.13) to it; if one has a vector ((s),(0,0,0,s,0,s,0,s,0)), the complete sublattice generates chiral partners for its massive excitations.

For any other choice of x,y and z one can find allowed spinors that do not automatically have chiral partners. If such spinors are present one has to add chiral partners, which leads to an enlargement of the sublattice (3.12). This leads to further constraints, etc. It is possible that one always ends up with a set of vectors (3.13), or with a vector which destroys the chirality of (3.11). It would be interesting to know if this is true, but it is not essential, because we will see that the constraint imposed by (3.13) already leads to large numbers of chiral theories.

The presence of the vectors (3.13) can also be rephrased as a condition on the spinor conjugacy classes. Because the inner product between (s) and (v) is 1/2 mod 1, we see that the number of (s) (or (c)) conjugacy classes overlapping with the (v)'s in (3.13) must be even. This implies that the $D_1$ factors can be grouped into triplets, so that for each vector on the lattice the sum of each of the triplet components is equal, mod 1, to the $D_5$ components of that vector. Conversely, if the latter condition is satisfied, the self-duality of the lattice implies the existence of vectors (3.13).

The same constraint has been obtained in [9] [10] from a different point of view. These authors observe that the world-sheet supersymmetry of the right-movers should not just be realized on the right-moving bosons and fermions with space-time indices, but also on the n right-moving internal bosons. Although it is not clear what the most general way of doing this is, a sufficient condition is this "triplet constraint". For non-chiral theories it is not a necessary condition for world-sheet supersymmetry; for example, Narain's compactifications [2] do not satisfy it in general.

Our condition imposes constraints on chiral theories only and is a necessary condition, which comes close to reproducing the sufficient condition of [10.] We cannot rigorously rule out other possibilities, such as $(D_1)^9$ lattices with a more general constraint (3.12), or entirely different lattices which do not admit a $D_1$ decomposition at all, but they appear implausible.

In the rest of this paper we will restrict ourselves to the triplet constraint, which ensures both world-sheet supersymmetry and space-time Lorentz invariance. This implies that we only have to consider lattices which admit a $D_1$ decomposition; this destroys the continuous infinity of Lorentzian self-dual lattices.

This can also be understood from a different point of view. As we have seen, the chiral partition function (i.e. the partition function multiplying the chiral ground state) should only depend on $\tau$, and not on $\bar{\tau}$. It should in fact be a modular function of weight $n-4$ (in $10-2n$ dimensions), and be holomorphic apart from possible poles at $\tau = i\infty$ (or $q = e^{i\pi\tau} = 0$) [17]. Functions of this kind are determined almost completely by their modular weight and pole structure, a fact employed a decade ago by Nahm [19] to obtain what we now call the partition function of the left moving sector of the heterotic string.

These arguments do not apply for partition functions which depend on q as well as $\bar{q}$, a situation which typically arises in string theories below ten dimensions. Such non-holomorphic partition functions are not classifiable, a fact most convincingly demonstrated by Narain's construction of continuous families of non-chiral lower-dimensional theories.

For chiral theories we are in a better position, because we can at least say something about the chiral partition function. Unfortunately this function is not very useful in four dimensions: because it has odd weight (namely $-1$), it vanishes. It is easy to see explicitly why it vanishes: for every lattice vector $(\vec{w}_L;\vec{u}_R,(s))$ (in $\Gamma_{22}\times(D_1)^9\times D_5$ notation) there is a vector $(-\vec{w}_L;-\vec{u}_R,(c))$, so that the two precisely cancel against each other in the chiral partition function (they are in fact CPT conjugates).

This is only a minor problem, which is easily circumvented by considering a more interesting object, the character valued partition function. (For a detailed discussion see [17]. ) It does not just contain information about the zeroth order traces (i.e. the dimensions) of the representations at all levels, but also about all higher order traces. It is easy to see that the cancellation described above will not occur for odd traces (unless only real representations appear, in which case we would not have a chiral theory). The coefficient function $C_k(q)$ of any odd trace of order k, except those proportional to $\mathrm{TrF}^2 - \mathrm{TrR}^2$, can now be shown to be a holomorphic modular function of weight $k-1$ (in four dimensions). In fact they cannot even have poles at $q=0$, because the tachyonic ground state of the left moving bosonic is in a singlet representation of the gauge group. It is known that any such function can be expressed completely in terms of the two Eisenstein functions $G_4$ and $G_6$. For example,

$$C_1 = a_1$$

$$C_3 = 0$$

$$C_5 = a_5 G_4$$

$$C_7 = a_7 G_6$$

$$C_9 = a_9 (G_4)^2$$

$$C_{11} = a_{11} G_4 G_6$$

$$C_{13} = a_{13} (G_4)^3 + b_{13} (G_6)^2 \quad \text{etc.}$$

The fact that $C_3 = 0$ implies that all anomalies cancel[1] [17]. The coefficients $a_n$ and $b_n$ are not fixed by modular invariance and have to be determined from the lowest level(s). The traces of all representations at higher levels (and hence to a large extent the representations themselves) are thus practically determined by the massless level. This does not tell us very much about the massless level itself, but it probably does restrict it, because it will not in general be possible to find representations at higher levels which produce precisely the right traces. It might be possible to prove general properties of the massless states in this way, but for the time being we will let the lattices take care of these subtle relations.

## 4. The spectrum

The theories we are considering can be constructed from even or odd Lorentzian self-dual lattices. Although the two constructions are completely equivalent, the even lattices have several practical advantages, so that in the rest of this paper we will only consider even lattices. One advantage is that the mass formula is symmetric in

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[1]Notice that in this case (but not in general) the anomaly cancellation occurs even at higher levels, among left-mover excitations which do not have right-moving partners.

left and right sectors; it is simply the one for the bosonic string,

$$1/8m_L{}^2 = 1/2\vec{w}_L^2 + N_L - 1 \tag{4.1}$$

$$1/8m_R{}^2 = 1/2\vec{w}_R^2 + N_R - 1 \tag{4.2}$$

$$m_L{}^2 = m_R{}^2 \tag{4.3}$$

$$m^2 = 1/2(m_L{}^2 + m_R{}^2), \tag{4.4}$$

where $(\vec{w}_L,\vec{w}_R)$ is a Lorentzian lattice vector, and $N_L$ and $N_R$ are the bosonic oscillator contributions. From now on, the word "mass" will be used to refer to the value of $1/8m^2$. For simplicity, we will concentrate on the case $D=10-2n=4$; $D=6,8$ and 10 are completely analogous. In four dimensions, the relevant even Lorentzian lattice is $\Gamma_{22;14}$; the right moving part should contain a $D_5$ sublattice; the remainder should be a $(D_1)^9$ lattice which can be split into 3 triplets, so that $\vec{w}_R$ decomposes as

$$\vec{w}_R = (\vec{x},\vec{y},\vec{z},\vec{v}_R) \equiv (\vec{u}_R,\vec{v}_R), \tag{4.5}$$

where $\vec{v}_R$ is a weight of $D_5$ and $\vec{x}$, $\vec{y}$ and $\vec{z}$ are 3-vectors. Because of the triplet constraint, the sum of the 3 entries of each of the three vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ should be equal the entries of $\vec{v}_R$ modulo 1.

The physical states are obtained by decomposing $D_5$ to $D_1 \times D_4$ and keeping only states without $D_4$ oscillator excitations and with a fixed $D_4$ vector or spinor weight (of length 1), for example $x_e=(1,0,0,0)$ and $x_e=(1/2,1/2,1/2,1/2)$. Hence, the massless states are characterized by $\vec{w}_L{}^2=0,2$ and $\vec{u}_R{}^2 \leq 1$ (see table 1). The lattice momenta plus Cartan subalgebra excitations produce states in representations of $SO(2)\times G_L \times G_R$, where $G_L$ is a rank 22 group corresponding to the left lattice and

$G_R$ a rank 9 group realized on the $(D_1)^9$ part of the right lattice. The transverse Lorentz group representation of a state is obtained by multiplying the lattice SO(2) representation (from the $D_1$ subalgebra of $D_5$) with SO(2) representations generated by the left and right moving uncompactified bosonic oscillators.

This construction is illustrated in table 1, where we summarize all states with masses less than or equal to zero that might occur in the four $D_5$ conjugacy classes. Right moving oscillators can be ignored, because they only contribute to massive states. The left movers can be divided in space-time oscillators ($N_L{}^S$) and internal ones ($N_L{}^I$), which generate the Cartan subalgebra of $G_L$. The mass difference between the ground state and the first excited state can be as small as 1/4, because one can add two spinor entries to the same triplet of $\vec{w}_R$ without violating the triplet constraint. Tachyons can only appear in the $D_5$ conjugacy class (v). The absence of tachyonic fermionic spinors is guaranteed by the triplet constraint.

In four dimensions (and also in eight) a change in the sign of a spinor weight changes the conjugacy class from (s) to (c) and vice versa. The lattice has a vector $(-\vec{w}_L;-\vec{w}_R)$ for every vector $(\vec{w}_L;\vec{w}_R)$. These two are each others' CPT conjugates, but each such pair should be counted as only one Weyl or Majorana spinor. This is most easily seen by counting gravitinos or gauginos in supersymmetric theories.

Another advantage of the even Lorentzian lattices is that all gauge invariances of the field theory can be traced back to roots of the lattice, i.e. vectors with either $\vec{w}_L{}^2=0$, $\vec{w}_R{}^2=2$ or $\vec{w}_L{}^2=2$, $\vec{w}_R{}^2=0$. In some cases we may conclude from the presence of such vectors that $D_5$ is actually embedded in a regular way in a larger simple Lie algebra. The following gauge particles may appear:

– gravitons, $B_{\mu\nu}$, dilaton: They owe their existence to the root lattice of $D_5$ (with $\vec{w}_L = \vec{u}_R = 0$) and are therefore always present.

- $G_L$ gauge bosons: Their presence is due to the $G_L \times D_5$ root lattice (with $u_R = 0$) and they are therefore also inevitable. There are at least 22 U(1) gauge bosons , plus one for every vector with $\vec{w}_L{}^2 = 2$.
- $G_R$ gauge bosons: their existence requires a vector $\vec{w}_R$ of length 2, with $\vec{w}_L = 0$.

Furthermore the $D_5$-component of $\vec{w}_R$ should be a vector. The presence of such a vector indicates that $D_5$ is a part of a larger algebra, which must be a $D_n$ algebra ($n > 5$). But then the $D_n$ spinors would decompose into $D_5$ spinors with opposite chirality, and we would get a non-chiral theory[1]. Thus chiral theories do not have $G_R$ gauge bosons. (This was a crucial assumption in sec.3, which has now been justified).

Finally, there may appear

- Gravitinos: to obtain gravitinos we need length 2 vectors $\vec{w}_R$ ($\vec{w}_L = 0$) with spinor components in $D_5$.

The only regular embedding of $D_5$ which yields such roots is in $E_6$, $E_7$ or $E_8$. It is easy to see that these choices lead to $N = 1,2$ and 4 supergravity respectively. Analogous results for other even dimensions are shown in table 2. (In two dimensions there are interesting additional possibilities because one can get spinors from triality

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[1]This is true in the absence of $G_R$ gauge symmetries. One might wonder whether the fermions could be in chiral representations of $G_R$. This is impossible because the gauged part of $G_R$ has no such representations: The $G_R$ gauge bosons originate from length 1 roots, so that the gauge group can only be a product of SO(3)'s [20]. In particular, this applies to models generalizing type II superstrings, which can straightforwardly be obtained by our construction.

rotated embeddings of $D_4$ in $D_n$.) This result also explains why only two of the ten-dimensional heterotic strings constructed from Niemeier lattices [8] have space-time supersymmetry. Some expected consequences of supersymmetry follow immediately, such as the presence of gauginos (the presence of $\vec{w}_L = 0$ for some $\vec{w}_R$ implies the presence of the entire $G_L$ root lattice combined with the same $\vec{w}_R$). Others, such as supersymmetry at higher levels, are less obvious, and depend on rather intriguing properties of exceptional groups[1].

## 5. Construction of lattices

Any even self-dual lattice constructed entirely out of $D_n$- factors can be mapped to another even self-dual lattice by changing the dimension of any $D_n$-factor by multiples of eight, keeping all conjugacy classes the same. Such a transformation leaves the lengths of all vectors unchanged mod 2, and all mutual dot-products mod 1, so that it does not affect self-duality. One may even subtract multiples of eight to make the dimension of a $D_n$-factor negative. This can be interpreted as a change of metric. For example, $(D_5)_R$ can be changed to $(D_3)_L$, where the subscripts refer to the left (negative metric) and right (positive metric) part of the lattice. We will make extensive use of such transformations.

[1] Thus, these may play in the covariant lattice approach including ghosts an important role in obtaining off-shell formulations of many supergravity theories.

In this way we can associate with any even self-dual ***Lorentzian*** lattice consisting entirely of $D_n$'s a left ***Euclidean*** lattice (obtained by mapping the right lattice to the left) or a right ***Euclidean*** lattice. Because of the triplet constraint, the lattices of chiral theories we consider consist only of $D_n$-factors on the right (as discussed in section 3), so that in all cases we can associate left Euclidean lattices with them.

Euclidean lattices have the advantage of having useful classifications. Lorentzian lattices do not: their classification is up to Lorentz transformations, which unfortunately do not leave the physics invariant. This Euclidean lattice classification is very successful in ten dimensions, where only 24-dimensional lattices have to be considered [8].

In four dimensions things are far more complicated. In the worst possible case we have a lattice $\Gamma_{22;14} = (\Gamma_{22})_L \times (D_5 \times (D_1)^9)_R$, which can be mapped to $(\Gamma_{22} \times D_3 \times (D_7)^9)_L$, a Euclidean lattice of dimension 88. A lower limit on the total number of such lattices is provided by the Siegel mass formula [21] [22]

$$\sum_{\Lambda} g(\Lambda)^{-1} = (8k)^{-1} B_{4k} \prod_{j=1}^{4k-1} (4j)^{-1} B_{2j}, \tag{5.1}$$

where the sum is over all even self-dual lattices of dimension 8k, and $g(\Lambda)$ is the order of the automorphism group of $\Lambda$. Because $g(\Lambda) \geq 1$ the right hand side is a lower limit of the number of lattices ($B_{2j}$ are the Bernoulli numbers). For $k=11$ this number is of order $10^{1500}$ ! The requirement that $\Lambda$ should contain $D_3 \times (D_7)^9$ with a triplet constraint will reduce the number considerably, but clearly this is not a viable approach towards classification. It only tells us that the number of chiral theories is finite, but most likely extremely large.[1]

[1] A more reasonable but less rigorous estimate can be made by observing that the 88-dimensional lat-

However, it is easy to construct many examples already from the 24-dimensional Niemeier lattices; a list of those may be found in [23.] To do so, one decomposes such a lattice in $D_n$-factors by using one of the following regular embeddings

$$
\begin{aligned}
&E_8 \supset D_8 \\
&E_7 \supset D_6 \times A_1 \\
&E_6 \supset D_5 \times U(1) \\
&A_n \supset [D_3]^k \times U(1)^{k-1} \qquad (k = [(n+1)/4]\,) \\
&A_2 \supset A_1 \times U(1)
\end{aligned}
\tag{5.2}
$$

After these decompositions one is left with factors $A_1$ and several $D_1$'s. An even number of $A_1$'s can be combined to $D_2$'s because $D_2 = A_1 \times A_1$ . In all cases we have considered all m remaining factors can be rotated to a $(D_1)^m$ lattice, and it is not unlikely that this is always possible, even for the Leech lattice. Notice that we are not doing anything to the lattices; we simply regard them as root-lattices plus weights of subalgebras of the original algebras. Some of the weights may have length two and this indicates the possibility of enlarging the algebras. One should always keep in mind that the Euclidean lattices are only used for classification. Their roots are not very relevant; the relevant roots are those of the associated Lorentzian lattices.

The Lorentzian lattices can be constructed as follows. A Euclidean lattice with p factors $D_n$ can be completely specified by a list of all conjugacy classes that appear. For self-dual lattices there are $2^p$ classes out of a total of $4^p$ on this list. The list can

---

tice has (at most) 32 factors, so that combinatorically their classification should be similar to the classification of even self-dual lattices of dimension 32 with $D_1$ lattices as building blocks. On the basis of such an estimate one would still expect a very large number of solutions.

be condensed to a set of generators, from which all others can be obtained by addition. For definiteness we will construct four-dimensional theories. Then the lattice must contain a $D_5$ factor, which becomes part of the right lattice. All other $D_n$'s can be decomposed, and put partly on the left and partly on the right. The space-time $D_5$ cannot be embedded in such a left-right split $D_n$, because the spectrum would then contain tachyons of mass $-1/2$ (see table 1. The ten-dimensional theories provide a nice illustration: the tachyon-free ones are precisely those with a separate $D_8$-factor.). Furthermore, as discussed in section 4, $D_5$ cannot be embedded in a larger $D_n$ which stays entirely on the right, or else the theory will not be chiral. Thus we should look for a separate $D_5$ (it may part of an $E_6$, but that is irrelevant at present).

The left-right decomposition of the remainder of the lattice should satisfy the triplet constraint. This is easy to achieve. One considers all possible ways of selecting 3 of the remaining $D_n$-factors. (Each factor may be selected more than once in one triplet). Then one makes a list of basic triplets, satisfying the constraint. Of course only the generators of the conjugacy classes have to be inspected. Finally one considers all possible ways of choosing 3 triplets from the list, where again each triplet may be chosen more than once. The triplet assignment may now be forgotten; the only relevant information is how many times a $D_n$-factor has been selected. If it has been selected m times, one can replace $(D_n)_R$ by $(D_m)_R \times (D_{m-n+8k})_L$, where k is adjusted so that the dimension is positive. The left dimension is now automatically 22 mod 8, and may or may not be adjustable to 22 by changing the values of k. If it cannot be adjusted, the lattice is of no interest.

Because many triplet assignments are possible, one generates many Lorentzian lattices from a given Euclidean one. Furthermore there are several simple tricks to construct even more Euclidean lattices. Of course one may add $E_8$ factors. One can "blow up" $D_n$-factors, by changing their dimension by multiples of 8. One can decompose $D_{n+m}$ to $D_n \times D_m$. However, there is no advantage in decomposing a $D_n$-lattice in smaller $D_n$ pieces. A more interesting change is splitting off a $D_4$, making a triality rotation on it and then blowing it up. This process destroys the length two vectors that hint at the fact that $D_4$ was once part of a larger algebra. Similarly one may recombine $A_1$ factors from several $D_2$'s and blow up the latter. One can always construct a mirror lattice by changing all $D_n$'s to negative dimensions (using mod 8 reductions), and changing the overall sign of the metric. Using combinations of these tricks one can change any $D_n$ with odd n to a $D_5$. Several of the Niemeier lattices can be obtained from the dimension 8 and 16 ones in this way.

## 6. Examples

We now illustrate the above-explained construction of Lorentzian lattices $\Gamma_{22;14}$ satisfying the triplet constraint, by means of some examples. We start with a Euclidean Niemeier lattice with an $E_6$-factor, so that we can expect to find some string theories with $N=1$ supersymmetry. There are two such lattices, namely $(E_6)^4$ and $A_{11} \times D_7 \times E_6$; in the following, we focus on the latter. The conjugacy classes which are present in addition to the roots can be found in [22]. They are generated by a weight of the representation (12,64,27) of $SU(12)\times SO(14)\times E_6$. This is a vector of length-squared four on the lattice. To decompose the lattice into $D_n$-factors one simply considers, for example, the regular embeddings

$$SU(12) \supset SU(4)\times SU(4)\times SU(4)\times U(1)\times U(1) \qquad (6.1)$$

and

$$E_6 \supset SO(10)\times U(1).$$

The SU(4) factors give $D_3$ lattices because $A_3 = D_3$. The three remaining U(1) factors can be rotated to a $(D_1)^3$ lattice. To see explicitly how that works, consider the decompositions of the (12) of SU(12) and the (27) of $E_6$ with respect to (6.1):

$$\begin{aligned}(12) = \;& (4,1,1,1/6\sqrt{3},1/6\sqrt{3}) \\ +\;& (1,4,1,1/4-1/12\sqrt{3},-1/4-1/12\sqrt{3}) \\ +\;& (1,1,4,-1/4-1/12\sqrt{3},1/4-1/12\sqrt{3})\end{aligned} \qquad (6.2)$$

$$(27) = (16,1/6\sqrt{3}) + (10,-1/3\sqrt{3}) + (1,2/3\sqrt{3})$$

Here the charges have been normalized so that they give the correct length-squared of the SU(12) and $E_6$ weights (namely 11/12 and 4/3 respectively). We can now define $(D_1)^3$ spinor weights as the following vectors on the $U(1)^3$ lattice:

$$\begin{aligned}\vec{s}_1 &= (1/6\sqrt{3},1/6\sqrt{3},1/6\sqrt{3}) \\ \vec{s}_2 &= (1/4-1/12\sqrt{3},-1/4-1/12\sqrt{3},1/6\sqrt{3}) \\ \vec{s}_3 &= (-1/4-1/12\sqrt{3},1/4-1/12\sqrt{3},1/6\sqrt{3}).\end{aligned} \qquad (6.3)$$

These vectors are mutually orthogonal and have length-squared 1/4. We can now express all other vectors on the $A_{11} \times D_7 \times E_6$ lattice as weights of

$$(D_3)^3 \times (D_1)^3 \times D_7 \times D_5, \tag{6.4}$$

by projecting on these three vectors. By decomposing the representation (12,64,27) completely one gets then a set of generators for the conjugacy classes of the lattice, which is in fact slightly redundant. A minimal set is

$$\begin{array}{c}(s,0,0,s,0,0,c,c)\\(0,s,0,0,s,0,c,c)\\(0,0,s,0,0,s,c,c)\\(0,0,0,s,s,s,0,s)\end{array} \tag{6.5}$$

By adding these vectors in all possible ways one generates 256 conjugacy classes (the addition rules for $D_n$, n odd are $(s)+(s)=(v)$, $(s)+(c)=(0)$, $(v)+(s)=(c)$, $(v)+(c)=(s)$, $(v)+(v)=(0)$ ). Notice that the result will be symmetric under simultaneous permutations of the first three entries (the $D_3$'s) and the second three (the $D_1$'s).

The last entry in the vectors (6.5) indicates the conjugacy class of the space-time factor $D_5$. We now have to form triplets of the remaining entries that add up to the last one, mod 1. The solutions can be specified by giving the vectors (3.13) which enforce the constraint. One has the following seven possibilities

$$\vec{t}_{1..3} = (v,0,0,v,0,0,v,v) \quad (+\text{ perm.})$$

$$\vec{t}_{4..6} = (0,v,v,v,0,0,0,v) \quad (+\ \text{perm.}) \qquad (6.6)$$
$$\vec{t}_7 \ = (0,0,0,v,v,v,0,v)$$

Here "perm" indicates the simultaneous permutations of the first and second three entries. These seven vectors have integral dot-products with all the generators, and by self-duality they must therefore be vectors on the Euclidean lattice. Now one has to select three triplets out of these seven. Valid choices are, for example, $3\times\vec{t}_1$, $2\times\vec{t}_1+\vec{t}_3$, $\vec{t}_4+\vec{t}_6+\vec{t}_7$ etc.

This determines the right lattice. For example, to obtain the triplet constraint corresponding to $3\vec{t}_1$ one maps the Euclidean lattice to the following Lorentzian lattice

$$(D_{3R})\times(D_{5L})\times(D_{5L})\times(D_{2L}\times D_{3R})\times(D_{7L})\times(D_{7L})\times(D_{4L}\times D_{3R})\times(D_{5R}).$$

The factors between parentheses indicate how the eight factors of the original lattice are distributed between left and right. By decomposing the vector $\vec{t}_1$ one obtains (among others) a set of three vectors of the form (3.13). The factors on the left lattice are determined mod 8, and we have chosen the minimal dimensions. In this case the total dimension of the left lattice is 30, which is too large. By considering the choice $3\times\vec{t}_7$ we get

$$(D_{5L})^3\times(D_{2L}\times D_{3R})^3\times(D_{1L})\times(D_{5R}).$$

This example yields a rank 22 gauge group $(SO(10)\times SO(4))^3\times U(1)$. In general one simply calculates the sum of the selected vectors $\vec{t_i}$, replacing the entries (v) by 1. The resulting vector specifies for each $D_n$ factor of the Euclidean lattice which part of it is put on the right lattice. The remainder (which may have positive or negative dimension) is moved to the left lattice; it can always be given a positive dimension by adding multiples of 8. Finally one has to check that the left lattice has dimension 22.

By going systematically through all possibilities and removing trivial permutations, we have obtained 21 different string theories from the Niemeier lattice $A_7\times D_{11}\times E_6$. Not all of these have supersymmetry. To get a supersymmetric theory one has to make sure that the $E_6$-factor of the Euclidean lattice remains intact. Because the U(1), which enlarges $D_5$ to $E_6$, is a linear combination of the three $D_1$'s, these must all be selected precisely once, since otherwise the length of the $E_6$ roots would be destroyed. There are 4 solutions (again up to permutations), namely

$$\begin{gathered}\vec{t_1}+\vec{t_2}+\vec{t_3}\\ \vec{t_1}+\vec{t_2}+\vec{t_6}\\ \vec{t_4}+\vec{t_5}+\vec{t_6}\\ \vec{t_1}+\vec{t_5}+\vec{t_6}\end{gathered} \tag{6.7}$$

The gauge groups associated with the first three solutions are respectively $SO(12)^3\times SO(8)$, $SO(14)^2\times SO(10)\times SO(6)$ and $SO(14)^3\times U(1)$. The "minimal" left lattice for the last solution is $(D_6)^2\times D_2$, which has dimension 14. We can enlarge the rank to 22 by extending any $D_n$-factor by eight, or by simply adding an $E_8$ factor. In any case the result will not be chiral, because none of the groups has complex representations. Of the first three supersymmetric models, two are at least potentially chiral, and we we will see further below that they do indeed have chiral fermions.

To obtain the complete spectrum of any of these theories one has to consider all 256 conjugacy classes, and decompose all $D_n$ factors in left and right parts. The decomposition rules for the conjugacy classes are

$$\begin{aligned}
(0) &\rightarrow (0,0) + (v,v) \\
(v) &\rightarrow (v,0) + (0,v) \\
(s) &\rightarrow (s,s) + (c,c) \\
(c) &\rightarrow (s,c) + (c,s)
\end{aligned} \tag{6.8}$$

It does not matter whether one uses these rules for a "Euclidean" decomposition $D_{m+n} \supset D_n \times D_m$, $m,n > 0$, or for a "Lorentzian" one where n or m can be negative. Obviously many sectors are generated, and a complete description of the theory would be rather lengthy. But if one is only interested in the massless sector (as one usually is) one can eliminate most of the conjugacy classes, because they have a minimal weight length larger than 2 on the left or the right lattice. As an example we consider the non-supersymmetric $(SO(10) \times SO(4))^3 \times U(1)$ model described above. The lattice conjugacy classes that give rise to chiral spinors are (in the order specified in (6.4))

$$\begin{aligned}
&(s,0,0,s,0,0,c,c) \quad + \text{c.c.} \\
&(0,s,0,0,s,0,c,c) \quad + \text{c.c.}
\end{aligned} \tag{6.9}$$

$$(0,0,s,0,0,s,c,c) \quad + \text{c.c.} \quad ,$$

where c.c. denotes the same set of vectors, but with opposite sign (so that (s) and (c) are interchanged). A vector and its opposite describe together just one physical state. One can immediately read off the representations of the chiral fermions (all taken to be lefthanded) for the group $SO(10) \times SO(10) \times SO(10) \times SO(4) \times SO(4) \times SO(4) \times SO(2)$

$$\begin{aligned} &4(16,1,1,2\ ,1,1,1/2) + 4(1,16,1,1,2\ ,1,1/2) + 4(1,1,16,1,1,2\ ,1/2) + \\ &4(16,1,1,2',1,1,1/2) + 4(1,16,1,1,2',1,1/2) + 4(1,1,16,1,1,2',1/2), \end{aligned} \tag{6.10}$$

where (2) and (2′) denote the inequivalent spinor representations of SO(4). The multiplicity of 4 is simply the dimension of the spinor representation of a factor $(D_3)_R$ which is not gauged. The spectrum is obviously chiral, and has 16 generations of (16)′s for each of the 3 SO(10) groups. The last entry gives the U(1) (=SO(2)) charge. There is no sign mistake: this U(1) factor is indeed not traceless, and it has anomalies with each of the SO(10) and SO(4) groups. This is in agreement with the theorem proved in [17], which states that because of modular invariance the anomaly is factorizable. For four dimensions the six-form from which the anomaly can be derived is predicted to be proportional to

$$(\mathrm{Tr}F^2 - \mathrm{Tr}R^2)\,\mathrm{Tr}F. \tag{6.11}$$

The factorization itself is a triviality for the theory under consideration, because it has only a U(1) anomaly. But the prediction goes further than that: if F and R are both in the vector representations of SO(N) groups, then $\mathrm{Tr}R^2$ and $\mathrm{Tr}F^2$ should appear with relative factor $-1$. Furthermore this should be true for each simple factor of the gauge group.

One can calculate the anomaly explicitly by using the index theorem in six dimensions. By the usual arguments [24] the anomaly of a Weyl-spinor is generated by

$$(\text{constant}) \times (1/8\ \mathrm{Tr}(F_r)^2 - 1/48\ \mathrm{Tr}R^2)\ \mathrm{Tr}F_r\ , \tag{6.12}$$

where $\mathrm{Tr}F_r$ denotes the trace over the gauge group representation r of the Weyl spinor. (TrF, used above, refers to the vector representation). Using the fact that for spinor representations of SO(N) groups

$$\mathrm{Tr}(F_s)^2 = 1/8\ \dim(s)\ \mathrm{Tr}F^2\ , \tag{6.13}$$

one can easily show that the anomaly of this model does indeed have the predicted form. The triplication of the SO(10) and SO(4) groups is essential to get the correct ratio between gauge and gravitational anomalies.

The appearance of such anomalous U(1)'s is not a new phenomenon. It has been observed before in [25] and [26] (the U(1)'s discussed in these two papers have different origins: those of the first paper come from isotropy groups of the compactifying spaces, whereas those of [26] are U(1) factors left over from the breaking of $E_8 \times E_8$). We expect that they are cancelled by the four-dimensional analogue of the Green-Schwarz mechanism.

Returning now to our example, we find that the massless scalars are in the following representations

$$
\begin{array}{ll}
6\,(10,1,1,1,1,1,\pm 1) & +\ \text{perm.} \\
6\,(1,1,1,4,4,1,0) & +\ \text{perm.} \\
6\,(10,10,1,1,1,1,0) & +\ \text{perm.}\,,
\end{array}
\tag{6.14}
$$

i.e., one has to include the permutations of the $SO(10) \times SO(4)$ factors and in addition to that a multiplicity factor of 6. Evidently the model is not supersymmetric, and it does indeed not have gravitinos.

The $N=1$ supersymmetric $SO(14)^3 \times U(1)$ theory discussed above has the following massless fermions in addition to the gravitino

$$
\begin{array}{ll}
2\,(64,1,1,1/2) & +\ \text{perm.} \\
2\,(64^*,1,1,1/2) & +\ \text{perm.} \\
(14,14,1,0) & +\ \text{perm.} \\
(14,1,1,\pm 1) & +\ \text{perm.}
\end{array}
\tag{6.15}
$$

This theory is chiral, but only because of the anomalous SO(2). Again the anomalies factorize properly because of the triplication. The other potentially chiral $N=1$ model turns out to be precisely the one discussed in [10].

The 21 models obtained from the Euclidean lattice $A_{11}\times D_7\times E_6$ with the specific embedding (6.1) are distributed as follows over the various possibilities:

| | | |
|---|---|---|
| 2 | Chiral, | $N=1$ supersymmetric |
| 4 | Non-chiral, | $N=1$ supersymmetric |
| 11 | Chiral, | not supersymmetric, tachyon free |
| 3 | Non-chiral, | not supersymmetric, tachyon free |
| 1 | Chiral, | not supersymmetric, with tachyons of mass[1] $-1/4$ |

Similarly, from the lattice $(E_6)^4$ we obtained seven non-chiral and two chiral $N=1$ supersymmetric models, as well as 27 chiral, tachyon-free non-supersymmetric ones.

The spectra of these models show some intriguing features. We find that the number of (16)'s of SO(10) (minus the number of $(16^*)$'s ) is always a multiple of 16. (Similarly, there are always 4k chiral (64)'s of SO(14)). Should this be a general phenomenon, then the prospects for these models are not very good. However, we do not know of any reason why this should be true in general.

It might be more attractive to look for models with a smaller gauge group realized on the massless fermions, such as $SU(3)\times SU(2)\times U(1)$. If such groups arise on $D_n$ lattices, they will not be easy to find. They can only appear when several $D_2$ or $D_1$ factors turn out to form a sublattice of the lattice of a larger algebra. To find such cases one has to look for additional length-2 vectors on the left lattice. It is not easy to look for such examples systematically, but they might very well exist.

---

[1]We remind the reader that by construction tachyons of mass $-1/2$ (the mass of the Neveu-Schwarz tachyon) cannot appear in these models.

An obvious but important observation is that the only representations of any simply laced group one can encounter in the massless fermion or scalar spectrum are those with weights of length two. Chiral fermions with simple groups can therefore only owe their chirality to the spinor representations of SO(14), SO(10) and SO(2), at least as long as one only considers SO(n) symmetries and not their possible enlargements (SO(6) spinors are anomalous; the anomalies of the SO(2) spinors can be cancelled by the Green-Schwarz mechanism). Furthermore an SO(10) spinor must appear in combination with a spinor representation of another orthogonal group in order to have the correct weight length. Despite the presumably gigantic number of models that may exist[1], the possibilities are thus still severely limited in comparison with field theory in four dimensions.

## 7. Conclusions

As anticipated in [7] [8], we have found that the covariant lattice construction, which provided a simple and elegant way of classifying all ten-dimensional string theories, has similar advantages for constructing chiral string theories below ten dimensions. We expect that all theories discussed in this paper have the same degree of consistency as the well-known ten-dimensional theories. Although the general class of theories we find has already been constructed fermionically in [9], the lattice approach gives far more immediate insight in their structure. Although the number

[1]With the help of a computer, it is easy to generate spectra of hundreds of theories within of a few moments.

of chiral theories of this type is finite, our results suggest that there exist very many of them, so that a complete enumeration appears impossible. Perhaps some interesting subclass can be classified completely.

It seems that not much is left of the once celebrated uniqueness of string theory. Of course string theory never really was unique even in ten dimensions, and it is already known for some time that the situation is much worse in four dimensions. Up to now, one may have taken comfort from the fact that four-dimensional theories are just compactifications of the ten-dimensional ones, at least if one believes that it is better to have one string theory with many vacua than many string theories. If this kind of uniqueness is what is desired, one would be better off if all fermionic strings could be shown to originate from the bosonic string, which seems the best candidate for a really unique theory. Our construction puts the ten- and lower-dimensional theories on equal footing in this respect.

Even if all that string theory could achieve would be a completely finite theory of all interactions including gravity, but with no further restrictions on the gauge groups and the representations, it would be a considerable success. But the situation is better than that; although gauge groups are not very much restricted except that in chiral models their rank cannot exceed 22, the representations are. The fact that weights of length larger than 2 cannot appear in the massless sector selects low-dimensional representations; therefore, it is impossible to obtain many models that have been considered in the past, such as those with large Higgs representations or color exotics. Furthermore, one is not free to select fermion and scalar representations in an arbitrary way, and couple them with arbitrary coupling constants.

In our models, there is a built-in mechanism for naturally producing several generations. The multiplicity occurring in the spectra (6.10) and (6.15) of our examples is a quite general phenomenon; it arises due to the possibility of assembling length-squared two vectors in the right-moving sector in different ways.

A rank 22 gauge group may seem excessively large in comparison with the standard model, but this problem can be dealt with in the same way as the "second" $E_8$ from the ten-dimensional heterotic string. There should be many cases where a large part of the gauge group does not act on the massless chiral fermions or where several parts of the gauge group act on several sets of fields separately. In fact, there is a slight tendency in favor to such a situation because of the limited weight length of massless states; we have indeed found examples where that is the case. Furthermore, it may be possible to reduce the gauge symmetry by dividing out discrete groups along the lines of [14] [27] [28]. It is obvious that there will exist even more string theories in four dimensions than the ones we can construct from self-dual lattices, or which have been constructed in [9] [10].

Of course we do not exclude the possibility that the kind of theories we construct here can in some way be regarded as compactifications of some or all ten-dimensional theories, but this will almost certainly be a rather epicyclical description compared to the one we have given here. In particular, there is no need anymore to study ten-dimensional field theories. If everything works as expected, one can directly calculate all relevant quantities in four dimensions using conformal field theory. After our recent excursion into higher dimensions, it may be difficult to get used to the fact that perhaps the world is four-dimensional after all.

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*Table 1: Massless Particles and Tachyons*

| Conjugacy classes $D_5 = D_1 \times D_4$ | | | Excitations | | | Mass | Particle type |
|---|---|---|---|---|---|---|---|
| $D_5$ | $D_1$ | $D_4$ | $1/2\vec{u}_R^{\,2}$ | $1/2\vec{w}_L^{\,2} + N_L^I$ | $N_L^S$ | $1/8m^2$ | |
| (0) | (0) | (0) Not physical | | | | | |
| | (v) | (v) | 0 | 1 | 0 | 0 | $G_L$ gauge bosons |
| | (v) | (v) | 0 | 0 | 1 | 0 | gravitons, $B_{\mu\nu}$, dilaton |
| (v) | (v) | (0) Not physical | | | | | |
| | (0) | (v) | 0 | 1/2 | 0 | −1/2 | scalar tachyon |
| | (0) | (v) | 1/4 | 3/4 | 0 | −1/4 | scalar tachyon |
| | (0) | (v) | 1/2 | 1 | 0 | 0 | massless scalar |
| | (0) | (v) | 1/2 | 0 | 1 | 0 | $G_R$ gauge bosons |
| (s) | (c) | (c) Not physical | | | | | |
| | (s) | (s) | 3/8 | 1 | 0 | 0 | massless fermion |
| | (s) | (s) | 3/8 | 0 | 1 | 0 | gravitino |
| (c) | Same as (s), but with opposite chirality. | | | | | | |

*Table 2: Conditions for Extended Supergravity and Chirality*

| Dimension | Embedding of space-time lattice | Supersymmetry | Chiral |
|---|---|---|---|
| D = 2 | $D_4 \subset D_4$ | None | possible |
| | $D_4 \subset D_n$, $5 \le n \le 16$ | None | |
| | $D_4 \subset D_n$, $5 \le n \le 16$ (*) | (2n−8,0) | yes |
| | $D_4 \subset E_6$ | (2,2) | |
| | $D_4 \subset E_7$ | (4,4) | |
| | $D_4 \subset E_8$ | (8,8) | |
| D = 4 | $D_5 \subset D_5$ | None | possible |
| | $D_5 \subset D_n$, $6 \le n \le 14$ | None | |
| | $D_5 \subset E_6$ | N = 1 | possible |
| | $D_5 \subset E_7$ | N = 2 | |
| | $D_5 \subset E_8$ | N = 4 | |
| D = 6 | $D_6 \subset D_6$ | None | possible |
| | $D_6 \subset D_n$, $7 \le n \le 12$ | None | |
| | $D_6 \subset E_7$ | N = 1 | yes |
| | $D_6 \subset E_8$ | N = 2 | |
| D = 8 | $D_7 \subset D_7$ | None | possible |
| | $D_7 \subset D_n$, $8 \le n \le 10$ | None | |
| | $D_7 \subset E_8$ | N = 1 | |
| D = 10 | $D_8 \subset D_8$ | None | possible |
| | $D_8 \subset E_8$ | N = 1 | yes |

The second column gives the maximal extension of the space-time lattice $D_{8-n} = D_{4-n} \times D_4$ within the right-moving part of the lattice $\Gamma_{16+2n;8+2n}$ ($D = 10-2n$). The entry "possible" means that the conditions for chirality we display are only necessary, but not sufficient; (*) indicates triality rotated embeddings. In type II theories one gets similar structures (with the same or opposite chirality) from the left-moving sector.