Matrix Factorizations and Open String TFT

W.Lerche, ZMP Hamburg, 12-2005

Joint work with
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(in preparation: with S. Govindarajan, H.Jockers, N.Warner)

Prior work by:
Kontsevich, Orlov, Kapustin-Li, Polishchuk

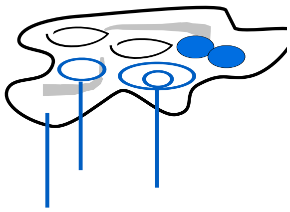
Closely related work by:
Hori, Ashok-Dell'Aquila-Diaconescu-Florea, Aspinwall-Katz,
Gaberdiel, Enger-Recknagel-Roggenkamp

Overview

- Motivation: study non-perturbative phenomena
  (quantum geometry of D-branes)
- Open string TFT, consistency conditions:
  (A_∞ relations, Cardy condition)
- New approach: boundary LG theory ....translates abstract mathematical notions into concrete physical terms
- Explicit computations: minimal models, torus, (CY)

D-brane worlds

Typical brane + flux configuration on a Calabi-Yau space



closed string (bulk) moduli t

open string (brane location + bundle) moduli u

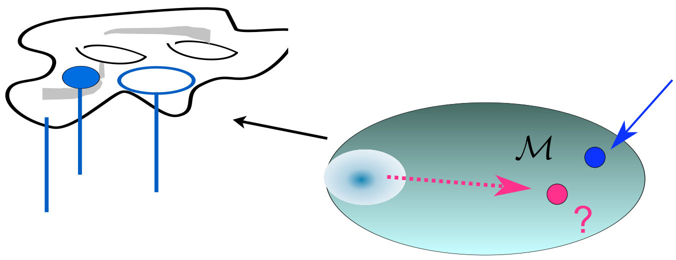
3+1 dim world volume with effective N=1 SUSY theory

What is the exact effective superpotential, the vacuum states, etc ?

$$\mathcal{W}_{\rm eff}(\Phi, t, u) \;= ?$$

Quantum geometry of D-branes

Classical geometry ("branes wrapping p-cycles", gauge bundle configurations on top of them) makes sense only at weak coupling/large radius:



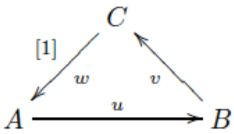
Quantum corrected geometry: (instanton) corrections wipe out notions of classical geometry

....well developed techniques (mirror symmetry)
for non-intersecting branes only !
and mostly for non-compact geometries.

The Derived Category Db(Coh(M))

Mathematicians (Kontsevich) tell us that the proper mathematical language for describing B-branes is the (bounded) **derived category** (of coherent sheaves on CY)

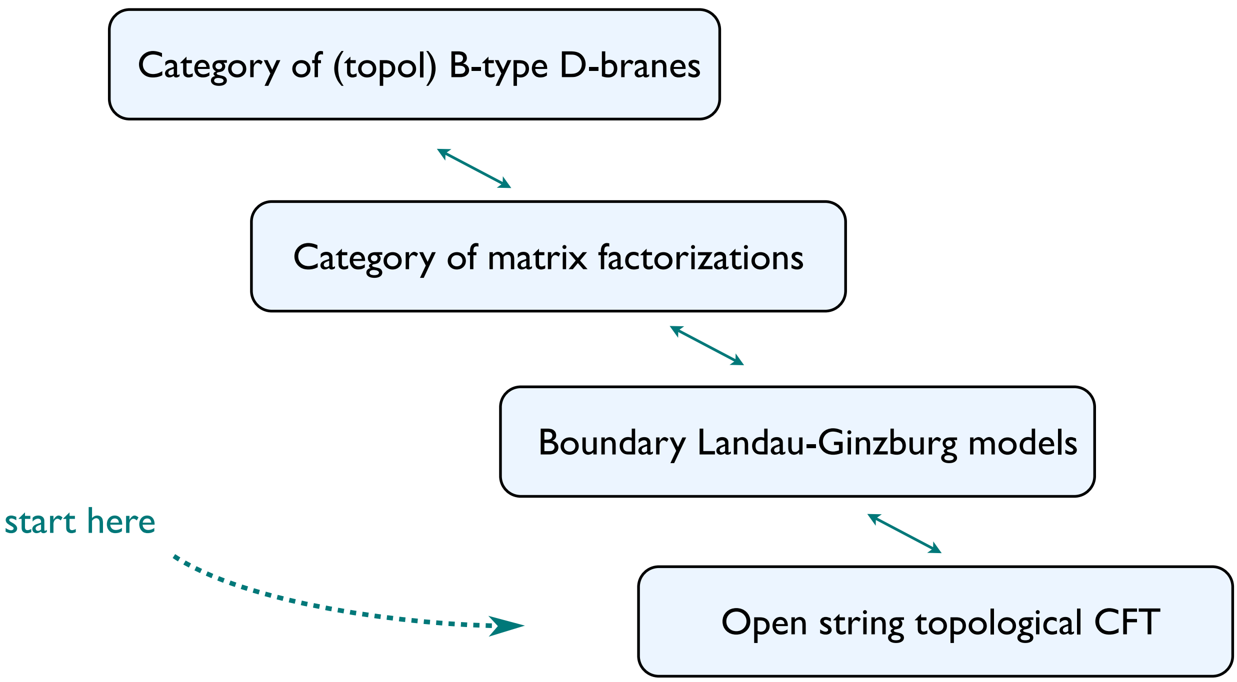
- more general than cohomology/ K-theory (RR U(1)charges)
- keeps track of brane positions
- describes bound state formation/tachyon condensation (triangulated category)



- treats branes and anti-branes on equal footing
- robust under continuous deformations (want: moduli dependence)

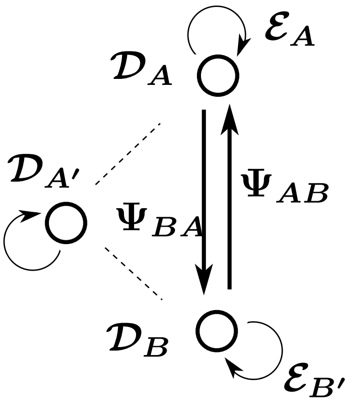
...we like to translate this language to one that is more familiar to physicists: **Landau-Ginzburg Theory**

Roadmap

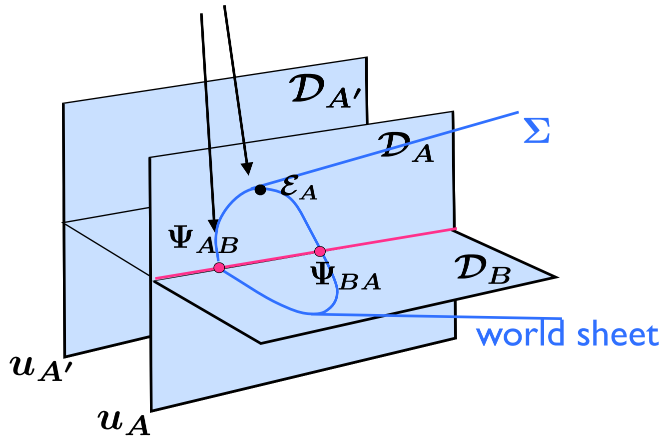


The category of topological D-branes

- objects: branes $\mathcal{D}$ $\longleftrightarrow$ boundary conditions
- morphisms (maps): $\mathcal{E}$ Ψ $\longleftrightarrow$ boundary preserving/changing open string vertex operators

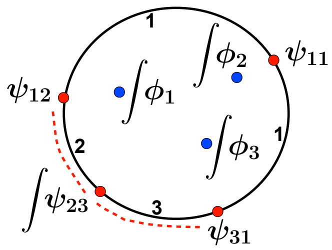


"Quiver" diagram



Open string TFT: twisted N=2 boundary SCFT

- A typical disk correlator looks like:



- Generating function:
  (t_i = bulk, u_a = boundary moduli)

$$\mathcal{W}_{\text{eff}}(t_i, u_a) = \left\langle e^{t_i \int_D \Phi_i^{(2)}} P e^{u_a \int_{\partial D} \Psi_a^{(1)}} \right\rangle$$
$$= \sum u_{a_m} ... u_{a_0} t_{i_n} ... t_{i_1} B_{a_0....a_m;i_1...i_n}(t)$$

where:

$$B_{a_0....a_m;t_1....t_n}(t) = \left\langle \Phi_{i_1} \Psi_{a_0} \, P \int \Psi_{a_1}^{(1)} \ldots \int \Psi_{a_m}^{(1)} \int \Phi_{i_2}^{(2)} \ldots \int \Phi_{i_n}^{(2)} \right\rangle$$
$$= \partial_{t_{i_n}} ... \partial_{t_{i_1}} \mathcal{F}_{a_1....a_n}(t)$$

Can show: infinitely many t-dependent cyclic prepotentials:

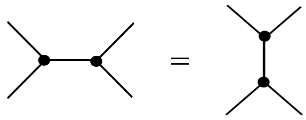
..in general not integrable wrto u

$$\left\{ \begin{array}{c} \mathcal{F}_{a_1}(t) \\ \mathcal{F}_{a_1 a_2}(t) \\ \mathcal{F}_{a_1 a_2 a_3}(t) \\ \mathcal{F}_{a_1 a_2 a_3 a_4}(t) \\ \vdots \end{array} \right.$$

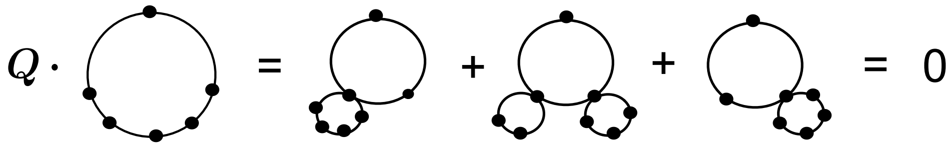
Open/closed top. string consistency conditions I

- Recap: WDVV equations in bulk TFT

$$\mathcal{F}_{ijm}\eta^{mn}\mathcal{F}_{nkl} \;=\; \mathcal{F}_{ikm}\eta^{mn}\mathcal{F}_{njl}$$



- Boundary TFT: Q-closedness and factorization



lead to "A∞ relations" for correlators $r_m(\Psi_{a_1}...\Psi_{a_m}) \;\equiv \Psi_{a_0}B^{a_0}_{a_1...a_m}$

$$\sum_{\substack{k,j=0\\k\leq j}}^{m}(-1)^{\tilde a_1+\ldots+\tilde a_k}r_{m-j+k}(\psi_{a_1}\ldots\psi_{a_k},r_{j-k}(\psi_{a_{k+1}}\ldots\psi_{a_j}),\psi_{a_{j+1}}\ldots\psi_{a_m})=0$$

with $r_0 \;=\; 0$ (no tadpoles): "strong" A∞ category

Kontsevich: D-branes indeed form a cyclic A∞ category

....but there is more.

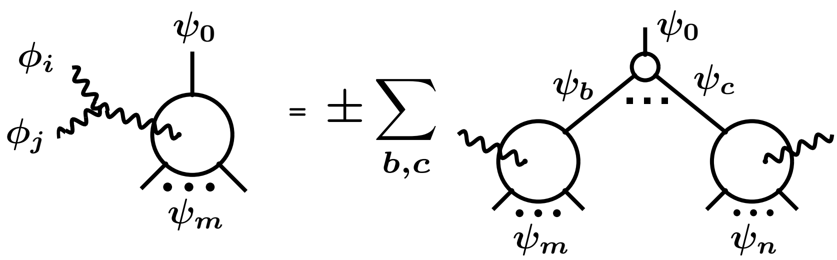
Open/closed top. string consistency condition II

- Beyond A∞ we have extra constraints, involving bulk operator insertions

....they deform $B^{a_0}_{a_1...a_m} \;\rightarrow\; B^{a_0}_{a_1...a_m}(t)$ and $r_0(t) \;\neq\; 0$ (weak A∞)

(deformation theory: "Hochschild complex")

- Bulk-boundary crossing symmetry:

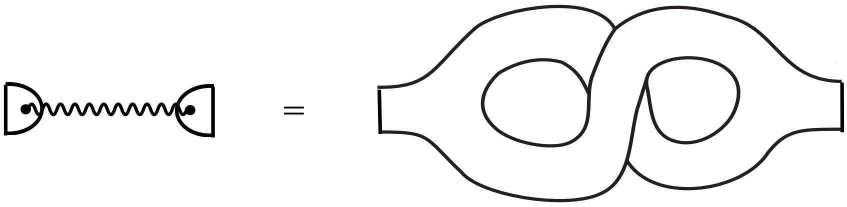


$$\partial_i\partial_j\partial_k\mathcal{F}(t)\;\eta^{kl}\;\partial_l\mathcal{F}_{a_0a_1\ldots a_m}(t) =$$

$$=\sum_{0\leq m_1\leq\ldots m_4\leq m}(-1)^s\;\mathcal{F}_{a_0\ldots a_{m_1}ba_{m_2+1}\ldots a_{m_3}ca_{m_4+1}\ldots a_m}(t)\;\partial_i\mathcal{F}^b{}_{a_{m_1+1}\ldots a_{m_2}}(t)\;\partial_j\mathcal{F}^c{}_{a_{m_3+1}\ldots a_{m_4}}(t)$$

Open/closed top. string consistency condition IIa

- Annulus factorization: topological Cardy condition



$$\partial_i \partial_j \partial_k \mathcal{F}(t)\ \eta^{kl}\ \partial_l \mathcal{F}_{a_0 a_1 \ldots a_m}(t) =$$

$$= \sum_{0 \le m_1 \le \ldots m_4 \le m} (-1)^s\ \mathcal{F}_{a_0 \ldots a_{m_1} b a_{m_2+1} \ldots a_{m_3} c a_{m_4+1} \ldots a_m}(t)\ \partial_i \mathcal{F}^b{}_{a_{m_1+1} \ldots a_{m_2}}(t)\ \partial_j \mathcal{F}^c{}_{a_{m_3+1} \ldots a_{m_4}}(t)$$

Open/closed top. string consistency conditions III

- These consistency conditions express abstract TFT sewing axioms [Moore-Segal, Getzler,..];

  presence of integrated insertions:
  ...issues of compactification of moduli space;

$$\sum_{k=2}^{m-1} (-1)^{s_k} \left\langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_2} \ldots \int \partial_{\tau_k} \psi_{a_k} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \right\rangle \tag{69}$$

$$= \sum_{k=2}^{m-1} (-1)^{s_k} \int_{\tau_1}^{\tau_m} d\tau_k \Big(\partial_{\tau_k} \langle \psi_{a_0} \psi_{a_1} \int_{\tau_1}^{\tau_k} \psi^{(1)}_{a_2} \int_{\tau_2}^{\tau_k} \psi^{(1)}_{a_3} \ldots \int_{\tau_{k-2}}^{\tau_k} \psi^{(1)}_{a_{k-1}} \psi_{a_k} \int_{\tau_k}^{\tau_{k+2}} \psi^{(1)}_{a_{k+1}} \ldots \int_{\tau_k}^{\tau_m} \psi^{(1)}_{a_{m-1}} \psi_{a_m} \rangle$$

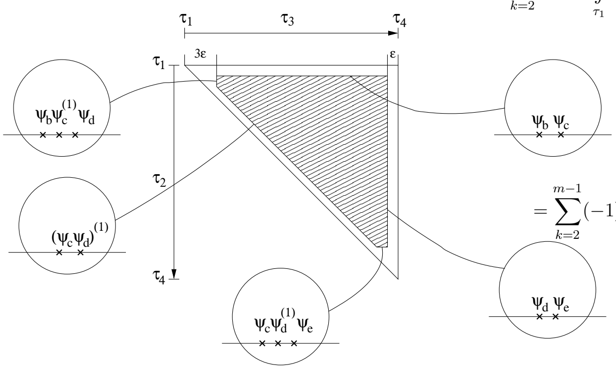
$$- \sum_{l=2}^{k-1} \langle \psi_{a_0} \psi_{a_1} \int_{\tau_1}^{\tau_k} \psi^{(1)}_{a_2} \ldots \Big[\psi^{(1)}_{a_l} \Big|_{\tau_l \to \tau_k} \int_{\tau_l}^{\tau_k} \psi^{(1)}_{a_{l+1}} \ldots \psi_{a_k} \Big] \int_{\tau_k}^{\tau_{k+2}} \psi^{(1)}_{a_{k+1}} \ldots \int_{\tau_k}^{\tau_m} \psi^{(1)}_{a_{m-1}} \psi_{a_m} \rangle$$

$$+ \sum_{l=k+1}^{m-1} \langle \psi_{a_0} \psi_{a_1} \int_{\tau_1}^{\tau_k} \psi^{(1)}_{a_2} \ldots \Big[\psi_{a_k} \ldots \int_{\tau_k}^{\tau_l} \psi^{(1)}_{a_{l-1}} \psi^{(1)}_{a_l} \Big|_{\tau_k \leftarrow \tau_l} \Big] \ldots \int_{\tau_k}^{\tau_m} \psi^{(1)}_{a_{m-1}} \psi_{a_m} \rangle \Big)$$

$$= \sum_{k=2}^{m-1} (-1)^{s_k} \Big(\langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_{k-1}} \Big[\psi_{a_k} \Big|_{\tau_k \to \tau_m} P \int \psi^{(1)}_{a_{k+1}} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Big] \rangle$$

$$- \langle \psi_{a_0} \Big[\psi_{a_1} P \int \psi^{(1)}_{a_2} \ldots \int \psi^{(1)}_{a_{k-1}} \psi_{a_k} \Big|_{\tau_1 \leftarrow \tau_k} \Big] P \int \psi^{(1)}_{a_{k+1}} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \rangle$$

$$- \sum_{l=2}^{k-1} \langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_2} \ldots \int \Big[\psi_{a_l} \Big|_{\tau_l \to \tau_k} P \int \psi^{(1)}_{a_{l+1}} \ldots \psi_{a_k} \Big]^{(1)} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \rangle \Big) ,$$



"associahedron"

Summary: open top. string factorization axioms

[Hofman, HLL]

WDVV: $\mathcal{F}_{ijm}\eta^{mn}\mathcal{F}_{nkl} \;=\; \mathcal{F}_{ikm}\eta^{mn}\mathcal{F}_{njl}$

A∞: $\sum_{\substack{k,j=0\\ k\le j}}^{m}(-1)^{\tilde a_1+\ldots+\tilde a_k} r_{m-j+k}(\psi_{a_1}\ldots\psi_{a_k}, r_{j-k}(\psi_{a_{k+1}}\ldots\psi_{a_j}),\psi_{a_{j+1}}\ldots\psi_{a_m}) = 0$

Crossing: $\partial_i\partial_j\partial_k\mathcal{F}(t)\;\eta^{kl}\;\partial_l\mathcal{F}_{a_0a_1\ldots a_m}(t) =$

$$= \sum_{0\le m_1\le\ldots m_4\le m}(-1)^s\;\mathcal{F}_{a_0\ldots a_{m_1}ba_{m_2+1}\ldots a_{m_3}ca_{m_4+1}\ldots a_m}(t)\;\partial_i\mathcal{F}^b{}_{a_{m_1+1}\ldots a_{m_2}}(t)\;\partial_j\mathcal{F}^c{}_{a_{m_3+1}\ldots a_{m_4}}(t)$$

Cardy: $\partial_i\mathcal{F}_{a_0\ldots a_n}\eta^{ij}\partial_j\mathcal{F}_{b_0\ldots b_m} =$

$$= \sum_{\substack{0\le n_1\le n_2\le n\\ 0\le m_1\le m_2\le m}}(-1)^{s+\tilde c_1+\tilde c_2}\;\omega^{c_1d_1}\;\omega^{c_2d_2}\;\mathcal{F}_{a_0\ldots a_{n_1}d_1b_{m_1+1}\ldots b_{m_2}c_2a_{n_2+1}\ldots a_n}\;\mathcal{F}_{b_0\ldots b_{m_1}c_1a_{n_1+1}\ldots a_{n_2}d_2b_{m_2+1}\ldots b_m}$$

- This is an (in general) infinite system of differential and algebraic equations... can we ever hope to (recursively) solve them explicitly for a given model ?

Apart from spectrum, we need extra input, in particular the three-point functions.... ⟹ Landau-Ginzburg theory

Landau-Ginzburg description of B-type D-branes

[K,O,K-L,B-H-L-S]

- Consider bulk LG model with superpotential:

$$\int_\Sigma d^2z d\theta^+ d\theta^-\; W_{LG}(\Phi) + \mathrm{cc.}$$

B-type SUSY variations induce boundary ("Warner")-term:

$$\int_\Sigma d^2z d\theta^+ d\theta^-(\bar Q_+ + \bar Q_-)\,W_{LG} = \int_\Sigma d^2z d\theta^+ d\theta^-(\theta^+\partial_+ + \theta^-\partial_-)W_{LG}$$
$$= \int_{\partial\Sigma} dx d\theta\, W_{LG}$$

- Restore SUSY by adding boundary fermions $\Pi = (\pi + \theta^+\ell)$

(... not quite chiral: $\bar D\,\Pi \;=\; E(\Phi)|_{\partial\Sigma}$)

via a boundary potential: $\delta S \;=\; \int_{\partial\Sigma} dx d\theta\,\Pi\,J(\Phi)$

Condition for SUSY: $J(\Phi)E(\Phi) \;=\; W_{LG}(\Phi)$

Matrix factorizations

- Physical open string spectrum: determined by the cohomolgy of the BRST operator:

$$\mathcal{Q} = \bar{\partial} + Q_\partial$$

$$Q_\partial = \pi J + \bar{\pi} E = \begin{pmatrix} & J \\ E & \end{pmatrix}$$

$$1/2\, Q_\partial \cdot Q_\partial = W_{LG} 1$$

- Generalization for n LG fields: need N=2^n boundary fermions, and

$$J \cdot E = E \cdot J = W_{LG}\, 1_{N\times N}$$

Category of Matrix factorizations is isomorphic to D(Coh(M)) = category of B-type D-branes ! [Orlov]

- Physical interpretation: N... Chan-Paton labels of space-filling D$\overline{\text{D}}$ pairs

Boundary potentials J,E are tachyon profiles that describe condensation to given B-type D-brane configuration [Kapustin-Li, Lazaroiu]

Toy example I: minimal models (type A_{k+2})

- Bulk superpotential (at "level k"):

$$W_{LG}(x) = \tfrac{1}{k+2}\, x^{k+2}$$

D0-branes M_l are described by all the possible polynomial factorizations:

$$M_\ell : \quad J(x) = x^{\ell+1}\,, \qquad E(x) = \tfrac{1}{k+2}\, x^{k-\ell+1}\,, \qquad \ell = -1, 0, \ldots, [k/2]$$

Precisely matches results obtained in BCFT !

- Same is true for the open string spectrum, described by matrices that belong to the non-trivial cohomology of the BRST operator:

$$Q_\partial = \begin{pmatrix} & x^{\ell+1} \\ \frac{1}{k+2}x^{k-\ell+1} & \end{pmatrix} \qquad \Psi : \{Q_\partial, \Psi\} = 0, \quad \Psi \neq \{Q_\partial, \Lambda\}$$

Physical spectrum: Q-cohomology

- Boundary preserving physical fields $\mathcal{E} \sim \mathrm{Hom}(M_\ell, M_\ell)$:
  x, ω = even/odd generators of boundary ring

$$\mathcal{E} \in \mathcal{R}_\partial \equiv C[x,\omega] \Big/ (x^{\ell+1} = 0,\ \omega^2 = x^{k-2\ell})$$

fields	deformation parameters	Q-exact
$\phi_i = \{1, x, \ldots, x^k\}$	$\{t_{k+2}, t_{k+1}, \ldots t_2\}$	$\partial_x W_{LG} \sim 0$
$\phi_a = \{1, x, \ldots, x^\ell\}$	$\{v_{(k+2)/2}, \ldots, v_{(k+2)/2-\ell}\}$	$\gcd(J,E) \sim 0$
$\psi_a = \omega \otimes \{1, x, \ldots, x^\ell\}$	$\{u_{\ell+1}, u_\ell, \ldots, u_1\}$	$\gcd(J,E) \sim 0$

- Boundary changing fields $\Psi_{\ell_1,\ell_2} \sim \mathrm{Hom}(M_{\ell_1}, M_{\ell_2})$ between M_{ℓ_1} and M_{ℓ_2}:

fields	parameters	Q-exact
$\phi_a^{\ell_1,\ell_2} = \beta^{\ell_1,\ell_2} \otimes \{1, x, \ldots, x^{\ell_{12}}\}$	$\{v_a^{[\ell_1,\ell_2]}\}$	$\gcd(J_i, E_i) \sim 0$
$\psi_a^{\ell_1,\ell_2} = \omega^{\ell_1,\ell_2} \otimes \{1, x, \ldots, x^{\ell_{12}}\}$	$\{u_a^{[\ell_1,\ell_2]}\}$	$\gcd(J_i, E_i) \sim 0$

$(\ell_{12} \equiv \min(\ell_1, \ell_2))$

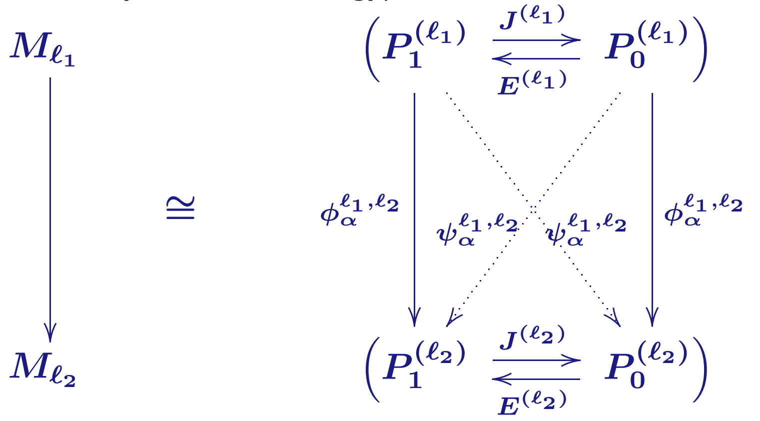
Kontsevich's triangulated category C_W

The LG model provides a concrete physical realization of a certain Z_2-graded category C_W: all maps have explicit LG representatives

- objects: D0 branes (composites out of D2$\overline{\text{D2}}$ branes): $\ell = 1, .., [k/2]$

$$M_\ell \cong \left(P_1^{(\ell)} \underset{E^{(\ell)}}{\overset{J^{(\ell)}}{\rightleftarrows}} P_0^{(\ell)} \right), \qquad J^{(\ell)} E^{(\ell)} = W$$

- morphisms (boundary Q-cohomology):



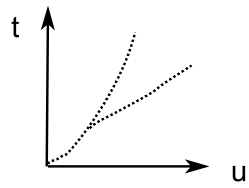
Deforming the minimal models

Consider infinitesimal perturbations:

$$\delta W_{LG}(x) = -\sum_{i=0}^{k} t_{k+2-i} x^i$$

$$\delta J(x) = -\sum_{a=0}^{\ell} u_{\ell+1-a} x^a$$

$$\delta E(x) = -x^{k-2\ell}\left(\sum_{a=0}^{\ell} u_{\ell+1-a} x^a\right)$$



Generic effects:

- Spoils factorization, so SUSY will be broken; may be restored along sub-manifolds.
- Along those, open string spectrum truncates (“boundary flow”)
- Branes may form composites

Applying the A∞ etc. consistency constraints

Disk correlators are uniquely determined once all constraint eqs are imposed!

Concise result for generating function:

$$\mathcal{W}_{\text{eff}}(u,t) = \oint W_{LG}(x,t)\ \log\det J(x,u) = \operatorname{Tr} V(X(u),t)$$

$$V'(x,t) \equiv W_{\text{LG}}(x,t)$$
$$X(u) = \text{diag}(\ldots x_i(u)\ldots)$$
$$\det J(x,u) = \prod(x - x_i(u))$$

with flat bulk LG potential [DVV]: (cf Kontsevich matrix model)

$$W_{LG}(t) = \tfrac{1}{k+2}\sum_{i=0}^{k} g_{k+2-i}(t)\ x^i$$

and boundary potential (tachyon profile):

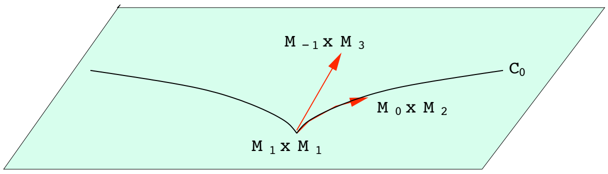
$$J = \begin{pmatrix} x^{\ell_1+1} - \sum_{\alpha=0}^{\ell_1} u^{[11]}_{\ell_1+1-\alpha} x^\alpha & -\sum_{\gamma=0}^{\ell_{12}} u^{[12]}_{\frac{1}{2}(\ell_1+\ell_2)+1-\gamma} x^\gamma \\ -\sum_{\gamma=0}^{\ell_{21}} u^{[21]}_{\frac{1}{2}(\ell_1+\ell_2)+1-\gamma} x^\gamma & x^{\ell_2+1} - \sum_{\alpha=0}^{\ell_2} u^{[22]}_{\ell_2+1-\alpha} x^\alpha \end{pmatrix}$$

Properties of the open/closed "moduli" space

$$\frac{\partial}{\partial u}\mathcal{W}_{\rm eff}(t,u) = \frac{\partial}{\partial u}\oint \log\,\det[J(x,u)]\cdot W_{LG}(x,t)$$
$$= -\oint {\rm Tr}\big[E\frac{\partial}{\partial u}J\big] \qquad E :\equiv W_{LG}/J$$
$$= 0 \quad \text{on factorization locus where } (E)_- = 0$$

matrix factorization locus = critical locus of effective superpotential!

... allows to systematically map out vacuum manifold and study composite formation ("topol. tachyon condensation") along it



Composite formation via boundary flows

- Boundary changing tachyon profile:

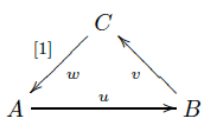
$$J = \begin{pmatrix} J_{\ell_1} & u_{12}\Psi_{12} \\ 0 & J_{\ell_2} \end{pmatrix}$$

- Physical realization of the "cone" construction:

triangle: $M_{\ell_1} \xrightarrow{u=u_{12}} M_{\ell_2} \longrightarrow C(u) \longrightarrow M_{\ell_1}[1]$

cone: $C(u) = \Big(P_1^{(\ell_1)} \oplus P_1^{(\ell_2)} \underset{E(u)}{\overset{J(u)}{\rightleftarrows}} P_0^{(\ell_1)} \oplus P_0^{(\ell_2)}\Big)$

- Reproduces flow patterns known from BCFT:



$$M_{\ell_1} \oplus M_{\ell_2} \xrightarrow{u_{12}\neq 0} M_{\ell+j+1} \oplus M_{\ell-j-1}$$

$$\Big(M_\ell\Big)^{\oplus N} \xrightarrow{\{u^{1,N}_{N-1},u^{2,N-1}_{N-2}\ldots u_1^{N/2,N/2+1}\}} \bigoplus_{\ell'} C_{N+2,\ell}{}^{\ell'} M_{\ell'}\ .$$

SU(2) fusion coeffs

Now on to a more interesting example .....

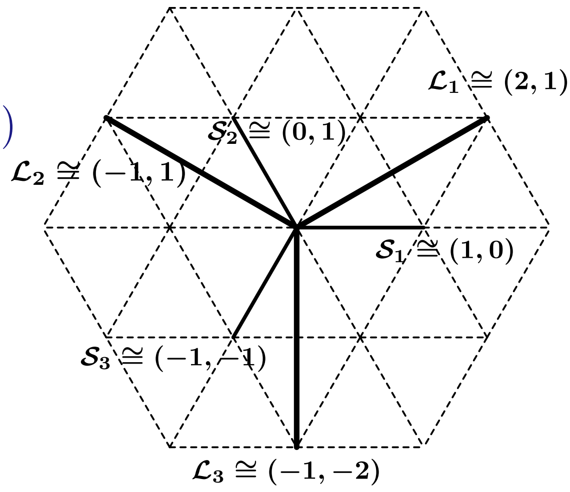
Toy model II: D-branes on the elliptic curve

- Simplest Calabi-Yau: the cubic torus

$$T_2 : \quad W_{LG} \equiv {x_1}^3 + {x_2}^3 + {x_3}^3 + a\, x_1 x_2 x_3 = 0$$

- B-type D-branes are composites of D2, D0 branes, characterized by $(N_2, N_0) = (\mathrm{rank}(V), c_1(V))$
  (NB: depending on Kähler parameters)

... these are mirror to A-type D1-branes with wrapping numbers $(p, q) = (N_2, N_0)$



- The "short diagonals" S are related to 2x2 factorizations, while the "long diagonals" L are described by 3x3 (4x4) factorizations

3x3 matrix factorization

- Simplest are the factorizations corresponding to the long diagonals L_i (i=1,2,3)

[Hori, Walcher]

$$J_i = \begin{pmatrix} \alpha_1^{(i)} x_1 & \alpha_2^{(i)} x_3 & \alpha_3^{(i)} x_2 \\ \alpha_3^{(i)} x_3 & \alpha_1^{(i)} x_2 & \alpha_2^{(i)} x_1 \\ \alpha_2^{(i)} x_2 & \alpha_3^{(i)} x_1 & \alpha_1^{(i)} x_3 \end{pmatrix}$$

$$E_i = \begin{pmatrix} \frac{1}{\alpha_1^{(i)}} {x_1}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_2 x_3 & \frac{1}{\alpha_3^{(i)}} {x_3}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_1 x_2 & \frac{1}{\alpha_2^{(i)}} {x_2}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_1 x_3 \\ \frac{1}{\alpha_2^{(i)}} {x_3}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_1 x_2 & \frac{1}{\alpha_1^{(i)}} {x_2}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_1 x_3 & \frac{1}{\alpha_3^{(i)}} {x_1}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_2 x_3 \\ \frac{1}{\alpha_3^{(i)}} {x_2}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_1 x_3 & \frac{1}{\alpha_2^{(i)}} {x_1}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_2 x_3 & \frac{1}{\alpha_1^{(i)}} {x_3}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_1 x_2 \end{pmatrix}$$

These satisfy $J_i E_i = E_i J_i \; = \; W_{LG} \mathbf{1}$
if the parameters satisfy the cubic equation themselves:

$$W_{LG}(\alpha_i) \;\equiv\; {\alpha_1}^3 + {\alpha_2}^3 + {\alpha_3}^3 + a(\tau)\,\alpha_1\alpha_2\alpha_3 \;=\; 0$$

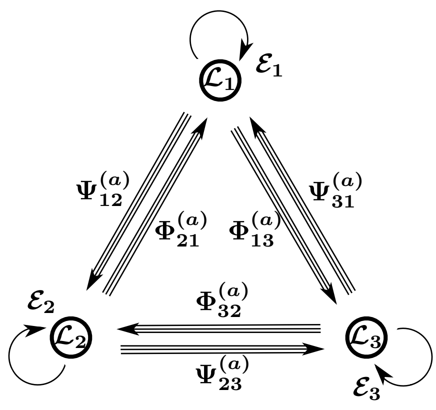
Thus the parameters parametrize the (jacobian) torus and can be represented by theta-sections:

$$\alpha_\ell^{(i)} \sim \Theta\left[\tfrac{1-\ell}{3} - \tfrac{1}{2} - \tfrac{1}{2} \,\middle|\, 3u_i, 3\tau\right]$$

u, τ ...flat coordinates of open/closed moduli space (natural in mirror A-model)

Open string BRST cohomology

Solving for the BRST cohomology yields explicit t,u-moduli dependent, "flat" matrix valued maps, eg (a=1,2,3):



- q=1 marginal operators corr. to brane locations

$$\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_i): \quad \mathcal{E}_i \;=\; \partial_{u_i} Q(u_i)$$

- q=1/3 tachyon operators

$$\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j): \quad \Psi_{ij}^{(a)} \;=\; \begin{pmatrix} 0 & F_{ij}^{(a)} \\ G_{ij}^{(a)} & 0 \end{pmatrix}$$

with eg, $F_{12}^{(1)} = \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & \zeta_2 \\ 0 & \zeta_3 & 0 \end{pmatrix}$ $G_{12}^{(1)} = \begin{pmatrix} \frac{\zeta_1}{\alpha_1^{(1)}\alpha_1^{(2)}} x_1 & \frac{\zeta_3}{\alpha_1^{(1)}\alpha_2^{(2)}} x_2 & \frac{\zeta_2}{\alpha_1^{(1)}\alpha_3^{(2)}} x_3 \\ \frac{\zeta_2}{\alpha_1^{(2)}\alpha_3^{(1)}} x_2 & \frac{\zeta_1}{\alpha_2^{(2)}\alpha_3^{(1)}} x_3 & \frac{\zeta_3}{\alpha_3^{(1)}\alpha_3^{(2)}} x_1 \\ \frac{\zeta_3}{\alpha_1^{(2)}\alpha_2^{(1)}} x_3 & \frac{\zeta_2}{\alpha_2^{(1)}\alpha_2^{(2)}} x_1 & \frac{\zeta_1}{\alpha_2^{(1)}\alpha_3^{(2)}} x_2 \end{pmatrix}$

and $\zeta_\ell \sim \Theta\left[\tfrac{1-\ell}{3} - \tfrac{1}{2} - \tfrac{1}{2} \,\middle|\, 3u_2 - 3u_1, 3\tau\right]$

Superpotential on brane intersection

Compute 3-point disk correlators = Yukawa couplings

$$\mathcal{W}_{\text{eff}} \sim C_{abc}(u_i,\tau)\, T_{13}^{(a)} T_{32}^{(b)} T_{21}^{(c)} + \ldots$$

$$\begin{aligned} C_{abc}(u_1,u_2,u_3) &= \langle \Psi_{13}^{(a)}(u_1,u_3)\Psi_{32}^{(b)}(u_3,u_2)\Psi_{21}^{(c)}(u_2,u_1)\rangle \\ &= \frac{1}{2\pi i}\oint \mathrm{Str}\Big[(\frac{dQ}{dW})^{\otimes\wedge 3}\Psi_{13}^{(a)}\Psi_{32}^{(b)}\Psi_{21}^{(c)}\Big] \end{aligned}$$

Final result:

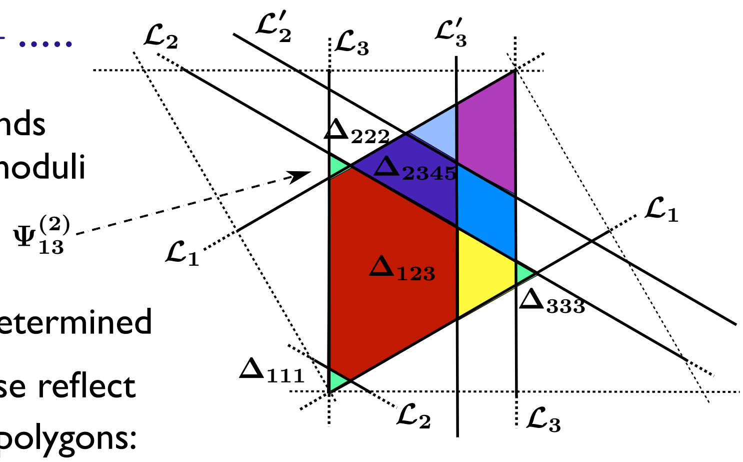
$$\begin{aligned} C_{111}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2} q^{3\xi_2{}^2/2} \sum_m q^{3m^2/2} e^{6\pi i m\xi} \\ C_{123}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2} q^{3\xi_2{}^2/2} \sum_m q^{3(m+1/3)^2/2} e^{6\pi i(m+1/3)\xi} \\ C_{132}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2} q^{3\xi_2{}^2/2} \sum_m q^{3(m-1/3)^2/2} e^{6\pi i(m-1/3)\xi} \end{aligned}$$

$(\xi \equiv u_1+u_2+u_3 = \xi_1+\tau\xi_2)$ (Cremades et al)

A-Model instantons

- Interpretation of q-series: In A model mirror language, these are contributions from triangular disk instantons whose world-sheets are bounded by the three D1-branes:

$$C_{abc} \sim e^{-S_{\text{inst}}} \sim q^{\Delta_{abc}} + \ldots..$$



- The u-dependence corresponds to position and Wilson line moduli
- Higher point functions are determined by the A∞ relations, and these reflect the composition of areas of polygons:

$$\langle \Psi_{12}(u_1,u_2)\Phi_{21}(u_2,u_1')\Psi_{12}(u_1',u_2')\Phi_{21}(u_2',u_1)\rangle$$
$$\sim \langle \Psi_{12}\Psi_{23}\Psi_{31}\rangle \langle \Phi_{32}\Phi_{21'}\Psi_{1'2}\Phi_{21}\Psi_{13}\rangle + \ldots$$

[HLN, to appear]

Homological mirror symmetry

- What's the underlying reason why this works ?
It is a reflection of the mirror symmetry between D0, D2 branes on the torus, and D1-branes on the dual torus.
- In mathematical terms, it reflects "homological mirror symmetry between the derived categegory of coherent sheaves on T_2, and the Fukaya category related to special lagrangian 1-cycles" [Kontsevich]
- This is in turn known to be tied to certain identities between theta-functions (addition formulas), which represent both the morphisms and the "Yukawa"-couplings [Polishchuk]:

$$\theta_a[u_1]\cdot\theta_b[u_2] \;=\; \sum \theta_{a-b+c}[u_1-u_2]\;\theta_{a+b+c}[u_1+u_2]$$

- In the LG approach, these theta-sections are carried by the matrix valued boundary fields in just the "right" combinations such as to reproduce the relevant addition formulas:

$$\Psi^{(a)}_{12}\cdot\Psi^{(b)}_{23} \;=\; \sum C_{abc}\,\Phi^{(c)}_{13}$$

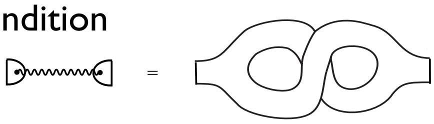
Beyond A_∞ ?

- disk correlators = theta-functions (in general, indefinite)

Thus they obey the heat equation:

$$\left[\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial u^2}\right] C_{a_0,\ldots,a_n}(u_i) \;=\; 0$$

Seems to arise from Cardy condition
(work in progress)

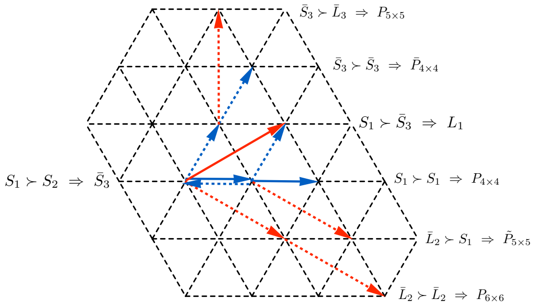


- Reflects reconstruction of closed string sector in terms of open string sector
- Interpretation in terms of "background independence" ?

Tachyon Condensation

(to appear)

- One can systematically apply the cone construction, and generate matrix factorization corresponding to branes with higher RR charges (rank(V),c1(V)) as composites out from a generating set



$$M_A \xrightarrow{\Psi} M_B \longrightarrow C(u) \longrightarrow M_A[1]$$

choice of tachyon determines outcome

- This is much more than just adding RR charges, due to the explicit moduli dependence of the matrices

- ... some interesting phenomena happen, eg bound states at threshold

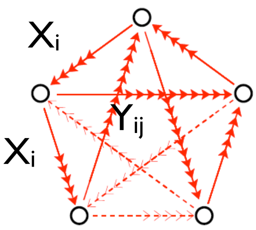
Conclusions and Outlook

- math: Cat of matrix factorizations $\longleftrightarrow$ D(Coh(M))
  phyz: Boundary LG theory $\longleftrightarrow$ Open string topological CFT

- Represent all quantities in a quiver diagram (objects and maps) by explicit moduli-dependent, matrix-valued operators

- Combined with mirror symmetry this allows to explicitly compute instanton-corrected superpotentials (in particular, for intersecting brane configs).

- Generalization to M = CY 3-folds, eg quintic:



$$\mathcal{W}_{eff} = C(t)\,\mathrm{Tr}XXY + B(t)\,\mathrm{Tr}(XXY)^2 + \ldots$$

t... Kähler modulus, interpolates between Gepner-point (BCFT) and large radius

... expect new results in enumerative geometry