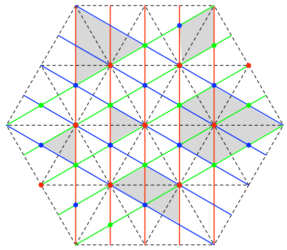
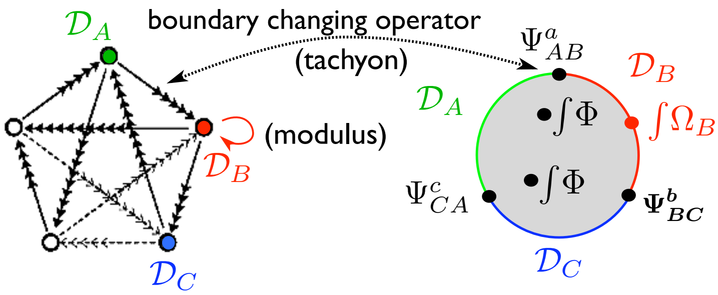
Overview of Part II

- Study "homological mirror symmetry" between A- and B-type of topological D-branes
- Matrix factorizations and D-branes on elliptic curve
- Practical application:
  effective superpotential for intersecting brane configurations
- Fun with instanton geometry



Motivation



Quiver diagram   Disk world sheet in TCFT

Superpotential ~ closed paths in quiver

$$\mathcal{W}_{eff} = T_a T_b T_c \underbrace{\langle \Psi^a_{AB}\Psi^b_{BC}\Psi^c_{CA}\rangle}_{C_{abc}(t,u)} + T_a T_b T_c T_d \underbrace{\langle \Psi^a_{AB}\Psi^b_{BC}\Psi^c_{CD}\Psi^d_{DA}\rangle}_{C_{abcd}(t,u)} + \ldots$$

tachyons

closed and open string moduli $\sim \mathrm{const} + \mathcal{O}(e^{-t}, e^{-u})$

instanton corrections

Relation with Part I

How to compute W_{eff} ?

Have seen that all TFT operators (~maps in category language) can be given an explicit matrix representation ....

Important ingredient: moduli dependence of all quantities

→ Make use of LG formulation of B-type branes in terms of matrix factorizations, in combination with mirror symmetry

Recap mirror symmetry

- For "every" Calabi-Yau X, there exists a mirror Y such that the Kähler and complex structure sectors are exchanged.

  Accordingly, it exchanges the topological A-model (which depends on Kahler moduli only), with the topological B-model (depending on complex structure moduli only).

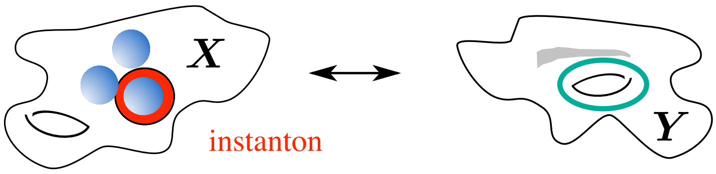
  The physical meaning of mirror symmetry is:

  Type IIA strings compactified on X are indistinguishable from Type IIB strings compactified on the mirror Y

Why is mirror symmetry useful?

- Mirror symmetry also maps even dimensional, holomorphic cycles into the lagrangian 3-cycles

Type IIA string | Type IIB string



size of **2-cycles** governed by Kahler param t_i | size of **3-cycles** governed by CS param

$$\int_{\Gamma_i^{2k}} (\omega_{1,1}^X)^k + \ldots \quad = \quad \int_{\Gamma_a^3} \Omega_{3,0}^Y$$

$$\sim\ t^k + \mathcal{O}(e^{-t}) \qquad\qquad \sim\ (\ln z)^k + \mathcal{O}(z)$$

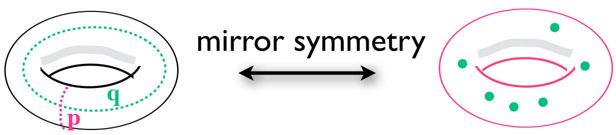
corrected | **exact**

... Do exact computation in topological B-model (LG-model), and via mirror symmetry this will give non-perturbative information about the A-model (sigma-model).

Homological mirror symmetry

- Mirror symmetry acts between full categories descr. A- and B-branes!

A-Model | B-Model



D1 branes on (p,q) cycles | (N_2,N_0) = (r,c_1) of "gauge bundle"

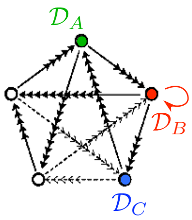
... Fukaya category of lagrangian cycles | ... derived category of coherent sheaves

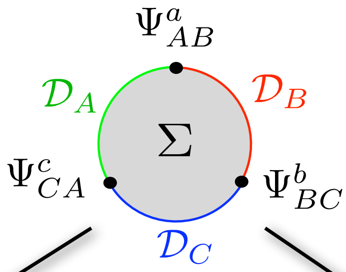
- There is much more to categories than just some diagrams with dots and arrows:

Besides RR charges (p,q)=(r,c_1), keep information about the moduli (eg positions) u of the branes

TFT correlation functions

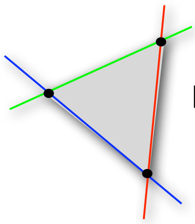
Disk amplitude for intersecting branes $C_{abc}(\tau; u_A, u_B, u_C) = \langle \Psi^a_{AB} \Psi^b_{BC} \Psi^c_{CA} \rangle$





A-Model
localizes on holomorphic maps:
world-sheet instantons $\Sigma \longrightarrow T_2$

B-Model
(Landau-Ginzburg model)
localizes on constant maps:
classical



Fukaya products

$C_{abc} \sim e^{-S_{\text{Inst}}}$

$S_{\text{Inst.}} \sim \text{Area}$

⟵ mirror symmetry ⟶

Massey products

$\lambda_m(\Psi^{\otimes m}) \;=\; \Psi_{a_0} C^{a_0}_{a_1 \ldots a_m}$

D-branes on the elliptic curve

- Simplest Calabi-Yau: the cubic torus

compl str modulus

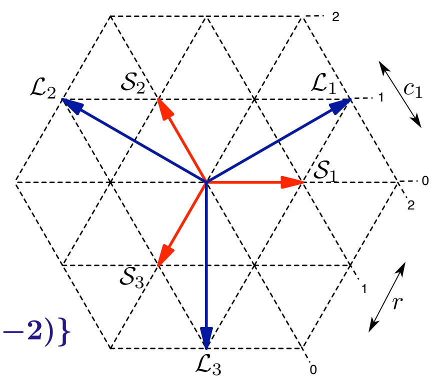
$$T_2 : \quad W_{LG} \;\equiv\; {x_1}^3 + {x_2}^3 + {x_3}^3 + a(\tau)\, x_1 x_2 x_3 \;=\; 0$$

- B-type D-branes are composites of D2, D0 branes, characterized by $(N_2, N_0; u) \;=\; (\text{rank}(V), c_1(V); u)$

  ... these are mirror to A-type D1-branes with wrapping numbers
  $(p, q) \;=\; (N_2, N_0)$

- The "short diagonals" S_A are related to 2x2 factorizations, while the "long diagonals" L_A are described by 3x3 factorizations

$(N_2, N_0)_{L_A} = \{(2,1), (-1,1), (-1,-2)\}$



3x3 matrix factorization

- Simplest are the factorizations corresponding to the long diagonals L_i (i=1,2,3)

$$J_i \;=\; \begin{pmatrix} \alpha_1^{(i)} x_1 & \alpha_2^{(i)} x_3 & \alpha_3^{(i)} x_2 \\ \alpha_3^{(i)} x_3 & \alpha_1^{(i)} x_2 & \alpha_2^{(i)} x_1 \\ \alpha_2^{(i)} x_2 & \alpha_3^{(i)} x_1 & \alpha_1^{(i)} x_3 \end{pmatrix}$$

$$E_i \;=\; \begin{pmatrix} \frac{1}{\alpha_1^{(i)}} {x_1}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_2 x_3 & \frac{1}{\alpha_3^{(i)}} {x_3}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_1 x_2 & \frac{1}{\alpha_2^{(i)}} {x_2}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_1 x_3 \\ \frac{1}{\alpha_2^{(i)}} {x_3}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_1 x_2 & \frac{1}{\alpha_1^{(i)}} {x_2}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_1 x_3 & \frac{1}{\alpha_3^{(i)}} {x_1}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_2 x_3 \\ \frac{1}{\alpha_3^{(i)}} {x_2}^2 - \frac{\alpha_3^{(i)}}{\alpha_1^{(i)}\alpha_2^{(i)}} x_1 x_3 & \frac{1}{\alpha_2^{(i)}} {x_1}^2 - \frac{\alpha_2^{(i)}}{\alpha_1^{(i)}\alpha_3^{(i)}} x_2 x_3 & \frac{1}{\alpha_1^{(i)}} {x_3}^2 - \frac{\alpha_1^{(i)}}{\alpha_2^{(i)}\alpha_3^{(i)}} x_1 x_2 \end{pmatrix}$$

These satisfy $J_i E_i = E_i J_i \;=\; W_{LG} 1$
if the parameters satisfy the cubic equation themselves:

$$W_{LG}(\alpha_i) \;\equiv\; {\alpha_1}^3 + {\alpha_2}^3 + {\alpha_3}^3 + a(\tau)\,\alpha_1\alpha_2\alpha_3 \;=\; 0$$

Thus the parameters parametrize the (jacobian) torus and can be represented by theta-sections:

$$\alpha_\ell^{(i)} \sim \Theta\left[\tfrac{1-\ell}{3} - \tfrac{1}{2} - \tfrac{1}{2} \,\middle|\, 3u_i, 3\tau\right]$$

$u,\ \tau$...flat coordinates of open/closed moduli space (Kahler moduli in mirror A-model)

Sections, bundles and matrices

- D-brane = bundle, sheaf V: $(N_2, N_0; u) = (r(V), c_1(V); u)$

How are these data encoded in the matrix factorization?

Associate bundles $\mathcal{E}$ with E,J (~bundles on "constituent branes")

$$\det E = W^r \qquad \mathcal{E}_E^{(r)} = \ker E$$
$$\det J = W^{n-r} \qquad \mathcal{E}_J^{(n-r)} = \ker J$$

Rougly, the Chern classes of $\mathcal{E}_J$ determine $(N_2, N_0; u)$

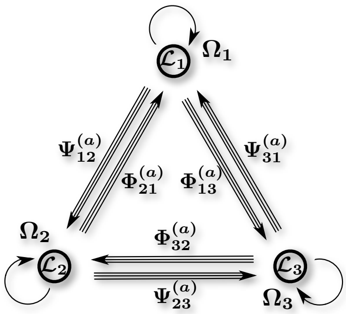
- Example: 3x3 factorization

$\det E = W$ line bundle

$\det J = W^2$ rank2 vector bundle

study transition function: $(r, c_1) = (2, -3)$

Back to 3x3: open string BRST cohomology



Solving for the BRST cohomology yields explicit t,u-moduli dependent matrix valued maps, eg (a=1,2,3):

- q=1 marginal operators corr. to brane locations

$$\mathrm{Ext}(\mathcal{L}_A, \mathcal{L}_A): \quad \Omega_A = \partial_{u_A} Q(u_A)$$

- q=1/3 tachyon operators

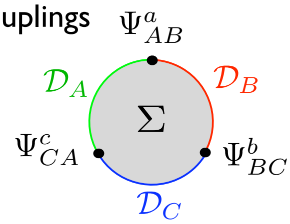
$$\mathrm{Ext}(\mathcal{L}_A, \mathcal{L}_B): \quad \Psi_{AB}^{(a)} = \begin{pmatrix} 0 & F_{AB}^{(a)} \\ G_{AB}^{(a)} & 0 \end{pmatrix}$$

with eg, $F_{12}^{(1)} = \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & \zeta_2 \\ 0 & \zeta_3 & 0 \end{pmatrix}$ $G_{12}^{(1)} = \begin{pmatrix} \frac{\zeta_1}{\alpha_1^{(1)}\alpha_1^{(2)}} x_1 & \frac{\zeta_3}{\alpha_1^{(1)}\alpha_2^{(2)}} x_2 & \frac{\zeta_2}{\alpha_1^{(1)}\alpha_3^{(2)}} x_3 \\ \frac{\zeta_2}{\alpha_1^{(2)}\alpha_3^{(1)}} x_2 & \frac{\zeta_1}{\alpha_2^{(2)}\alpha_3^{(1)}} x_3 & \frac{\zeta_3}{\alpha_3^{(1)}\alpha_3^{(2)}} x_1 \\ \frac{\zeta_3}{\alpha_1^{(2)}\alpha_2^{(1)}} x_3 & \frac{\zeta_2}{\alpha_2^{(1)}\alpha_2^{(2)}} x_1 & \frac{\zeta_1}{\alpha_2^{(1)}\alpha_3^{(2)}} x_2 \end{pmatrix}$

and $\zeta_\ell \sim \Theta\left[{}^{\frac{1-\ell}{3} - \frac{1}{2}}_{-\frac{1}{2}} \Big| 3u_2 - 3u_1, 3\tau\right]$

Superpotential on brane intersection I

- Compute 3-point disk correlators = Yukawa couplings



$$\mathcal{W}_{eff} = T_a T_b T_c \underbrace{\langle \Psi^a_{AB} \Psi^b_{BC} \Psi^c_{CA} \rangle}_{C_{abc}(\tau, u_i)} + \ldots$$

$$C_{abc}(\tau, u_1, u_2, u_3) = \langle \Psi^a_{13}(u_1, u_3) \Psi^b_{32}(u_3, u_2) \Psi^c_{21}(u_2, u_1) \rangle$$

Use super-residue formula (from localization in LG model) for our matrix-valued, moduli-dependent operators:

$$= \frac{1}{2\pi i} \oint Str\Big[(\frac{dQ}{dW})^{\otimes \wedge 3} \Psi^{(a)}_{13} \Psi^{(b)}_{32} \Psi^{(c)}_{21} \Big]$$

Superpotential on brane intersection II

- Make heavy use of theta-function identities such as the addition formula:

$$\theta_a[u_1]\cdot\theta_b[u_2] \;=\; \sum \theta_{a-b+c}[u_1-u_2]\;\theta_{a+b+c}[u_1+u_2]$$

(math: expresses product in Fukaya-category)

- Final result: theta functions

$$\begin{aligned}
C_{111}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2}q^{3\xi_2{}^2/2}\sum_m q^{3m^2/2}e^{6\pi i m\xi}\\
C_{123}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2}q^{3\xi_2{}^2/2}\sum_m q^{3(m+1/3)^2/2}e^{6\pi i(m+1/3)\xi}\\
C_{132}(\tau,\xi) &= e^{6\pi i\xi_1\xi_2}q^{3\xi_2{}^2/2}\sum_m q^{3(m-1/3)^2/2}e^{6\pi i(m-1/3)\xi}
\end{aligned}$$

$$(\xi \equiv u_1+u_2+u_3 = \xi_1+\tau\xi_2)$$

What's the interpretation of the q-series?

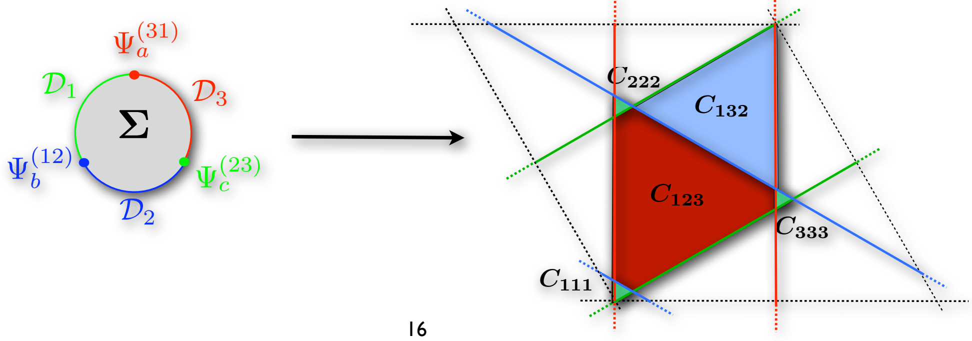
The topological A-Model: instantons

- Interpretation of q-series: In A model mirror language, these are contributions from triangular disk instantons whose world-sheets are bounded by the three D1-branes:

$$C_{abc} \sim e^{-S_{\rm inst}} \sim q^{\Delta_{abc}} + ...$$

(the u-dependence corresponds to position and Wilson line moduli)

Count maps: $\Sigma \longrightarrow T_2$



Complete effective potential (long diag branes)

$$\mathcal{W}_{\rm eff}(\tau,u,T) \;=\; \sum_{N=1}^{6} T^{(a_1)}\ldots T^{(a_N)}\, C^{(N)}_{a_1,\ldots,a_N}(\tau,u_1,\ldots,u_N)$$

↑ tachyons ↑ moduli

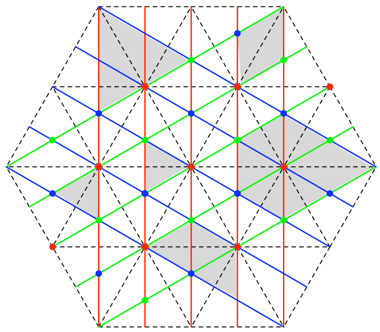
- B-model: difficult to compute higher N-point Massey products with N>3!

  For (flat) elliptic curve, A-model is simpler....

- Generically, N-point functions get contributions from N-gonal instantons

General structure:
indefinite theta-functions summing over all lattice translates, positive areas

$$\sum_{m,n}{}' q^{mn} \equiv \left(\sum_{m,n>0} - \sum_{m,n<0}\right) q^{mn}$$



Polygons and instantons

N=4: trapezoids

$$\mathcal{T}_{ab\bar{c}\bar{d}}(\tau,u_i) \;=\; \delta^{(3)}_{a+b,\bar{c}+\bar{d}}\,\Theta_{trap}\left[\begin{array}{c}[b-\bar{c}]_3\\ [\bar{d}-\bar{c}+3/2]_3\end{array}\right](3\tau|3(u_1+u_2+u_4),3(u_1-u_3))$$

$$\Theta_{trap}\left[\begin{array}{c}a\\ b\end{array}\right](3\tau|3u,3v) \;=\; \sum_{m,n}{}' q^{\frac{1}{6}(a+3n)(a+3n+2(b+3m))}e^{2\pi i\left((a+3n)(u-1/6)+(b+3m)v\right)}$$

N=4: parallelograms

$$\mathcal{P}_{a\bar{b}c\bar{d}}(\tau,u_i) \;=\; \delta^{(3)}_{a+c,\bar{b}+\bar{d}}\;\Theta_{para}\left[\begin{array}{c}[c-\bar{b}]_3\\ [\bar{d}-c]_3\end{array}\right](3\tau|3(u_1-u_3),3(u_4-u_2))$$

$$\Theta_{para}\left[\begin{array}{c}a\\ b\end{array}\right](3\tau|3u,3v) \equiv \sum_{m,n}{}' q^{\frac{1}{3}(a+3n)(b+3m)}e^{2\pi i\left((b+3m)u+(a+3n)v\right)}$$

N=5: pentagons

$$\wp_{a\bar{b}\bar{c}\bar{d}\bar{e}}(\tau,u_i) = \delta^{(3)}_{a,\bar{b}+\bar{c}+\bar{d}+\bar{e}}\;\Theta_{penta}\left[\begin{array}{c}[-b-c-d]_3\\ [e+c+d]_3\\ [c-d+\frac{3}{2}]_3\end{array}\right](3\tau|3(u_5-u_2),3(u_1-u_4),3(u_3+u_2+u_4))$$

$$\Theta_{penta}\left[\begin{array}{c}a\\ b\\ c\end{array}\right](3\tau|3u,3v,3w) \equiv \sum_{m,n,k}{}' q^{\frac{1}{3}(a_>+3(n+k))(b_>+3(m+k))-\frac{1}{6}(c+3k)^2}e^{2\pi i\left((a_>+3(n+k))u+(b_>+3(m+k))v+(c+3k)(w-1/6)\right)}$$

N=6: hexagons

$$\mathcal{H}_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}}(\tau,u_i) = \delta^{(3)}_{0,\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}}\Theta_{hexa}\left[\begin{array}{c}[-b-c-d]_3\\ [c+d+e]_3\\ [c-d+\frac{3}{2}]_3\\ [a-f+\frac{3}{2}]_3\end{array}\right](3\tau|3(u_5-u_2),3(u_1-u_4),3(u_3+u_2+u_4),3(-u_6-u_1-u_5))$$

$$\Theta_{hexa}\left[\begin{array}{c}a\\ b\\ c\\ d\end{array}\right](3\tau|3u,3v,3w,3z) \equiv \sum_{m,n,k,l}{}' q^{\frac{1}{3}(a+3n)(b+3m)-\frac{1}{6}(c+3k)^2-\frac{1}{6}(d+3l)^2}e^{2\pi i\left((a+3n)u+(b+3m)v+(c+3k)(w-1/6)+(d+3l)(z+1/6)\right)}$$

$$\sum_{m,n,k,l}{}' = \sum_{m,n\geq 0}^{\infty}\sum_{k\geq 0}^{<k_{max}}\sum_{l\geq 0}^{<l_{max}} - \sum_{m,n\leq -1}^{-\infty}\sum_{k\leq -1}^{>k_{min}}\sum_{l\leq -1}^{>l_{min}}$$

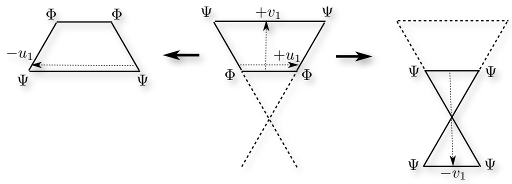
Global properties of open string moduli space

- Indefinite theta-fcts: singularities due to colliding branes

  eg., rewrite trapezoidal function in terms of Appel function:

$$\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau|3u,3v) \;=\; e^{2\pi i v b}\sum_{n\in Z}\frac{q^{\frac{1}{6}(a+3n)(a+2b+3n)}e^{2\pi i(a+3n)(u-1/6)}}{1-q^{a+3n}e^{6\pi i v}}$$

- analytic continuation



Area becomes negative: resum instantons in terms of different geometry

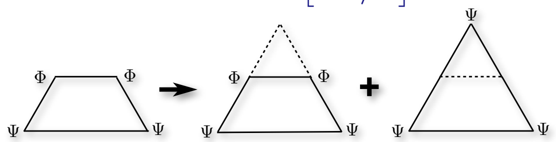
"instanton flop"

Global properties of open string moduli space

- monodromy

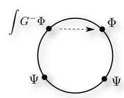
$$\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3(u\pm\tau),\,3v) \;=\; e^{\mp 6\pi i v}\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3u,\,3v)$$
$$\mp\, e^{-2\pi i(u-\frac{1}{6})(b-\frac{3}{2}\pm\frac{3}{2})}e^{2\pi i v(b-\frac{3}{2}\mp\frac{3}{2})}q^{-\frac{1}{6}(b-\frac{3}{2}\pm\frac{3}{2})^2}\Theta\left[\begin{array}{c} a+b \\ -3/2 \end{array}\right](3\tau\,|\,3u)$$

induces "homotopy transformation", modular anomaly of eff action (compensate by non-lin field redef)



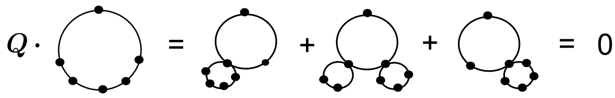
$$\mathcal{T}_{ab\bar{c}\bar{d}} \;\rightarrow\; \mathcal{T}_{ab\bar{c}\bar{d}} + f_{\bar{c}\bar{d}}{}^{e}\Delta_{abe}$$

contact term



Open/closed top. string consistency conditions

- How can we be sure that these expressions are correct ?
  Make use of Q-closedness and factorization constraints



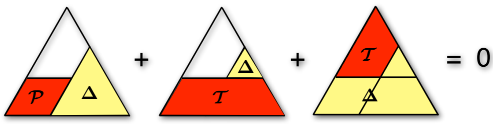
These lead to A∞ relations for correlators resp. Fukaya products:

$$\sum_{\substack{k,j=0\\k\leq j}}^{m}(-1)^{\tilde a_1+\ldots+\tilde a_k}\lambda_{m-j+k}(\psi_{a_1}\ldots\psi_{a_k},\lambda_{j-k}(\psi_{a_{k+1}}\ldots\psi_{a_j}),\psi_{a_{j+1}}\ldots\psi_{a_m})=0$$

$$\lambda_m(\Psi_{a_1}\ldots\Psi_{a_m})\ \equiv\Psi_{a_0}C^{a_0}_{a_1\ldots a_m}$$

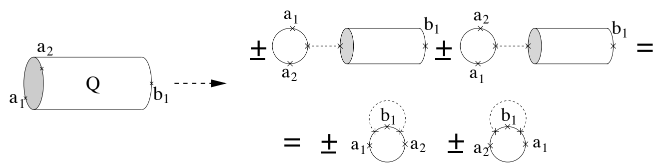
....here: simple interpretation in terms of instanton geometry:

$$\sum\mathcal{P}_{a_1\bar c a_4\bar a_5}\ \Delta_{c a_2 a_3}+\sum\mathcal{T}_{a_1a_2\bar c\bar a_5}\ \Delta_{c a_3 a_4}+\sum\Delta_{a_1a_2c}\ \mathcal{T}_{\bar c a_3 a_4\bar a_5}=0$$



Quantum A∞ relation for the annulus

- There are analogous factorization relations in higher genus, eg:



$$\sum_{c,d}((-)^{\tilde a_1+\tilde d\tilde a_2}\ C^{0,1}_{a_1ca_2}\ \eta^{cd}\ C^{0,2}_{d|\ b_1}+(-)^{\tilde a_1+\tilde a_2}\ C^{0,1}_{a_1a_2c}\ \eta^{cd}\ C^{0,2}_{d|\ b_1})$$

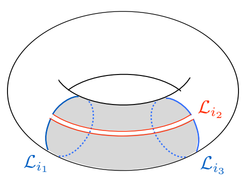
$$=\sum_{c,d}((-)^{\tilde a_1+\tilde b_1(\tilde d+\tilde a_2)}\eta^{cd}C^{0,1}_{a_1cb_1da_2}+(-)^{\tilde a_1+\tilde a_2+\tilde b_1\tilde d}\eta^{cd}C^{0,1}_{a_1a_2cb_1d})$$

- In concrete case, it boils down to an identity between disk and and annulus correlators:

$$\partial_{u_3}\mathcal{A}_{\Omega|\,\cdot}\ \equiv\ \partial_{u_3}\sum_{3\,n\neq0,m}{}'\,q^{nm}e^{6\pi i n(u_1-u_3)}$$

$$=\sum_{c=1}\partial_{u_3}\mathcal{P}_{a\bar c c\bar a}$$

This maps disk and annulus instantons into each other!



Conclusions and Outlook

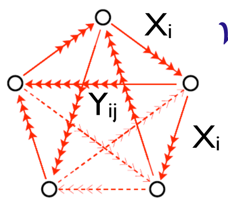
- math: Cat of matrix factorizations ⟷ D(Coh(M))

  phyz: Boundary LG theory ⟷ Open string topological CFT

- Represent all quantities in a quiver diagram (objects and maps) by explicit moduli-dependent, matrix-valued operators

- Combined with mirror symmetry this allows to explicitly compute instanton-corrected superpotentials (in particular, for intersecting brane configs).

- Generalization to M = CY 3-folds, eg quintic:



$$\mathcal{W}_{eff} = C_{XXY}(t)\,\mathrm{Tr}XXY + C_{XXYXXY}(t)\,\mathrm{Tr}(XXY)^2 + \ldots$$

t... Kähler modulus, interpolates between Gepner-point (BCFT) and large radius

... new results in enumerative geometry