Topological Landau-Ginzburg Models and 2d Gravity

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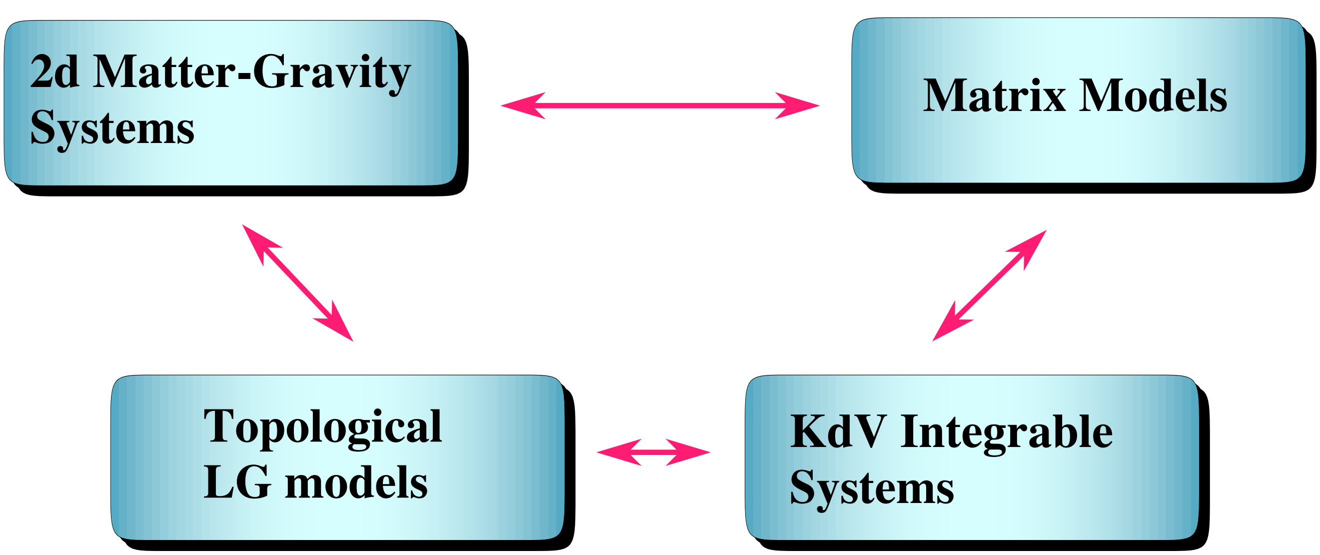
Motivation

There is some evidence for an intrinsic, general relationship between 2d conformal matter coupled to gravity, topological gravity, N=2 LG models, and integrable systems:



Topological LG Theory

(d=2 N=2 superspace)

$$\mathcal{L} \;=\; \int d^4\theta\, K(X_i, \bar{X}_i) + (\int d^2\theta\, W(X_i) + h.c.)$$

''almost'' free fields: $X_i(Z_1)\cdot X_j(Z_2) \;=\; -\delta_{ij}\log(Z_{12})$

$$(X = x + \theta^- \psi - \tfrac{1}{2}\theta^+\theta^-\partial x)$$

Super stress tensor $\mathcal{T} \;=\; \sum_i \left[\omega_i X_i D_- \Psi_i + (\omega_i - 1)(D_- X_i)\Psi_i\right]$

commutes with screener $\left[Q_W, \mathcal{T}\right] \;=\; 0\;,\;\; Q_W \;=\; \oint d\theta^- dz\, W(X_i)$

if superpotential is
quasi-homogenous: $W(\lambda^{\omega_i} X_i) \;=\; \lambda\, W(X_i)$

and has central charge : $c^{N=2} \;=\; 3\sum_i (1 - 2\omega_i)$

Under the topological twist $T \;\to\; T + \frac{1}{2}\partial J$

this becomes a TFT with $c^{N=2} \;\to\; c^{top} = 0$

Physical states

are determined in terms of the cohomology of $Q_s + Q_W,\; Q_s \equiv \oint d\theta^- dz\,\mathcal{T}$
and can be represented as a polynomials in terms of the LG fields.
This ''primary chiral ring'' $\mathcal{R}_x$ is finite because of:

$$\begin{aligned}\Phi(x) \;&=\; p_i(x)\partial_{x_i} W(x) \\ &=\; \left\{Q_s, \bar{\psi}_i p_i(x)\right\} \\ &=\; \left\{Q_s, \Lambda_\Phi\right\} \;=\; \text{null}\end{aligned}$$

$(\delta_s\bar{\psi} = \partial W$

For $W = x^{k+2}$, one has the N=2 minimal models with $c^{N=2} = \dfrac{3k}{k+2}$
and $\mathcal{R}_x \;=\; \left\{1, x, x^2, x^3, \ldots, x^k\right\}$

Topological Gravity (DVV)

Fields: Liouville (ϕ, ψ) + conjugates (π, χ) + ghosts (b, c), (β, γ)

$$
\begin{aligned}
Q_{BRST} &= Q_v + Q_s \\
Q_v &= \oint \left[c\ (T_l + \tfrac{1}{2} T_{gh}) + \gamma (G_l^- + \tfrac{1}{2} G_{gh}^-) \right] \\
Q_s &= \oint \left[G_l^+ + G_{gh}^+ \right]
\end{aligned}
$$

(from Hamiltonian reduction of SL(2|2) WZW model)

Physical operators:

''gravitational descendants'' $\sigma_n \equiv (\sigma_1)^n$ where

$$
\begin{aligned}
\sigma_1 &= \left\{ Q_s, \left[Q_v, \phi \right] \right\} - c.c. \\
&= \left\{ Q_s, \partial c + \ldots \right\} - c.c \\
&= \left\{ Q_s, \Lambda_\sigma \right\} - c.c \qquad \text{naively null !}
\end{aligned}
$$

Need to impose ''equivariant cohomology'':

$$
|\text{state}\rangle \in \mathcal{H} \longleftrightarrow (b_0 - \bar{b}_0)|\text{state}\rangle = 0
$$

Here: $$
\begin{aligned}
(b_0 - \bar{b}_0)|\,\Lambda_\sigma\,\rangle &= (b_0 - \bar{b}_0)(c_0 - \bar{c}_0 + \ldots)|\,0\,\rangle \\
&\neq 0
\end{aligned}
$$

thus $|\,\sigma_1\,\rangle \neq Q_s|\text{phys state}\rangle$

$\Longrightarrow$ σ_1 is physical

Top Matter coupled to top Gravity

Tensor product: $$Q_{BRST} \;=\; Q^{(m)}_{BRST} + Q^{(gr)}_{s} + \text{[illegible]}$$
$$\mathcal{R}_{x,\sigma} \;=\; \{x^l,\, l=0,\ldots,k\} \;\otimes\; \{(\sigma_1)^n\}$$

Similarity transf (EYK): $$S \;=\; \exp\big[\oint c(G^-_{(m)} + G^-_{(l)} + \tfrac{1}{2}G^-_{(ghost)})\big]$$

$$S\,\big(Q_{BRST}\big)\,S^{-1} \;=\; Q^{(m)}_{s} + Q^{(gr)}_{s} \;\equiv\; Q_s$$

BRST cohomology is isomorphic to cohomology of N=2 supercharge Q_s --> gravity factors out !

Equivariance condition then becomes

$$(b_0 - \bar{b}_0)\,|\text{state}\rangle \;=\; 0 \;\longrightarrow\; (b_0 + G^-_0 - c.c)\,|\text{state}\rangle \;=\; 0$$

Reconsider
$$\begin{aligned} \Phi(x) &= p(x)\partial_x W(x) \\ &= \{Q_s, \bar{\psi}p(x)\} \\ &= \{Q_s, \Lambda_\Phi\} \end{aligned}$$

$\downarrow$ test

$$\begin{aligned} \{b_0 + G^-_0 - c.c, \Lambda_\Phi\} &= \{G^-_{0(m)} - c.c,\, \bar{\psi}p(x)\} \\ &= \partial_x p(x) \\ &\equiv: \mathcal{K}_1(\Phi) \end{aligned}$$

Recursion operator (contact algebra)
$$\Phi \;\underset{\sigma_1}{\overset{\mathcal{K}_1}{\leftrightarrows}}\; \mathcal{K}_1(\Phi)$$

$$\begin{aligned} \mathcal{K}_1(\Phi) &= \text{null} \longrightarrow \Lambda_\Phi \text{ is phys} \longrightarrow \Phi \text{ is null} \\ \mathcal{K}_1(\Phi) &= \text{phys} \longrightarrow \Lambda_\Phi \text{ is null} \longrightarrow \Phi \text{ is phys} \end{aligned}$$

$\Rightarrow$ When requiring equivariant cohomology, all x^n become physical, except for:

$$\Phi(x) \;=\; x^{k+1+n(k+2)} \;=\; \sigma_n(\partial_x W(x))$$

In particular, interpret

$$W(x) \;=\; x^{k+2} \;=\; \sigma_1$$

Prove via

$$\left(x^{k+2} - \sigma_1\right) \;=\; \left\{Q_s,\, \bar\psi x - \partial c - \ldots\right\} - c.c$$

$\downarrow$ test

$$\left\{b_0 + G_0^-,\, \bar\psi x - \partial c - \ldots\right\} \;=\; 1 - 1 \;=\; 0$$

$\Rightarrow$ $\left(x^{k+2} - \sigma_1\right)$ is null in equivariant cohomology

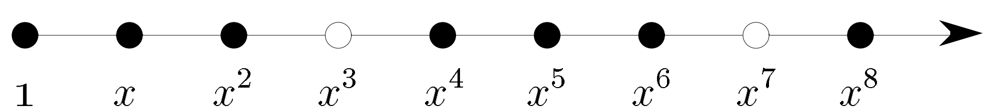
The matter-gravity system can be represented entirely in terms of the topological Landau-Ginzburg theory !

$$\mathcal{R}_{x,\sigma} \;=\; \left\{x^l \;\mathrm{mod}\;\; [x^{k+1+n(k+2)} = 0]\,\right\}$$

(Example: pure topological gravity associated with "trivial" ($c^{N=2} = 0$) superpotential:

$$\mathrm{W(x)} = \mathrm{x}^2\;, \qquad \mathcal{R}_\sigma = \left\{x^{2l}\right\}$$

)



"gravitational chiral ring"

Same pattern as KdV flows of the underlying matrix models !

⟹ Can compute all correlators of top. matter+gravity in terms of N=2 LG theory.

Non-trivial are the correlators in the perturbed theory

$$W(x,g(t)) \;=\; x^{k+2} - \sum_{i=1}^{k} g_{i+2}(t) x^{k-i}$$

as functions of t = flat coordinates of deformation space
= KdV times

Theory is solved once the Landau-Ginzburg couplings g(t) are known. They are determined by the Gauss-Manin system

$$-\frac{\partial}{\partial t_{i+2}} W(x,g(t)) \;=\; \frac{\partial}{\partial_x}\Omega_{k+1-i}(x,g(t))$$

with Hamiltonians $\Omega(x,g)_i \;=\; \big(W(x,g)\big)_{+}^{\frac{i}{k+2}}$

These eqs are indeed the same as the KdV flow equations in the dispersionless limit , under the substitution D-> x

superpotential W(x,t) ⟷ Lax operator $L(D,t) \equiv D^{k+2} - \sum_{i=1}^{k} g_{i+2}(t) D^{k-i}$

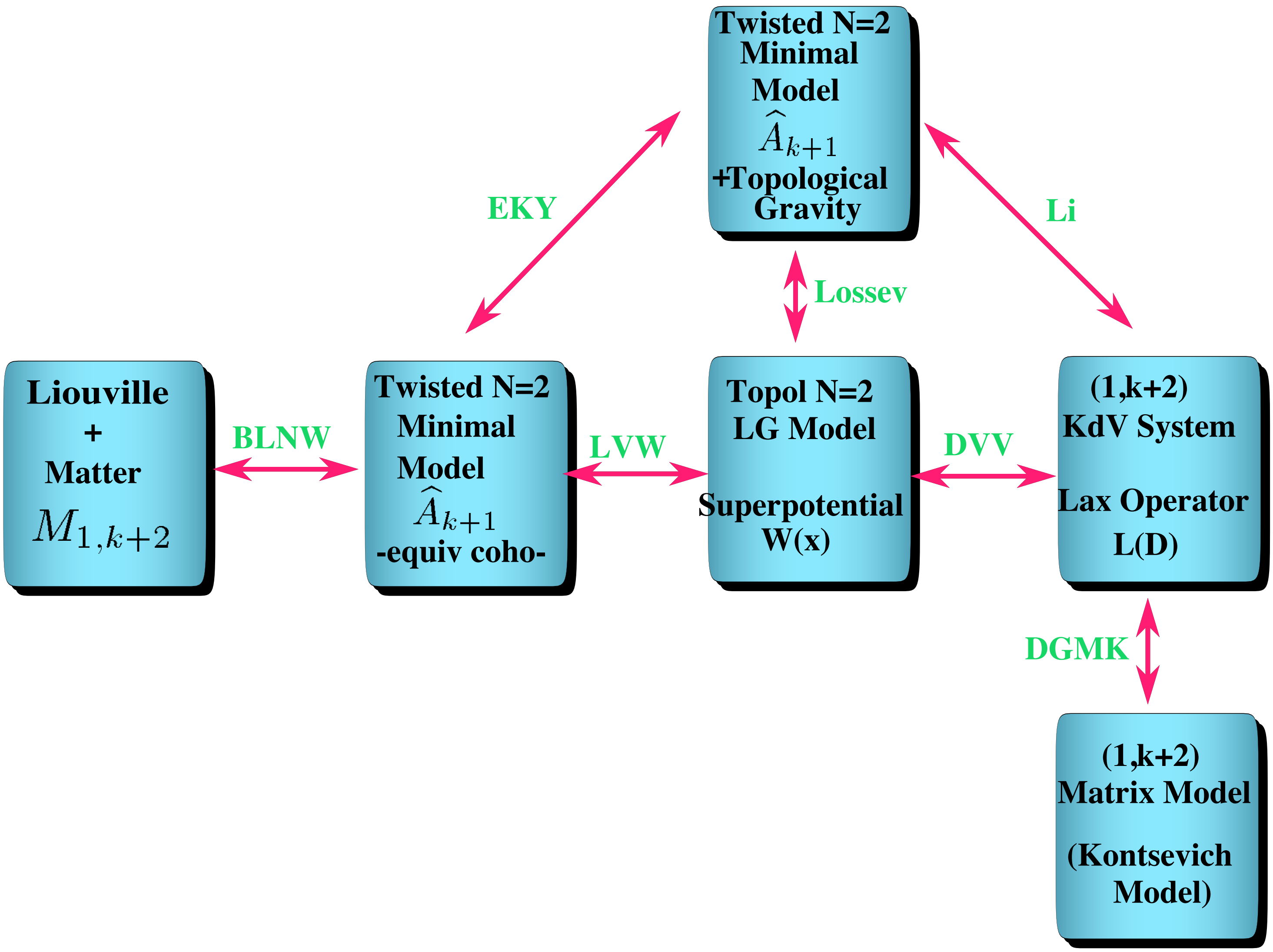
In terms of the flat fields $\Phi_{n,i}(x,t) \;\equiv\; \sigma_n(\phi_i) \;=\; \partial_x \Omega_{i+n(k+2)+1}(x,t)$
the correlators are completely determined in terms of W(x,t):

$$\big\langle \sigma_{n_1}(\phi_{i_1}) \ldots \sigma_{n_l}(\phi_{i_l}) \big\rangle_g (t) \;=\; \int \frac{dx}{\partial_x W} \Phi_{n_1,i_1}(x,t) \ldots \Phi_{n_l,i_l}(x,t) \big(\partial_x^2 W(x,t)\big)^{g-1}$$

They satisfy well-known recursion relations, and have an interpretation in terms of the topology of the moduli space of Riemann surfaces (Witten)

Because of the relation to KdV, they agree with the correlators of the corresponding matrix models

General Scheme :



Recapitulation

Topolog. gravitational structure is built into N=2 LG theory

Ingredients that are intrinsic to LG model
(defined by superpotential $W(x) = x^{k+2}$):

1) Chiral algebra:

$$Q_W \;=\; \oint d\theta^- dz\, W(X_i) \;\longleftarrow\; \mathcal{T}(X,\bar{X}) \;=\; \{J, G^+, G^-, T\}$$

2) Usual physical state cond: $G_0^+|\Phi\rangle \;=\; 0\,, \qquad |\Phi\rangle \;\neq\; G_0^+|\tilde{\Phi}\rangle$
----> primary chiral fields

3) Equivariant physical state cond: $G_0^-|\Phi\rangle \;=\; 0$
----> chiral gravitational descendants

Obvious Questions:

* How is this for arbitrary LG theories ? That is,

* What is the gravitational structure that is intrinsic to a given general, multi-field superpotential W(x) ?

* What is the geometrical interpretation of the correlators ?

* What is the structure of the underlying integrable systems ?

* What is the analog of the ‘‘General Scheme’’ ?

Consider special class: LG theories associated with N=2
CFT based on cosets $G/H = SU(n)_k/U(n-1)$

Superpotentials: $-\log\Big[\sum_{i=1}^{n-1}(-\lambda)^i x_i\Big] =: \sum_{k=-n+1}^{\infty} \lambda^{n+k}\, W_k^{(n)}(x_1,\ldots,x_{n-1})$

$$\left(\begin{array}{lll} \text{eg, n=3:} & W_0^{(3)} = \frac{1}{3}{x_1}^3 - x_1x_2\ , & c^{N=2}=0,\ \text{(trivial)} \\ & W_3^{(3)} = \frac{1}{3}{x_2}^3 - {x_1}^4x_2 - 2{x_1}^2{x_2}^2\ , & c^{N=2}=3 \end{array}\right)$$

Compute from this the chiral algebra supercurrents via

$$\Big[\oint d\theta^- dz\, W(X_i),\ \mathcal{T}_i\Big] = 0$$

Find that $\mathcal{T}_i = \{V_i, U_i^+, U_{i+1}^-, W_{i+1}\},\ i=1..(n-1)$ generate a <u>$N=2\,W_n$algebra</u>

<u>Physical states</u>: chiral ring in terms of LG Wn-eigenpolynomials $\Phi^{l_1\ldots l_{n-1}}(x_i)$

$$\mathcal{R}_x = \big\{\Phi^{l_1\ldots l_{n-1}}(x_1,...,x_{n-1}),\ \textstyle\sum l_i \le k\big\}$$

Natural: coupling to top. Wn-gravity (with n-1 Liouville fields and ghost systems) :

$$\mathcal{R}_{x,\sigma} = \mathcal{R}_x \otimes \big\{(\sigma_1^{(1)})^{l_1}(\sigma_1^{(2)})^{l_2}\ldots(\sigma_1^{(n-1)})^{l_{n-1}}\big\}$$

where $(\sigma_1^{(i)}) \equiv \big\{Q_s, \big[Q_v, \phi^{(i)}\big]\big\} - c.c.$

Physical state conditions:

$$\big(Q_s^{(m)} + Q_s^{(gr)} + Q_v\big)\,|state\rangle = 0\ , \qquad |\text{state}\rangle \neq \delta_{BRST}|\widetilde{\text{state}}\rangle$$

$$\big(b_0^{(1)} - \bar b_0^{(1)}\big)|\text{state}\rangle = \underbrace{\ldots = \big(b_0^{(n-1)} - \bar b_0^{(n-1)}\big)}_{\text{new}}|\text{state}\rangle = 0$$

Automorphism : $S \;=\; \exp\big[\frac{1}{2\pi i}\oint dz\big(\sum c_j(U^-_{m+l,j+1} + \frac{1}{2}U^-_{ghost,j+1})\big)\big]$

$$S\,(Q_v + Q_s)\,S^{-1} \;=\; Q_s$$

modified phys state conditions in ‘‘matter picture’’:

$$\big(b_0^{(i)} + (U^-_{i+1})_0 - h.c.\big)\,|\mathrm{state}\rangle \;=\; 0\;, \quad i = 1,\ldots,n-1$$

↓ evaluate completely in LG sector

For LG polynomials $\Phi(x_i) \;=\; \big\{Q_s, \sum p_i(x)\psi_i\big\} \;\equiv\; \big\{Q_s, \Lambda_\Phi\big\}$
the physical state conditions can be expressed in terms of the recursion operators ($[\mathcal{K}_i, \mathcal{K}_j] \;=\; 0$)

$$\mathcal{K}_i(\Phi) \;=\; \oint dz\, z^i\, U^-_{i+1}(z)\cdot\Lambda_\Phi(0)$$

which map LG polynomials into LG polynomials, eg., for n=3 (t=k+3):

$$\mathcal{K}_1(\Phi) \;=\; \sum \partial_{x_i} p_i(x)$$

$$\mathcal{K}_2(\Phi) \;=\;$$

$$\Big[\,(t^2+4t-24)\,\frac{\partial}{\partial x_1} + 2\,(3t-8)\,x_2\,\frac{\partial^2}{\partial x_1 \partial x_2} + (3t-8)\,x_1\,\frac{\partial^2}{\partial {x_1}^2}\Big]\,p_1\,(x_1,x_2)$$

$$+\Big[\,(t^2-6t+12)\,\frac{\partial}{\partial x_2} + (t+2)\,x_2\,\frac{\partial^2}{\partial {x_2}^2} - 2\,(t-5)\,x_1\,\frac{\partial^2}{\partial x_1 \partial x_2}$$

$$-\,(5t-18)\,\frac{\partial^2}{\partial {x_1}^2}\Big]\,p_2\,(x_1,x_2)$$

....‘‘inverse’’ gravitational descendants

....presumably appear in W-geometric contact terms

⇨ study spectrum

Result: when imposing equivariant cohomology (via the $\mathcal{K}_i(\Phi)$) on the the SU(n)/U(n-1) LG models, all eigenpolynomials $\Phi^{l_1 \ldots l_{n-1}}(x_i)$ become physical, except those for which

$$\Big(\sum (l_i+1)\lambda_i\Big)\cdot \alpha_j \;=\; 0 \text{ mod } k+n$$

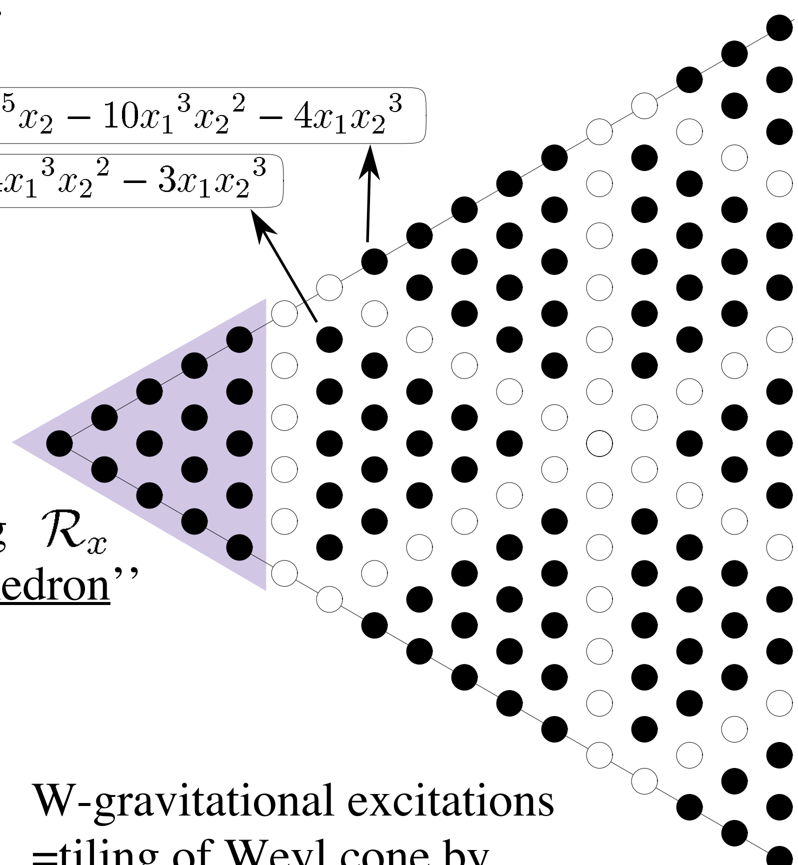
fundamental weights (λ_i) — positive roots of SU(n) (α_j)

⇒ Physical states are 1:1 to the highest weight representations of affine-sl(n).
The matter subring is 1:1 to the subset of <u>integrable</u> highest weight representations.

eg., for SU(3), k=4:

$$\sigma_1 \;=\; {x_1}^7 - 6{x_1}^5 x_2 - 10{x_1}^3{x_2}^2 - 4x_1{x_2}^3$$

$$\sigma_2 \;=\; {x_1}^5 x_2 - 4{x_1}^3{x_2}^2 - 3x_1{x_2}^3$$



Matter chiral ring $\mathcal{R}_x$
=‘‘<u>Newton polyhedron</u>’’

W-gravitational excitations
=tiling of Weyl cone by
copies of Newton polyhedron

What are the correlation functions?

⇨ determined by new type of integrable hierarchy

Recall ordinary gravity:

One LG variable x <--> KdV with‘‘ one generator’’

Flow equations in matrix form: $\partial_{t_i}\Omega_j(g(t)) \;=\; \partial_{t_j}\Omega_i(g(t))$

Matrix hamiltonians: powers of a single generator

$$\Omega_i(t) \;=\; \big(L_1\big)^i_+ \;=\; \big(\Omega_1\big)^i + \mathcal{O}(t)$$

where $L_1 \;=\; \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ z & 0 & 0 & \dots & 0 \end{pmatrix}$ = chiral ring structure constant
(z: quantum deformation
sl(2) -> Heisenberg algebra)

Ω_1 defined by "superpotential spectral equation":

$$W(\Omega_1(g(t)),t) \;\equiv\; \big(\Omega_1\big)^{k+2} - \sum g_i(t)\big(\Omega_1\big)^{k+2-i} \;=\; z\mathbf{1}$$

Generalization to Wn- gravity:

(n-1) LG variables x <--> Integr system with ‘‘(n-1) generators’’

Matrix hamiltonian generators are defined via multi-field LG superpotentials:

$$\Omega_{i_1\dots i_{n-1}} \;=\; \big((L_1)^{i_1}\dots(L_{n-1})^{i_{n-1}}\big)_+$$
$$W(\Omega_1(t),...\Omega_{n-1}(t),t) \;=\; z\mathbf{1}$$

L_i are (deformations of) the chiral ring structure constants of the underlying chiral rings $\mathcal{R}!_x$

Resume

Some sort of extended 2d topological gravitational structure is generically built into any (?) 2d N=2 LG model

Ingredients intrinsic to given LG model, ie., to superpotential W(x):

1) Extended chiral algebra is determined by W:

$$Q_W \;=\; \oint d\theta^- dz\, W(X_i) \longleftarrow \mathcal{T}_{\rangle}(X,\bar{X}) \;=\; \left\{V_i, U_i^+, U_{i+1}^-, W_{i+1}\right\}$$

2) Usual physical state cond: $U_{i,0}^+|\Phi\rangle \;=\; 0$

----> primary chiral fields that make up the Newton polyhedron

3) extra, equivariant physical state cond: $U_{i,0}^-|\Phi\rangle \;=$ [illegible]

Open Questions:

* What is the geometrical interpretation of the correlators ?

* What is the precise analog of the ‘‘General Scheme’’ ?
(new kinds of matrix models?)