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Solitons in Integrable, N=2 Supersymmetric Landau-Ginzburg Models†

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ABSTRACT

We review the properties, and soliton structure, of a class of quantum integrable $N = 2$ supersymmetric field theories that can be obtained by a particular perturbation of certain $N = 2$ superconformal field theories. These integrable theories are remarkable in that they have an exactly known effective Landau-Ginzburg superpotential, and this enables us to determine much about the soliton spectrum. We also discuss some features of other integrable perturbations of the $N = 2$ supersymmetric minimal models.

1. Introduction

Our intention here is primarily to give an overview of what one can learn about $N = 2$ supersymmetric quantum integrable models by employing Landau-Ginzburg methods. A secondary purpose is to discuss what we believe to be a very interesting class of $N = 2$ superconformal theories. This class of theories goes by the unprepossessing name of *SLOHSS* models‡ . These theories were originally described in terms of cosets[1], but, as one might expect from the title of this lecture, they also have a Landau-Ginzburg description[2,3]. In addition to this, the *SLOHSS* models also have a Toda description, and a related free field description[4,5]; they are almost certainly super-W minimal models[6], and very probably have a simple solvable lattice description[7]. On top of these facts, the most relevant chiral primary perturbations of these models provide $N = 2$ supersymmetric, integrable field theories, and these will be the main topic of this lecture. In particular, we will describe the chiral soliton spectrum of these theories, and give a very simple characterization of

∗ Talk presented by N.P. Warner.

‡ This stands for some permutation of the words: supersymmetric, simply-laced, level one, and hermitian symmetric space.

the quantum numbers of the solitons in terms of projections of a regular geometric figure called the *soliton polytope*. The discussion here is based primarily upon the work in a recent paper[8] and upon earlier material[9,4].

Apart from, and somewhat independent of, this, we have included some as yet unpublished ideas about particular perturbations of the $N=2$ supersymmetric A-D-E minimal models.

2. The *SLOHSS* Models

The *SLOHSS* models form a subclass of the general $N=2$ super-coset models of Kazama and Suzuki[1]. To obtain the latter models one applies the super-GKO construction to Kählerian cosets G/H'. Specifically, one considers coset models of the form $\frac{G\times SO(d)}{H'}$, where $d=\dim\big(G/H'\big)$; H' is embedded into both G and $SO(d)$, and $H' = H\times U(1)$, with the $U(1)$ inducing the Kähler structure on G/H'. The Kähler structure enables one to decompose the single supercharge of the super-GKO construction, into the two $N=2$ supercharges. The *SLOHSS* spaces are obtained by making the following further restrictions:

(i) G is simply-laced.
(ii) G has level one.
(iii) G/H' is a hermitian symmetric space.

This leaves one with three infinite series of models, and two exceptionals♮ :

$$G/H \;=\; \frac{SU(n+m)}{SU(n)\times SU(m)\times U(1)}\ ,\quad \frac{SO(n+2)}{SO(n)\times U(1)}\quad (n \text{ even})\ , \\ \frac{SO(2n)}{SU(n)\times U(1)}\ ,\quad \frac{E_6}{SO(10)\times U(1)}\ ,\quad \text{or}\quad \frac{E_7}{E_6\times U(1)} \tag{2.1}$$

The chiral primary fields◇ of the $N=2$ supersymmetric coset models have been computed[2]. These fields from a ring, $\mathcal{R}$, with multiplication induced from the operator product expansion. For *SLOHSS* models, this ring can be characterized by a Landau-Ginzburg potential, $W_0(\phi_i)$. That is, there are chiral primary fields, ϕ_i, that generate the ring $\mathcal{R}$, and a quasihomogeneous function $W_0(\phi_i)$ such that $\mathcal{R}$ is the local ring of W_0•. The fact that W_0 is quasihomogeneous means that there are some rational numbers ω_i such that:

$$W_0(\lambda^{\omega_i}\ \phi_i) \;=\; \lambda\, W_0(\phi_i).$$

♮ Note that $\frac{SU(n+1)}{SU(n)\times U(1)}$ describes the $N=2$ minimal series of type A_{n+1}, whereas $\frac{SO(n+2)}{SO(n)\times U(1)}$ (n even) gives the series D_{n+2}. Note also that there exists no *SLOHSS* model based on E_8.

◇ These are primary fields, ϕ, that are annihilated by half the supercharges: $G^+_{-1/2}\phi = \widetilde{G}^+_{-1/2}\phi = 0$, where $G^\pm(z)$ and $\widetilde{G}^\pm(\bar z)$ are the four supercurrents.

• The local ring of W_0 is defined as the quotient $\mathcal{R}\ \equiv \mathcal{P}/\mathcal{J}$ where $\mathcal{P}$ is the ring of power series in ϕ_i and $\mathcal{J}\equiv\{\frac{\partial W_0}{\partial\phi_i}\}$ is the ideal of $\mathcal{P}$ generated by the partials $\frac{\partial W_0}{\partial \phi_i}$ of W_0.

The conformal weights of ϕ_i satisfy $h_i = \bar{h}_i = \frac{1}{2}\omega_i$. One can now take the view that an $N=2$ *SLOHSS* theory can be described by going to the infra-red fixed point of an $N=2$ Landau-Ginzburg theory with a superpotential W_0[10,2].

It is relatively straightforward, though a little laborious, to obtain the superpotential $W_0(\phi_i)$ for a given coset G/H' explicitly. One can compute this superpotential by exploiting the fact that the chiral ring, $\mathcal{R}$, is isomorphic to the cohomology ring $H^*(G/H')$ [2], and the latter can be computed by using the invariant differential forms on the coset[11]. These observations lead to the following algorithm for computing W_0: Let ψ_A, $A = 1, \ldots, r$ be the independent (irreducible) casimirs of H' and let χ_A, $A = 1, \ldots, r$ be those of G. The number of these casimirs is equal to $\mathrm{rank}(G) = \mathrm{rank}(H')$, and their degrees are $m'_A + 1$ and $m_A + 1$, where m'_A and m_A are the exponents of H' and G respectively. (We adopt the convention that a $U(1)$ factor in H' has an exponent equal to zero, and a casimir of degree one, *i.e.* the trace.) The casimirs of G are necessarily H' invariant, and so one can decompose the χ_A into polynomial functions of the ψ_B. The ring $\mathcal{R}$ consists of all polynomials generated by the ψ_B modulo the ideal $\mathcal{J}$ generated by the polynomials $\chi_A(\psi_B)$. Some of the χ_A have terms that are linear in some ψ_C, and these particular χ_A can be succesively set to zero and thereby used to eliminate some of the ψ_C. Let ϕ_i be the subset of the ψ_A that remain after this process. The ring $\mathcal{R}$ can then be generated by these ϕ_i's modulo the remaining χ_A's. The somewhat surprising empirical fact is that one can 'integrate' this ring to obtain a quasihomogeneous function W_0, whose local ring is $\mathcal{R}$. We stress that this is an empirical fact, and, as yet, we have no deep understanding of why this can be done.

3. Perturbed *SLOHSS* Models

The idea now is to construct integrable models by using conformal perturbation theory[12]. In the $N=2$ superconformal field theories described above there is a unique lowest dimensional chiral primary field, which will be denoted by ϕ_1. Consider the field theory obtained by making a perturbation of the conformal "action" by:

$$\lambda \int G^-_{-1/2}\widetilde{G}^-_{-1/2}\ \phi_1\ d^2z \ + \ \bar{\lambda}\int G^+_{-1/2}\widetilde{G}^+_{-1/2}\ \bar{\phi}_1\ d^2z\ , \tag{3.1}$$

where $\bar{\phi}_1$ is the anti-chiral conjugate of ϕ_1. Since conformal field theories do not necessarily have actions, one must interpret the foregoing as a statement about modifying the hamiltonian, or, equivalently, one defines the new correlation functions pertubatively by making insertions of the form of Eq.(3.1).

One fairly obvious fact is that the perturbed model is still $N=2$ supersymmetric. What is far less obvious is that the model is also (quantum) intergrable. One can establish this by using conformal perturbation theory[12], but doing this directly from the coset description is extremely laborious and is only really feasible for the minimal models[9]. The most effective method to demonstrate integrability is to pass to the Toda, or to the free field, formulation of the underlying conformal model. By making a simple extension of arguments that are used for W minimal models[13], it can be shown[4] that the W algebra generators of G extend to integrals of motion in the perturbed *SLOHSS* models. These models have a super-W structure[6], and one can show[14] that each of the W generators of G is, in fact, the top component of a super-multiplet of currents. Thus, there are conserved currents with spins $m_A + 1$, and associated charges, $q^{(m_A)}$, with spin m_A, where, once again the m_A are the exponents of G. While it has not been established beyond the sine-Gordon models, there are probably infinitely many conserved charges, whose spins are congruent to the m_A mod g, where g is the (dual) Coxeter number of G.

3.1. The Ground States of the Perturbed Model

The perturbed conformal field theory has an effective Landau-Ginzburg potential:

$$W(\phi_i) \ = \ W_0(\phi_i) \ + \ \lambda \phi_1 \ . \tag{3.2}$$

To determine the ground states of this theory, one has to solve $\frac{\partial W}{\partial \phi_i} = 0$. This is most easily accomplished by reverting to the group theoretic characterization of the ring, $\mathcal{R}$. Indeed, in terms of the casimirs, χ_A, of G, one has to solve:

$$\chi_r \ = \ \text{const.}\ \lambda \ , \quad \chi_A \ = \ 0 \ , \quad A = 1, \ldots, r-1 \ , \tag{3.3}$$

where χ_r is the casimir of maximal degree, $g = m_r + 1$. It is now very convenient to consider χ_A restricted to the Cartan subalgebra (CSA) of G, and think of χ_A as a function acting on some basis elements of the CSA. Indeed, χ_A is a homogeneous polynomial of degree $m_A + 1$ on the CSA, and this polynomial is $W(G)$-invariant, where $W(G)$ is the Weyl group of $G^\star$. Let ξ denote a vector in the CSA. Consider a Coxeter element† ,s, of $W(G)$. This element has order g, and acting on the CSA, s has eigenvalues $exp(2\pi i m_j/g)$, $j = 1, \ldots, r$. Let $\xi_{(1)}$ be an eigenvector of s with eigenvalue $exp(2\pi i/g)$. Now observe that because χ_A is $W(G)$ invariant, we have:

$$\chi_A(\xi_{(1)}) \ = \ \chi_A(s(\xi_{(1)})) \ = \ \chi_A(e^{2\pi i/g}\ \xi_{(1)}) \ = \ e^{2\pi i (m_A+1)/g} \chi_A(\xi_{(1)}) \ ,$$

where the last identity follows from the homogeneity of χ_A. However the foregoing implies that $\chi_A(\xi_{(1)}) \equiv 0$, $A = 1, \ldots, r-1$. Moreover $\chi_r(\xi_{(1)}) \neq 0$ since the vanishing of all the casimirs would imply $\xi_{(1)} \equiv 0$[15]. Observe that all the $W(G)$ images of $\xi_{(1)}$ also satisfy Eq.(3.3). Finally, note that because the Landau-Ginzburg fields are all H'-casimirs, all the $W(H)$ images of $\xi_{(1)}$ yield the same ground

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$\star$ Conversely, a $W(G)$-invariant polynomial on the CSA can be extended to a casimir on G.

$\dagger$ A Coxeter element can be written as a product $r_1 r_2 \ldots r_r$, where r_i is the Weyl reflection in the simple root α_i. A Coxeter element depends on the choice of a system of simple roots, and the upon the ordering of the r_i in the foregoing product, but all Coxeter elements are conjugate, and any such element will suffice here.

state. This gives a one-to-one* mapping between the ground states of the perturbed model and the cosets of $W(G)/W(H)$. It is interesting to note that in the conformal theory there is also a natural one-to-one association of Ramond ground states and the cosets of $W(G)/W(H)$ [2,8]. Presumably these associations of Weyl cosets with ground states in the conformal and perturbed conformal theories are related, but this is not obvious from the two constructions.

One should also note that the perturbed superpotential in Eq.(3.2) is completely resolved (morsified), and only has massive perturbations.

3.2. Soliton Structure and Soliton Masses

Suppose that the two-dimensional space time is $\mathbb{R} \times \mathbb{R}$ with coordinates (σ, t). We want to find the minimum energy configurations, $\phi_i(\sigma, t)$, subject to the boundary conditions:

$$\phi_i(\sigma = \pm\infty) \ = \ \phi_i^{\pm} \ , \tag{3.4}$$

where $\phi_i^{\pm}$ are two of the solutions to: $\frac{\partial W}{\partial \phi_i} = 0$. In particular, we wish to determine the fundamental *chiral* solitons; that is, those single soliton states that are annihilated by half of the supercharges in the pertubed theory‡ .

At is this point it is helpful to use the following physical picture as a guide: The conformal field theory has $\mu = \left|\frac{W(G)}{W(H)}\right|$ degenerate Ramond ground states that can be mapped into each other via operator product with the chiral primary fields. The perturbed theory also has μ distinct degenerate ground states, and we are now seeking the chiral solitons that link these ground states. Such chiral solitons presumably have the chiral primary fields as their conformal progenitors♮ .

The commutation relations of the perturbed superalgebra can be computed[8] and one finds that the mass, M, of a solitonic state satisfies a Bogomolny bound:

$$M \ \geq \ |\Delta W| \ = \ | \ \text{const.} \ \Delta\phi_1| \ , \tag{3.5}$$

where $\Delta W \equiv W(\phi_i^+) - W(\phi_i^-)$ is the topological charge of the soliton, and this charge is proportional to $\Delta\phi_1 \equiv \phi_1^+ - \phi_1^-$. The bound in Eq.(3.5) can also be established by semi-classical arguments[16,9]. The chiral solitons are precisely those solitons that saturate this bound, and because of this one can argue that the chiral solitons are fundamental. That is, the chiral solitons are generally not multi-soliton states. Therefore, to determine the mass spectrum of the (fundamental) chiral

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∗ To completely establish this one needs the theorem that the values of the casimirs on ξ uniquely specify ξ up to Weyl images[15].

‡ The perturbed theory has four supercharges: $Q^{\pm}$and $\tilde{Q}^{\pm}$ and two appropriately chosen linear combinations [9,8]of these charges annihilate chiral solitons.

♮ It would be nice to establish this interpretation rigorously, and in particular see how operators and states behave as one continuously deforms the theory away from the conformal point.

solitons, one needs to solve the problem of which pairs of ground states are connected by such solitons. It is simplest to first state the solution to this problem and then justify it. To do this we need to introduce the *soliton polytope*, $\mathcal{P}$.

For a hermitian symmetric space G/H', there is a canonical representation, V, (called Ξ in our earlier work[8]) of G such that (i) the representation is miniscule, *i.e.* all the weights of V have the same length, and (ii) the highest weight space $|\lambda>$ is fixed by H'. (In fact $\lambda = \frac{2}{g}(\rho_G - \rho_H)$, where ρ_G and ρ_H are the Weyl vectors of G and H respectively.) For example, for the grassmanians:

$$G_{m,n} \equiv \frac{SU(n+m)}{SU(n) \times SU(m) \times U(1)} , \tag{3.6}$$

the representation, V, is the m index anti-symmetric tensor of $SU(m+n)$. The important point is that, literally by definition, the weights of V are in one-to-one correspondence with the cosets of $W(G)/W(H)$. Thus we may associate ground states of the integrable model with the vertices of a regular geometric figure whose vertices are the weights of V. This is the soliton polytope, $\mathcal{P}$. If λ_1 and λ_2 are two weights of V whose corresponding ground states can be linked by a chiral soliton, then, from the results above, it is elementary to see that the mass of this soliton is given by:

$$M = M_0 \, |\xi_{(1)} \cdot (\lambda_1 - \lambda_2)| , \tag{3.7}$$

where M_0 is some overall constant and $\xi_{(1)}$ is the eigenvector of the Coxeter element introduced earlier.

The characterization of chiral solitons is now elementary: two ground states are connected by a chiral soliton if and only if the corresponding weights, λ_1 and λ_2, on $\mathcal{P}$ differ by a root of G. While there is not a completely rigorous proof of this statement, there is very strong evidence that it is true[8]. There are some physical arguments, of which perhaps the best is based upon the relationship between the conformal theory and the perturbed theory[8]. However, perhaps the most compelling evidence is the fact that the foregoing characterization of fundamental chiral solitons satisfies a vast number of consistency conditions provided by resonances. Since this analysis leads to the soliton charges for the higher spin integrals of the motion, we will describe it here in some detail.

3.3. Higher Spin Conserved Charges

Consider any three vacua on $\mathcal{P}$ that are connected by a triangle of chiral solitons. Label these solitons by a,b and c. Project this soliton triangle into the the complex plane defined by $\xi_{(1)}$. Then from Eq.(3.7) one sees that the sides of the triangles have lengths equal to the masses M_a, M_b and M_c of the three solitons. Let the angles be labelled by θ_a, θ_b and θ_c as shown in figure 1. Then some trivial trigonometry shows that:

$$M_a \, e^{i(\theta_a - \pi)} + M_b \, e^{-i(\theta_b - \pi)} = M_c . \tag{3.8}$$

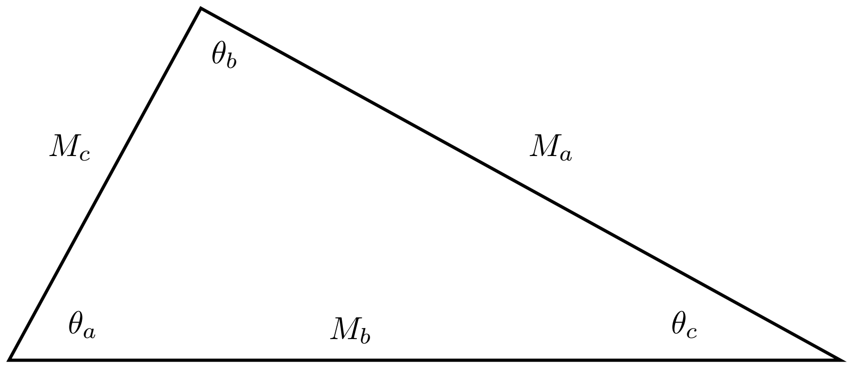

Figure 1. The mass projection of the soliton triangle.

If one now imagines scattering soliton a and soliton b against each other, then there is a resonance to create soliton c at rest when soliton a and soliton b have rapidities $i(\theta_a - \pi)$ and $-i(\theta_b - \pi)$ respectively. Now recall that there are conserved charges, $q^{(m_A)}$, whose spins are m_A, $A = 1, \dots, r$ (with $q^{(1)}$ being the mass). Let $q_a^{(s)}$, $q_b^{(s)}$ and $q_c^{(s)}$ be the spins s charges of the solitons a, b and c. The resonance implied by the soliton triangle of a, b and c imposes the following constraint upon the $q^{(s)}$ charges:

$$q_a^{(s)}\ e^{is(\theta_a - \pi)} \ + \ q_b^{(s)}\ e^{-is(\theta_b - \pi)} \ = \ q_c^{(s)}\ . \tag{3.9}$$

Considering every soliton triangle in the polytope provides a highly overdetermined system for all the spin s charges of the solitons (see figure 2, for example). The solution to this system of equations is also provided by the soliton polytope.

Consider, once again, the Coxeter element of the Weyl group of G acting on the CSA. Let $\xi_{(m_A)}$ be an eigenvector with eigenvalue $exp(2\pi i m_A/g)$. Project the soliton triangles onto the complex plane defined by $\xi_{(s)}$. The remarkable fact about the soliton polytope is that this new triangle has interior angles $s\theta_a$, $s\theta_b$ and $s\theta_c$ mod π[8,17,18]. Thus, modulo signs and an overall scale, one can identify $q_a^{(s)}$, $q_b^{(s)}$ and $q_c^{(s)}$ with the side lengths of this projection of the soliton triangle. Moreover, one can orient the sides and thereby give these lengths a sign so that these signed lengths exactly satisfy Eq.(3.9).

Thus the soliton polytope encodes all the information about all of the charges of the chiral solitons. The fact that there is a solution to such a highly over determined system of resonance constraints also provides good evidence that the original characterization of chiral solitons is correct.

3.4. Further Comments on the Integrable SLOHSS Models

The actual set of numerical values of the higher spin charges of the chiral solitons is not altogether surprising: Depending upon the soliton, the spin s charge, $q_a^{(s)}$, is always some component of the eigenvector of the Cartan matrix of G with eigenvalue $2 - 2cos(\frac{s\pi}{2})$. (The details of how these eigenvectors emerge in the projections of the polytopes may be found elsewhere[8,17].) Thus one sees further evidence of the relationship to affine Toda theory.

It is also important to remind oneself that the Landau-Ginzburg model is supersymmetric. This means that each soliton described above is, in fact, a (shortened) supermultiplet of four solitons, two "bosonic" and two "fermionic". In the corresponding affine Toda theory, this supersymmetry appears only at the quantum level, and requires a special choice of background charge a particular value for the coupling constant. This coupling constant is also purely imaginary, just as in the non-supersymmetric affine Toda theories that describe integrable perturbations of non-supersymmetric conformal field theories.

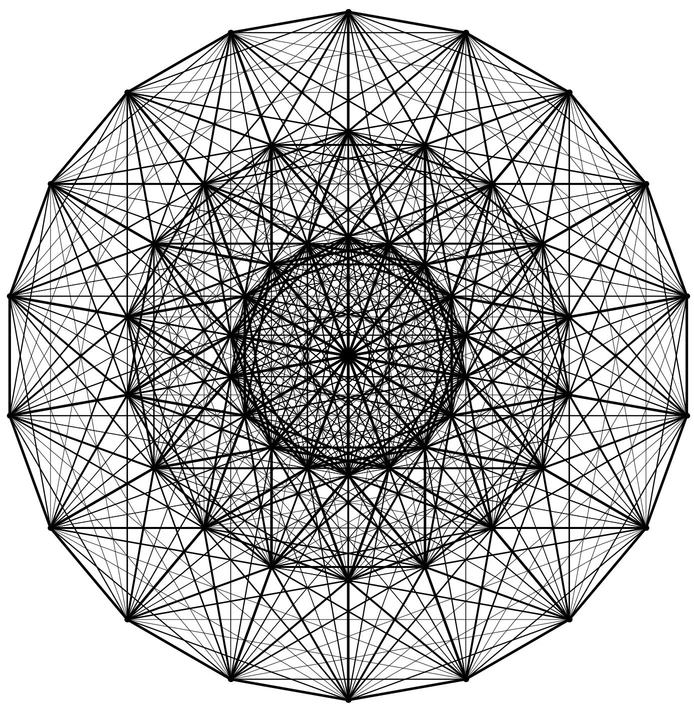

Figure 2. The soliton polytope for the $E_7/E_6 \times U(1)$ model is obtained by taking $V \equiv \mathbf{56}$ of E_7. This diagram shows the $\xi_{(1)}$ (or mass) projection of the polytope; the dots are the images of the vacuum states. The central dot in the diagram represents two vertices. The lengths of the soliton lines are proportional to the soliton masses. There are 756 solitons and 4032 soliton triangles. The system of equations (3.9) is thus overdetermined by more than a factor of 5.

It is known that the classical Toda theory, with a purely imaginary coupling constant, has solitonic solutions with physically sensible quantum numbers[19]. These interpolate between the vacuum states, which lie on the weight lattice of the corresponding group. At the quantum level, it will be necessary to truncate the soliton spectrum to provide a unitary Hilbert space. For the simplest affine Toda model, the sine-Gordon model, this quantum group truncation[20] can be viewed as effectively reducing the infinite-well potential to a Landau-Ginzburg potential with a finite number of wells, or as reducing the weight space of $SU(2)$ to a finite weight diagram of some representation of $SU(2)$. We conjecture that the G/H' Landau-Ginzburg solitons correspond precisely to such classical Toda solitons, after some form of (perhaps affine) quantum group truncation. This truncation should effectively reduce the whole weight space of G to the finite weight diagram of the representation V.

The work described above characterizes all the chiral solitons. A consistent scattering matrix for these (supermultiplets of) solitons has only been computed for the simplest (minimal) models[9]. It would be interesting to find S-matrices for some of the more complicated Landau-Ginzburg models. There is, however, a problem to be solved before one can do this. Simple kinematic arguments[9]show that the chiral solitons cannot (except in the type A minimal models) form a closed scattering theory. One needs to add new states. One can make educated guesses as to what these states should be, and kinematic consistency can often be restored by adding some new "breather" states. It would be nice to have some method of determining the complete spectrum and then finding the S-matrix.

Another interesting issue is raised by the circuitous manner in which the higher spin charges were determined. The masses of the solitons were obtained from the superalgebra and a Bogomolny bound involving a topological charge. The higher spin charges were then deduced from resonance consistency. Since the higher integrals of motion come from the super-W algebra generators, one might hope to obtain new Bogomolny bounds, and perhaps some new topological charges, and thereby determine the higher spin conserved charges of the solitons directly from the perturbed super-W algebra. The fact that the answers are so beautifully encoded in the soliton polytope suggests that there may also be some simple underlying geometry to the perturbed super-W algebra.

Finally, it should be remembered that the most relevant chiral primary perturbation is not the only perturbation of the $N = 2$ super-coset models that leads to a quantum integrable $N = 2$ supersymmetric field theory. For the minimal A-series there are two other such perturbations[9,4,21]. (One of these perturbations will be discussed in the next section.) There are also indications[22] that there may be other perturbations of the $SLOHSS$ models that give rise to integrable theories. It is certainly of interest to determine the soliton spectrum and S-matrices for these models.

4. Real Resolutions of A-D-E Singularities, Solitons and Fusion Rules

Our purpose here is to make some, hopefully, amusing observations concerning perturbed A-D-E minimal models[⋆]. These observations will be discussed in more

⋆ Note that only the A and D series minimal models are $SLOHSS$ models.

detail elsewhere. In contrast to perturbing by the most relevant chiral field, as above, we now perturb the minimal models with the (F-component of the) unique, *least* relevant operator, ϕ_{top}, of dimension $c/6$. As we will see, this type of perturbation also leads to interesting soliton structure[†] . For the A-series, it is known that this perturbation leads to an integrable theory[9,4,24,21], and non-trivial integrals of motion have been constructed. Similar results can probably be established also for the D and E series.

An important point to realize is that perturbing $N=2$ theories by insertions of the form $e^{-\sum \lambda_\ell(\int \phi_\ell + h.c.)}$ in the correlation functions amounts to deforming the operator algebra in a way that can be characterized by an effective superpotential[25], $W(\phi,\lambda) = W_0 + \sum_{\phi_k \in \mathcal{R}} \ g_k(\lambda_\ell)\ \phi_k$. Here, the fields ϕ_k denote generic elements of the chiral ring, and the coupling constants g_k are particular, non-trivial[*] functions of the perturbation parameters (or "flat coordinates"), λ_ℓ, and can be determined by using the techniques described in[25,26,27].

For the minimal A series the effective superpotential for the perturbation with ϕ_{top} is

$$A_n: \quad W \;=\; \tfrac{2}{n+1}\lambda^{\frac{n+1}{2}}\, T_{n+1}(\lambda^{-1/2}\phi/2) \;=\; \tfrac{1}{n+1}\phi^{n+1} - \lambda\,\phi^{n-1} + O(\lambda^2)\ , \qquad (4.1)$$

where T_n are Chebyshev polynomials. The corresponding potential $|\frac{\partial W}{\partial \phi}|^2$ is a multi-well potential with n zeros along the real-ϕ axis. If one calculates the values of W at all the critical points, one finds that it takes only two values. Thus all the chiral solitons have the same mass ($M = |\Delta W|$). This is perfectly consistent with the conjecture that this model is equivalent to a quantum truncated, $N=2$ supersymmetric sine-Gordon model[22]. (This conjecture is also supported by the structure of the quantum integrals of motion.)

The potential in Eq.(4.1) received recently attention[28,29,22,30] because of the remarkable fact that for $\lambda=1$, the structure constants ${c_{\ell m}}^n$ of the deformed chiral ring, in a basis consisting of fields $\Phi_\ell(\phi,\lambda) \equiv -\frac{\partial W(\phi,\lambda)}{\partial \lambda_\ell}$, coincide with the fusion coefficients ${N_{\ell m}}^n$ of $SU(2)_{k=n-1}$ WZW models. This is because one finds that:

$$\Phi_\ell(\langle\phi\rangle_a, 1) \;=\; \frac{S_{\ell a}}{S_{1a}}\ , \qquad (4.2)$$

where $S_{\ell a}$ is the modular transformation matrix of the $SU(2)$ characters, and a labels the n vacuum states, $\frac{\partial W}{\partial \phi}(\langle\phi\rangle_a) \equiv 0$.

† There is an analogous, relevant perturbation of general *SLOHSS* models and one can construct S-matrices for the solitons[23].

∗ As remarked earlier, the most relevant chiral primary perturbation leads to a simple effective superpotential $W(\phi,\lambda) = W_0 + \lambda\phi_1$.

Apparently, similar properties hold also for other minimal models, where the dependence on the flat coordinate λ is[25,31 ‡] :

$$
\begin{aligned}
D_n: \qquad W &= (-1)^{n-1}\tfrac{1}{2}\phi_1{\phi_2}^2 + \tfrac{1}{n-1}(-\lambda)^{n-1}\,T_{n-1}(1-(2\lambda)^{-1}\phi_1) \\
&= (-1)^{n-1}\tfrac{1}{2}\phi_1{\phi_2}^2 + \tfrac{1}{2n-2}{\phi_1}^{n-1} - \lambda{\phi_1}^{n-2} + O(\lambda^2) \qquad (4.3)\\
E_6: \qquad W &= \tfrac{1}{3}{\phi_1}^3 + \tfrac{1}{4}{\phi_2}^4 - \lambda\phi_1{\phi_2}^2 + \tfrac{1}{2}\lambda^3{\phi_2}^2 - \tfrac{1}{12}\lambda^4\phi_1 \qquad (4.4)\\
E_7: \qquad W &= \tfrac{1}{3}{\phi_1}^3 + \tfrac{1}{3}\phi_1{\phi_2}^3 - \lambda{\phi_1}^2\phi_2 + \tfrac{4}{9}\lambda^3{\phi_1}^2 - \tfrac{1}{9}\lambda^4\phi_1\phi_2 + \tfrac{1}{81}\lambda^6\phi_1 + \tfrac{1}{4374}\lambda^9
\end{aligned}
$$

(The perturbed E_8 potential has not yet been computed). We find that for these models:

$$
\Phi_\ell(\langle\phi\rangle_a,\lambda) \;=\; \lambda^{\frac{m_\ell-1}{2}}\,\frac{q_a^{(m_\ell)}}{q_a^{(1)}}\ , \qquad (4.6)
$$

where $q_a^{(m_\ell)}$ are once again the ubiquitous eigenvectors (with a particular normalization) of the corresponding A-D-E Cartan matrix, and m_ℓ are the exponents of the corresponding group. This equation seems to be a consequence of the well-known fact that the intersection matrix in the integral homology of the level surfaces associated with the A-D-E singularities is given by the appropriate Cartan matrix. For the A series one finds that the eigenvectors $q_a^{(m_\ell)}$ are equal to the entries of the modular transformation matrix $S_{\ell a}$. However, this is not true for the D and E theories. Therefore, Eq.(4.6) does not seem to diagonalize fusion rules of the corresponding $SU(2)$ theory in all cases. But we find again that all structure constants are integers for particular values of λ, and whether this has an interesting interpretation is an open question.

We also find that certain genus g amplitudes of the topological matter models are given in terms of the Frobenius-Perron eigenvector of the Cartan matrix,

$$
\begin{aligned}
S(g) \equiv \langle\ \Phi_{top}(\phi,\lambda)^{1-g}\ \rangle_{(g)} \;&=\; \sum_a \Big(\frac{\Phi_{top}(\langle\phi\rangle_a,\lambda)}{det[\partial_{\phi_i}\partial_{\phi_j}W(\langle\phi\rangle_a,\lambda)]}\Big)^{1-g}\\
&=\; \sum_a (q_a^{(1)})^{2-2g} \;=\; \text{integer}
\end{aligned}
$$

These integers (which also show up in the context of Toda mass matrices[32]) appear to be the dimensions of Friedan-Shenker vector bundles of the topological minimal models on Riemann surfaces. For example, we have for E_6: $S(g) = 2^{g-1}(2^g+1)[(3+\sqrt{3})^{g-1} + (3-\sqrt{3})^{g-1}]$, and the first few values of $S(g)$ are: $1, 6, 60, 864$.

It is clear from Eq.(4.6) that Φ_ℓ evaluated on the vacuum states are real numbers (for real λ), and in particular, that $W(\phi,\lambda)$ evaluated on the vacuum states

‡ We disagree with some of the coefficients given in the second reference.

is real. Potentials with this special property have been, in fact, thoroughly investigated in the mathematical literature (see, for example, a paper of Gusein-Zade[33], where indeed the potentials of Eqs.(4.1), (4.3) and (4.4) appear explicitly). Thus the flat coordinate associated with the top element of the chiral ring provides what is called a *real resolution* of a singularity.

A most striking property of these real resolutions is that the soliton structure is given by the Dynkin diagram of the corresponding A-D-E group. That is, the vacuum states are associated with the nodes of the Dynkin diagram, whereas the solitons are associated with the links[♮]. For an example, see figure 3. It also turns out that when one of these superpotentials, W, is evaluated at one of its critical points, then W only takes one of two possible values. Thus, in a given model, the fundamental solitons all have the same mass.

This gives support to the conjecture that these particular perturbations of the D and E theories yield integrable models. These models might also be in some way related to the integrable lattice models discussed by Pasquier[34]. Indeed, our Eq.(4.6) is more or less the same as Eq.(23) of Pasquier[34].

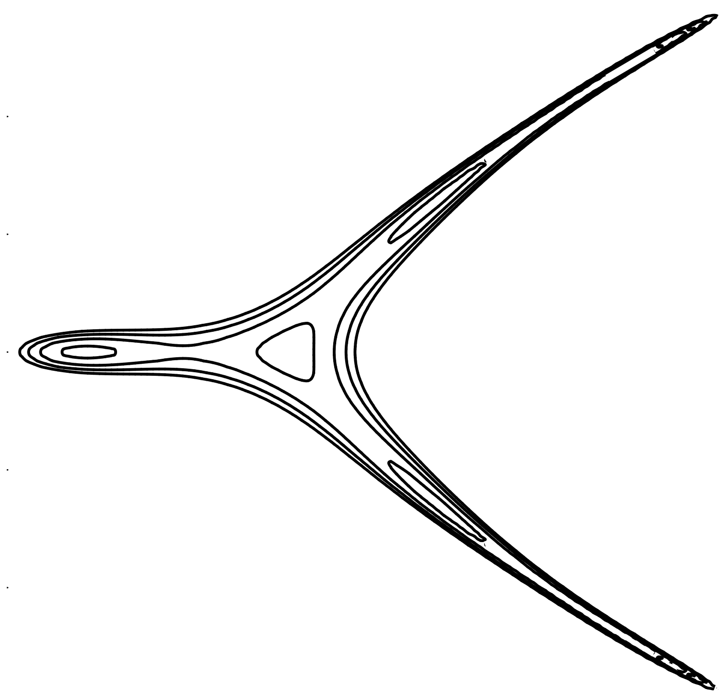

Figure 3. Contour plot of the potential $|\nabla W(\phi_1, \phi_2)|^2$ obtained from the superpotential in Eq.(4.4). It clearly displays the structure of the E_6 Dynkin diagram.

♮ The model A_n has thus two interpretations: the vacuum states and solitons are associated either with some weight diagram of $SU(2)$ (corresponding to a quantum truncation of the $N=2$ sine-Gordon model), or with the Dynkin diagram of A_n.

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