CALT-68-1703
DOE RESEARCH AND
DEVELOPMENT REPORT

Polytopes and Solitons in Integrable, N=2 Supersymmetric Landau-Ginzburg Theories*

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ABSTRACT

We find that the solitons of a class of $N=2$ supersymmetric, two dimensional Landau-Ginzburg models can be characterized by higher dimensional polytopes, and that certain projections of these give the various quantum numbers of the solitons. Our results are also of direct relevance to Toda theory.

* This research was supported in part by DOE contract DE-AC0381ER40050 and in part by NSF under Grant No. PHY89-04035, supplemented by funds from the National Aeronautics and Space Administration.

† Alfred P. Sloan Foundation Fellow.

April 26, 1996

1. Introduction

It was observed in [1] that there is a Landau-Ginzburg theory corresponding to each $N=2$ superconformal coset model [2] based on a hermitian symmetric space, G/H, where G is simply laced and has Kac-Moody level equal to one. Such $N=2$ coset models will henceforth be referred to as SLOHSS models. Explicitly, these cosets are $G/H = \frac{SU(n+m)}{SU(n)\times SU(m)\times U(1)}$, $\frac{SO(n+2)}{SO(n)\times U(1)}$ (n even), $\frac{SO(2n)}{SU(n)\times U(1)}$, $\frac{E_6}{SO(10)\times U(1)}$ or $\frac{E_7}{E_6\times U(1)}$ * . In [3] it was described how any such SLOHSS theory based on some coset G/H could be related to a conformally invariant H-Toda theory (tensored with a free boson). In addition it was shown that the perturbation by (the F-component of) the most relevant, chiral primary field yields a quantum integrable, $N=2$ supersymmetric, massive field theory which, in turn, is related to an affine G-Toda theory. Since any SLOHSS model possesses a Landau-Ginzburg description, it follows that the massive theory obtained by perturbation can be given a Landau-Ginzburg description. In particular, under the perturbation, the multiply degenerate Ramond ground states of the conformal Landau-Ginzburg model resolve into distinct vacua.

It is our purpose to determine in this paper the structure of these ground states and of the various solitons linking them‡ . These soliton sectors give rise to a new class of exactly solvable scattering theories, whose S-matrices are closely related to those of Toda theories. We will work in the spirit of Zamolodchikov [4], but will utilize specific properties of the $N=2$ structure of the foregoing models. We will show that the solitons are related to the edges of certain higher dimensional polytopes, and we will apply the techniques of [5] to determine the exact soliton masses. From this and the bootstrap equations the entire soliton spectrum is completely determined, along with the

* Note that $\frac{SU(n+1)}{SU(n)\times U(1)}$ describes the $N=2$ minimal series (with type A_{n+1} modular invariant), and $\frac{SO(n+2)}{SO(n)\times U(1)}$ (n even) gives the minimal series of type D_{n+2}. Note also that there exists no SLOHSS model based on E_8.

‡ Some preliminary results have been mentioned in [3].

eigenvalues of the integrals of motion evaluated on each soliton. The various quantum numbers of the solitons are just given by particular projections of the polytopes onto two dimensional planes. This gives a geometrical solution of the bootstrap equations. The projections also contain the elements of the Toda perturbation expansion in the form of dual diagrams [6].

In section 2, we will review the structure of the SLOHSS models, giving some more details of our previous work [1]♮ , and in section 3 we will discuss the Landau-Ginzburg structure. The vacua and solitons of the perturbed SLOHSS models will then be discussed in section 4, and the results of [5] will be extended to Landau-Ginzburg potentials in more than one variable. We will discuss the soliton charges of the higher spin integrals of motion in section 5. In section 6, we give some comments on the relation of our results to Toda theories.

2. SLHOSS Models

The $N=2$ supersymmetric coset models were introduced in [2] and are obtained by the GKO construction applied to $\frac{G\times SO(2d)}{H}$, where $d\equiv \dim_c G/H$, $\mathrm{rank}(G)=\mathrm{rank}(H)$, $H=H_0\times U(1)$, and the $U(1)$ factor defines a Kähler structure on G/H. The group H acts on the tangent space of G/H, and thus has the obvious embedding into $SO(2d)$. By way of notation, let ℓ be the rank of G, g and h the dual Coxeter numbers◇ of G and H, respectively. Let also $W(G)$ and $W(H)$ denote the Weyl groups of G and H. The Lie algebras of G and H will be denoted by $\mathcal{G}$ and $\mathcal{H}$, respectively, the simple roots of $\mathcal{G}$ will be denoted by $\alpha_i, i=1,\ldots,\ell$ and the highest root by $\theta\equiv\sum_{i=1}^{\ell} q_i\alpha_i$ for some positive integers q_i. The roots of $\mathcal{G}$ and $\mathcal{H}$ will be denoted by $\alpha,\beta,\ldots$ and $\alpha',\beta',\ldots$, while those roots of $\mathcal{G}$ that are not roots of $\mathcal{H}$ will be denoted by $\overline{\alpha},\overline{\beta},\ldots$. The sets $\Delta(G)_\pm$, $\Delta(H)_\pm$ will denote the positive (negative) roots of $\mathcal{G}$ and $\mathcal{H}$, while $t_\pm\equiv\Delta(G)_\pm\backslash\Delta(H)_\pm$. The currents of $\mathcal{G}$ in the Cartan-Weyl basis will be denoted by $H^i(z)$ and $J^\alpha(z)$. To describe the $SO(2d)$ factor it is convenient to introduce free complex fermions $\lambda^{\overline{\alpha}}$ which satisfy $\lambda^{\overline{\alpha}}(z)\lambda^{\overline{\beta}}(w)\sim 0$ if $\overline{\alpha}+\overline{\beta}\neq 0$ and $\lambda^{-\overline{\alpha}}(z)\lambda^{+\overline{\alpha}}(w)\sim\frac{1}{z-w}$, $\overline{\alpha}\in t_+$. The Cartan subalgebra of currents of $\mathcal{H}$ will be denoted by $h^i(z)$, and one finds [2]

$$h^i(z) \;=\; H^i(z)+\sum_{\overline{\alpha}\in t_+}\overline{\alpha}^i : \lambda^{+\overline{\alpha}}\lambda^{-\overline{\alpha}} : (z)\ . \tag{2.1}$$

Considered as a current algebra, if $\mathcal{G}$ has level k, then $\mathcal{H}$ has has level $k+g-h$.

Restricting to hermitian, symmetric spaces, the $N=2$ superconformal generators have the form

$$\begin{aligned} G^\pm(z) &= \sum_{\overline{\alpha}\in t_+}\lambda^{\pm\overline{\alpha}}(z)J^{\mp\overline{\alpha}}(z)\\ J(z) &= j_f(z)-\tfrac{1}{k+g}\,j_b(z)\end{aligned}\tag{2.2}$$

where

$$\begin{aligned} j_f(z) &= \sum_{\overline{\alpha}\in t_+} :\lambda^{\overline{\alpha}}\lambda^{-\overline{\alpha}}:(z)\\ j_b(z) &= \sum_{\overline{\alpha}\in t_+}\overline{\alpha}\cdot h(z) \;=\; \Big(\sum_{\overline{\alpha}\in t_+}\overline{\alpha}\cdot H(z)\Big)+g\,j_f(z)\ .\end{aligned}\tag{2.3}$$

The energy momentum tensor is given by the well-known combination of Sugawara tensors. For each model, (2.2) generates an $N=2$ algebra with central charge $c=\frac{3d}{g+1}$.

The coset model Hilbert spaces, $\mathcal{H}_\lambda^{\Lambda,\tilde\Lambda}$, are obtained by the usual decomposition method, and the labels $\Lambda,\tilde\Lambda$ and λ are highest weight labels of $G, SO(2d)$ and H at levels $k,1$ and $k+g-h$, respectively. The Ramond and Neveu-Schwarz sectors of the coset model correspond to the R and NS sectors of the $SO(2d)$ factor. As discussed in [1], spectral flow in the Kac-Moody algebras will yield identifications amongst the $\mathcal{H}_\lambda^{\Lambda,\tilde\Lambda}$ spaces. For simply laced G

♮ Some related and subsequent work may also be found in [7].
◇ If H is the product of simple and $U(1)$ factors, then h is to be thought of as a vector in the obvious manner. The dual Coxeter number of a $U(1)$ factor is defined to be zero.

at level one ($k=1$) there are no fixed point problems; moreover, the spectral flow identifications mean that to get exactly one representative of each equivalence class of the $\mathcal{H}_{\lambda}^{\Lambda,\tilde{\Lambda}}$, one only needs to consider the spaces with $\Lambda \equiv 0$, i.e., the singlet representation of the Kac-Moody algebra of $\mathcal{G}$. Henceforth, we will restrict to SLOHSS models.

The fields of interest in the Landau-Ginzburg description are the chiral, primary fields, and those satisfy

$$G^{+}_{-1/2}\Phi \;=\; 0, \qquad G^{-}_{1/2}\Phi \;=\; 0 \;. \tag{2.4}$$

By spectral flow in the $N=2$ algebra, they are related in a one-to-one fashion to the ground states of the Ramond-sector. To construct these fields explicitly, it is actually more convenient to construct first the corresponding Ramond ground states. Knowing the structure of the Ramond ground states will also prove useful when we come to discuss the ground states of the perturbed models.

Representations of the Ramond ground states of a SLOHSS model can be obtained by first restricting ones attention to the Ramond ground states of $SO(2d)$ (see [1] for a proof that this is sufficient). Thus one considers the decomposition of an $SO(2d)$ spinor ground state, with highest weight label $\tilde{\Lambda}$, into H represenations. In [1] it was shown that for $\Lambda \equiv 0$, the coset model Ramond ground states thereby obtained have an highest weight labels, λ, of H of the form:

$$\lambda(w) \equiv w(\rho_G) - \rho_H \;, \tag{2.5}$$

where $w \in W(G)$ is chosen so that $w(\rho_G)$ is a highest weight of H (it then automatically follows that $\lambda(w)$ is also a highest weight of H). The chirality, or eigenvalue of $(-1)^F$, of this ground state is equal to the sign of the determinant of w in (2.5). There is also a one-to-one correspondence between the Ramond ground states and the elements of the coset $\frac{W(G)}{W(H)}$, so that, in particular, the total number of Ramond ground states is $\mu = \frac{|W(G)|}{|W(H)|}$. An equivalent

characterization of the Weyl group elements $w \in W$, employed in (2.5) is that all have the property, that for every $\alpha \in \Delta_+(H)$ one has $w^{-1}(\alpha) \in \Delta_+(G)$. We will refer to such a $w \in W(G)$ as being H-positive, and note that each coset of $\frac{W(G)}{W(H)}$ has an unique H-positive coset representative.

While the foregoing implicitly defines the Ramond ground states, it is very useful to obtain a more explicit representation in the coset model.

For every element $w \in W(G)$, let $\Delta_\pm(w)$ be such that $\alpha \in \Delta_\pm(w)$ if and only if $w^{-1}(\alpha) \in \Delta_\pm(G)$. Observe that $\Delta_+(w)$ and $\Delta_-(w)$ are disjoint and their union is the whole of $\Delta(G)$. Therefore

$$w(\rho_G) \;\equiv\; \frac{1}{2} \sum_{\alpha \in \Delta_+(G)} \pm\alpha \;,$$

where the sign is dictated by $\alpha \in \Delta_\pm(w)$. The number of elements in $\Delta_-(w)$ is called the length* of w, and is usually denoted by $\ell(w)$. Let $v_{\overline{\alpha}}$, $\overline{\alpha} \in t_+$, denote the components of a weight vector of the spinor representation of $SO(2d)$. The highest weight vector has $v_{\overline{\alpha}} = \frac{1}{2}$ for all $\overline{\alpha}$. Let w be an H-positive element of $W(G)$, and define $u_{\overline{\alpha}} = \pm\frac{1}{2}$ for $\overline{\alpha} \in \Delta_\pm(w)$. It follows that $u_{\overline{\alpha}}$ is a spinor weight, and the corresponding weight in the torus defined by (2.1) is nothing other than $\lambda = w(\rho_G) - \rho_H$. We have thus identified the fermionic state from which the corresponding coset Ramond ground state is made. One should also observe that the $SO(2d)$ fermion charge of the ground state is $\frac{1}{2}d - \ell(w)$, and thus from (2.2) and (2.3), the $N=2$ $U(1)$ charge of this ground state is $J_0(\lambda(w)) = \frac{1}{2(g+1)}(d - 2\ell(w))$. A simpler formula for the length $\ell(w)$ of an H-positive Weyl element w is obtained by noting that $(\rho_G - \rho_H)\cdot\alpha = \pm\frac{1}{2}g$ for $\alpha \in t_\pm$ and $(\rho_G - \rho_H)\cdot\alpha = 0$ for $\alpha \in \Delta(H)_\pm$, and hence $\ell(w) = \frac{d}{2} - \frac{2}{g}(\rho_G - \rho_H)\cdot w(\rho_G)$. Consequently, we have

$$J_0(\lambda(w)) \;=\; \frac{2}{g(g+1)}\,(\rho_G - \rho_H)\cdot w(\rho_G) \;. \tag{2.6}$$

* The length of w may also be defined as the minimum value of M such that $w = r_{i_1}\ldots r_{i_M}$ and the r_i are the fundamental Weyl reflections generated by the simple roots α_i.

By spectral flow, we can use the foregoing construction to generate representations of the chiral, primary fields (2.4). Indeed, for any H-positive w, consider the operator

$$\Phi_w(z) \ = \ \prod_{\overline{\alpha}\in\Delta_-(w)} \left(\lambda^{\overline{\alpha}}(z)\right) \ , \tag{2.7}$$

where the ordering of the product is a matter of irrelevant choice. The $N=2$ $U(1)$ charge and conformal weight of this state are $\frac{\ell(w)}{g+1}$ and $\frac{\ell(w)}{2(g+1)}$, respectively. This field is thus the product of a chiral, primary field, and some H-primary field[†] .

It was shown in [1] that the ring $\mathcal{R}$ of the chiral, primary fields has Poincaré polynomial

$$P(t) \ \equiv \ \mathop{\mathrm{Tr}}_{\mathcal{R}}[t^{(g+1)J_0}] \ = \ \prod_{j=1}^{\ell}\left(\frac{1-t^{m_j+1}}{1-t^{m'_j+1}}\right) \ , \tag{2.8}$$

where m_j and m'_j are the exponents of G and H (for the $U(1)$ factor in H we take $m'_1=0$). The degrees of the algebraically independent Casimirs of G are simply m_j+1 (and similarly for H).

It was also shown in [1] that $\mathcal{R}$ has a close relationship to a particular fundamental representation, Ξ, of G. The highest weight of Ξ is defined by the fundamental weight of G corresponding to the node of the G-Dynkin diagram that defines the embedding of the $U(1)$ factor, that is, the highest weight is given by $\frac{2}{g}(\rho_G-\rho_H)$. For a SLOHSS model, the possible choices of the $U(1)$ factor correspond to the Dynkin nodes which have Kac weight equal to one, i.e., the allowed representations Ξ are the level one representations of affine-G. Specifically, for $G=SU(n)$, Ξ can be any fundamental, antisymmetric tensor representation. It turns out that the Poincaré series (2.8) can be written as a certain $U(1)$-character valued trace over Ξ. This implies, in particular, that the number of primary, chiral fields is given by

$$\begin{aligned}\mu \ \equiv \ dim\,\mathcal{R} \ &= \ \frac{|W(G)|}{|W(H)|} \\ &= \ dim\,\Xi\end{aligned} \tag{2.9}$$

† The H-state of corresponding to this operator is simply the lowest weight state in the H-ground state representation with highest H-weight: $\lambda\equiv w(\rho_G)-\rho_H$.

3. Chiral Rings and Landau-Ginzburg potentials of SLOHSS Models

From a case by case analysis, based on the explicit form of (2.8), it was argued in [1] that all of these rings $\mathcal{R}$ of chiral, primary fields correspond to local rings of quasihomogenous, isolated singularities [8], and hence, to Landau-Ginzburg superpotentials W_0 [9]. Consequently, all SLOHSS models have also a description in terms of $N=2$ supersymmetric Landau-Ginzburg models,

$$\mathcal{L} \ = \ \int d^2\theta^+ d^2\theta^- \ K(\Phi_A,\overline{\Phi}_A) + (\int d^2\theta^- \ W_0(\Phi_A) + h.c.) \tag{3.1}$$

for some Kähler potential K. The generic form of these superpotentials was explicitly given for the Grassmannian models [1][*] . Another result that was noticed in [1] is that the chiral, primary fields are in one-to-one correspondence with the Lie-algebra cohomology, and in particular the primary chiral ring $\mathcal{R}$ of a given model based on G/H was shown to be isomorphic[‡] to the Dolbeault cohomology ring of G/H. More precisely, if $\frac{q}{g+1}$ denotes the $N=2$ $U(1)$ charge of a chiral, primary field, then Φ_q can be thought of as an element of $H^{q,q}(G/H,\mathbb{R})$.

∗ Grassmannian potentials and chiral rings have also been considered by [7].

‡ It was argued in [1] that these rings were isomorphic at least up to some deformation in the moduli. It is however possible to show, by using explicit representations of the chiral, primary fields, that these rings are completely isomorphic.

An extremely convenient characterization of these cohomology rings is provided by the Chern classes of vector bundles over G/H. Indeed, given a representation V of H, one can define a vector bundle over G/H by defining $\tilde{E} = \{(g,v) : g \in G, v \in V\}$ and taking the total space of the bundle to be $E = \tilde{E}/\sim$, where $\sim$ is the equivalence relation

$$(g,v) \ \sim \ (g\,h, h\,v) \quad \text{for all } h \in H \ .$$

The Chern classes of all such vector bundles then generate the cohomology ring of G/H (for an exposition, see for example [10,11]). Note however that if V can be extended to a representation of G, then one can globally trivialize the bundle by using the g action on V. One finds that the cohomology ring, $H^{*,*}(G/H, \mathbb{R})$, is the free polynomial algebra in the Chern classes of the bundles defined by H-representations, but with vanishing relations generated by all the Chern classes of H representations that can be combined into G-representations.

In practice, the foregoing observation gives rise to the following characterization of the chiral, primary ring and the corresponding Landau-Ginzburg superpotential. Introduce variables ξ, which may be viewed as coordinates on the Cartan subalgebra of G (or H). The independent Casimirs of G (H, respectively), when restricted to the Cartan subalgebra, give rise to homogenous polynomials $p_{m_i+1}(\xi)$ (respectively $p'_{m'_j+1}(\xi)$), $i,j = 1,\ldots,\ell$, of degrees $m_i + 1$ (respectively, $m'_j + 1$) and which are invariant under $W(G)$ (respectively, $W(H)$). Conversely, such homogenous, Weyl invariant polynomials completely characterize the corresponding Casimir invariants. (As a parenthetic comment, these polynomials generate the Chern classes by replacing the ξ by the 2-form Cartan subalgebra eigenvalues of the curvature tensors). Note that the exponent corresponding to the $U(1)$ factor of H is $m'_1 \equiv 0$, and that the corresponding linear combination of variables ξ that defines the $U(1)$ is given by $p_1{}'(\xi)$. The corresponding generator in the cohomology is the first Chern class of the line bundle defined by non-trivial one dimensional representations of this $U(1)$.

We can introduce new variables $x_j \equiv p'_{m'_j+1}(\xi)$, and observe that because $p_{m_i+1}(\xi)$ is invariant under $W(G)$, it is invariant under $W(H)$. Hence each p_{m_i+1} can be written as a polynomial in the x_j. The chiral, primary ring is then

$$\mathcal{R} \ = \ \mathbb{C}[x_1,\ldots,x_\ell]/\mathcal{I} \ , \tag{3.2}$$

where $\mathcal{I}$ is the ideal generated by $p_{m_i+1}(\xi) \equiv p_{m_i+1}(x_j)$. Thus we have the vanishing relations

$$p_{m_i+1}(\xi) \ \equiv \ p_{m_i+1}(x_j) \ = \ 0 \ . \tag{3.3}$$

These vanishing relation fall into two categories. If $p_{m_i+1}(\xi)$ has the same degree as some $p'_{m'_k+1}(\xi)$, then it yields a "trivial" vanishing relation in that it gives an identity which enables one to eliminate x_k in favour of $x_j's$ of lower degree. If $p_{m_i+1}(\xi)$ does not have the same degree as some $p'_{m'_k+1}(\xi)$, then it represents a "non-trivial" vanishing relation (that we are going to relate to the vanishing relations $\nabla W = 0$ of the Landau-Ginzburg theory). Suppose that one eliminates all the $x_j's$ that can be eliminated by trivial vanishing relations. Relabel the remaining $x_j's$ as Φ_A; these will be associated with the independent Landau-Ginzburg fields with $U(1)$ charges $\omega_A \equiv \frac{m'_A+1}{g+1}$. The remarkable empirical observation is that for the SLOHSS models in question, for every variable Φ_A of degree m'_A+1[⋆] there is a vanishing relation $p_{g-m'_A}(\xi) \equiv p_{g-m'_A}(\Phi_B) = 0$ of degree $(g - m'_A)$. Moreover, one can integrate (certain combinations of) the vanishing relations so as to obtain a Landau-Ginzburg potential, $W_0(\Phi_A)$, of degree $g+1$. As usual, these Landau-Ginzburg potentials are quasihomogenous functions

$$W_0(\lambda^{\omega_A}\Phi_A) \ = \ \lambda\, W_0(\Phi_A) \ , \qquad \lambda \in \mathbb{C} \ , \tag{3.4}$$

with an isolated singularity at the origin (with multiplicity μ given in (2.9)).

⋆ Henceforth, the degree of an element of the ring will the degree as a polynomial in the variables ξ. This degree is equal to $(g+1)$ times the $U(1)$ charge of the corresponding Landau-Ginzburg field.

The associated local rings, $\mathcal{R} = \frac{\mathbb{C}[\Phi_A]}{\nabla W = 0}$, are isomorphic to the foregoing cohomology and chiral rings. One immediate consequence of the quasihomogeneity is obtained by differentiating with respect to λ and then setting $\lambda \equiv 1$:

$$W_0(\Phi_A) \;=\; \sum_A \omega_A \, \Phi_A \, \frac{\partial W_0}{\partial \Phi_A} \; . \tag{3.5}$$

Here, the set of derivatives, $\frac{\partial W_0}{\partial \Phi_A}$ is algebraically equivalent to the set of vanishing relations $p_{g-m'_A}(\Phi_B)$.

At this point we think it appropriate to give an explicit example, and we will consider the coset model $\frac{E_6}{SO(10)\times U(1)}$. Here, $\{m_i+1\} = \{2,5,6,8,9,12\}$ and $\{m'_j+1\} = \{1,2,4,5,6,8\}$. We will first compute the Chern classes of the non-trivial bundles, and to keep track of the degrees of the forms, we introduce a dummy variable t and define the total graded Chern form for any $H \equiv SO(10)\times U(1)$ representation v as

$$Ch(v) \;\equiv\; \det{}_v(1+t\Omega) \;\equiv\; \sum_{j=0}^{dim\, v} t^j \, p'_j(\xi) \; . \tag{3.6}$$

(Here, Ω denotes the curvature 2-form with values only in the Cartan subalgebra.) We consider in particular the H-representations that occur in the decomposition of the $\mathbf{27}$ of E_6, $\mathbf{27} \to \mathbf{10}_{-2} \oplus 16_1 \oplus \mathbf{1}_4$:

$$\begin{aligned} Ch(\mathbf{10}_{-2}) \;&=\; \prod_{n=1}^{5} (1+t[\xi_n - 2\xi_0])(1-t[\xi_n+2\xi_0]) \\ Ch(\mathbf{16}_1) \;&=\; \prod^{16} (1+t[\pm\tfrac{1}{2}\xi_1 \pm \tfrac{1}{2}\xi_2 \ldots \pm \tfrac{1}{2}\xi_5 + \xi_0]) \\ Ch(\mathbf{1_4}) \;&=\; 1 + 4t\,\xi_0 \quad . \end{aligned} \tag{3.7}$$

(with an even number of "$-$" signs in $Ch(\mathbf{16}_1)$). We now expand

$$Ch(\mathbf{27}) \;=\; Ch(\mathbf{10}_{-2})\, Ch(\mathbf{16}_1)\, Ch(\mathbf{1}_4) \;=\; \sum_{k=0}^{27} t^k \, p_k(\xi) \; , \tag{3.8}$$

and express the p_k in terms of the independent Casimirs of H, $x_{m'_j+1} \equiv$

$p'_{m'_j+1}(\xi)$:

$$\begin{aligned} x_j \;&=\; \sum_{n=1}^{5} (\xi_n)^j \; , \quad j = 2,4,6,8, \\ x_5 \;&=\; \prod_{n=1}^{5} \xi_n \\ x_1 \;&=\; \xi_0 \quad . \end{aligned} \tag{3.9}$$

For the independent E_6 Casimirs one obtains:

$$p_2(x_j) \;=\; -36{x_1}^2 - 3x_2$$

$$p_5(x_j) \;=\; 144{x_1}^5 - 24{x_1}^3 x_2 + 3x_1{x_2}^2 - 6x_1x_4 + 12x_5$$

$$\begin{aligned} p_6(x_j) \;=\; & -4680{x_1}^6 - 1062{x_1}^4 x_2 - \tfrac{177}{2}{x_1}^2{x_2}^2 - \tfrac{23}{8}{x_2}^3 - 15{x_1}^2 x_4 + \tfrac{5}{4}x_2x_4 \\ & - 60x_1x_5 - x_6 \end{aligned}$$

$$\begin{aligned} p_8(x_j) \;=\; & 25830{x_1}^8 + 7098{x_1}^6 x_2 + \tfrac{3027}{4}{x_1}^4{x_2}^2 + \tfrac{363}{8}{x_1}^2{x_2}^3 + \tfrac{171}{128}{x_2}^4 + \tfrac{555}{2}{x_1}^4 \ldots \\ & + \tfrac{105}{4}{x_1}^2x_2x_4 - \tfrac{15}{32}{x_2}^2x_4 - \tfrac{35}{32}{x_4}^2 + 1740{x_1}^3x_5 + 75x_1x_2x_5 - 6{x_1}^2x_6 \ldots \\ & - \tfrac{1}{2}x_2x_6 + \tfrac{15}{8}x_8 \end{aligned}$$

$$\begin{aligned} p_9(x_j) \;=\; & 28560{x_1}^9 - 1008{x_1}^7x_2 + 42{x_1}^5{x_2}^2 + 35{x_1}^3{x_2}^3 + \tfrac{105}{16}x_1{x_2}^4 - 924{x_1}^5 \ldots \\ & - 70{x_1}^3x_2x_4 - \tfrac{105}{4}x_1{x_2}^2x_4 + \tfrac{35}{4}x_1{x_4}^2 + 840{x_1}^4x_5 + 420{x_1}^2x_2x_5 \\ & + \tfrac{35}{2}{x_2}^2x_5 - 7x_4x_5 - 112{x_1}^3x_6 + 28x_1x_2x_6 - 21x_1x_8 \end{aligned}$$

$$\begin{aligned} p_{12}(x_j) \;=\; & 177660{x_1}^{12} + 97902{x_1}^{10}x_2 + \tfrac{36063}{4}{x_1}^8{x_2}^2 + \tfrac{1635}{4}{x_1}^6{x_2}^3 + \tfrac{7569}{64}{x_1}^4 \ldots \\ & - \tfrac{577}{128}{x_1}^2{x_2}^5 - \tfrac{15}{1024}{x_2}^6 + \tfrac{15705}{2}{x_1}^8x_4 + \tfrac{6087}{2}{x_1}^6x_2x_4 - \tfrac{7299}{16}{x_1}^4{x_2}^2x_4 \ldots \\ & + \tfrac{1741}{32}{x_1}^2{x_2}^3x_4 + \tfrac{1307}{1536}{x_2}^4x_4 + \tfrac{7527}{16}{x_1}^4{x_4}^2 - \tfrac{795}{32}{x_1}^2x_2{x_4}^2 - \tfrac{423}{256}{x_2}^2 \ldots \\ & + \tfrac{85}{128}{x_4}^3 + 94104{x_1}^7x_5 + 13050{x_1}^5x_2x_5 + \tfrac{1551}{2}{x_1}^3{x_2}^2x_5 - \tfrac{59}{8}x_1{x_2}^3 \ldots \\ & + 219{x_1}^3x_4x_5 + \tfrac{243}{4}x_1x_2x_4x_5 + 6{x_1}^2{x_5}^2 - \tfrac{19}{2}x_2{x_5}^2 - 948{x_1}^6x_6 \\ & + 1041{x_1}^4x_2x_6 - \tfrac{313}{4}{x_1}^2{x_2}^2x_6 - \tfrac{61}{48}{x_2}^3x_6 - \tfrac{25}{2}{x_1}^2x_4x_6 + \tfrac{25}{24}x_2x_4x_6 \\ & - 50x_1x_5x_6 + \tfrac{1}{3}{x_6}^2 - \tfrac{4257}{4}{x_1}^4x_8 + \tfrac{561}{8}{x_1}^2x_2x_8 + \tfrac{97}{64}{x_2}^2x_8 - \tfrac{45}{32}x_4x_8 \end{aligned}$$

Demanding $Ch(\mathbf{27}) \equiv 1$ defines the vanishing relations $p_{m_i} = 0$, the trivial ones of which determine

$$\begin{aligned} x_2 &= -12\, {x_1}^2 \\ x_5 &= \tfrac{1}{2} x_1 \left(-144 {x_1}^4 + x_4\right) \\ x_6 &= {x_1}^2 \left(4608 {x_1}^4 - 60 x_4\right) \\ x_8 &= \tfrac{1}{12}\left(393984 {x_1}^8 - 2016 {x_1}^4 x_4 + 7 {x_4}^2\right) \ . \end{aligned}$$

The remaining relations, $p_9 = 0$ and $p_{12} = 0$, give the equations of motions of the Landau-Ginzburg theory, up to quasihomogenous combinations. More precisely, putting $\Phi_1 = x_1$ and $\Phi_3 = x_4$ and eliminating the x_i above, we have

$$\begin{aligned} \frac{\partial}{\partial \Phi_3} W_0(\Phi_1, \Phi_3) &= p_9(\Phi_1, \Phi_3) \\ \frac{\partial}{\partial \Phi_1} W_0(\Phi_1, \Phi_3) &= p_{12}(\Phi_1, \Phi_3) - 180\, {\Phi_1}^3 p_9(\Phi_1, \Phi_3) \ . \end{aligned} \qquad (3.10)$$

Only this particular combination can be integrated to give the superpotential. By reparametrization, it can be represented by a normal form that has the lowest number of terms:

$$W_0\left[\tfrac{E_6}{SO(10)\times U(1)}\right] \ = \ {\Phi_1}^{13} + {\Phi_1}^9 \Phi_3 + a\, \Phi_1 {\Phi_3}^3 \ , \qquad a = -(\tfrac{5}{13})^2 \qquad (3.11)$$

The corresponding local ring has dimension $\mu = 27 = dim \Xi$, the dimension of the fundamental representation of E_6.

With a similar, but more extensive calculation, we can also obtain the Landau-Ginzburg superpotential for the $\frac{E_7}{E_6 \times U(1)}$ theory:

$$\begin{aligned} W_0\left[\tfrac{E_7}{E_6\times U(1)}\right] \ &= \ {\Phi_1}^{19} + {\Phi_3}^2 \Phi_6 + \Phi_1 {\Phi_6}^2 + a\, {\Phi_1}^{14} \Phi_3 + b\, {\Phi_1}^{10} \Phi_6 \\ & a = 37(\tfrac{19}{2791})^{3/4} \ , \quad b = -21(\tfrac{19}{2791})^{1/2} \quad (\mu = 56) \ . \end{aligned} \qquad (3.12)$$

Note that the modular parameters, a and b, are fixed to very specific values.

4. Ground States, Solitons and Polytopes

We now wish to consider the models obtained by perturbing the SLOHSS models with the (F-component of the) most relevant chiral, primary field. The corresponding Landau-Ginzburg superpotentials are

$$W(\Phi_A) \ = \ W_0(\Phi_A) - \lambda \Phi_1 \ , \qquad (4.1)$$

where W_0 is the quasihomogenous potential of the given conformal theory, λ is the perturbation parameter and Φ_1 the lowest dimensional, chiral primary field (with $h = \frac{1}{2(g+1)}$). The field Φ_1 corresponds to the first Chern class of the $U(1)$ line bundle over the hermitian, symmetric space, G/H. In terms of the Cartan subalgebra variables, ξ, $\Phi_1(\xi)$ is simply the projection of ξ on the $U(1)$ factor, which is given explicitly by $(\rho_G - \rho_H)\cdot\xi$.

As the bosonic potential of (3.1) is given by $V = \overline{\nabla W}(K'')^{-1}\nabla W$, the vacuum states of the theory are given by expectation values of Φ_A that satisfy the "equations of motion"

$$\frac{\partial}{\partial \Phi_A} W(\Phi_A) \ = \ 0 \ . \qquad (4.2)$$

We will show below that under the perturbation (4.1), the μ-fold multi-critical points of W_0 resolve into μ quadratic critical points of W, so that we have μ distinct vacuum states $\Phi^0_{A,i}$, $i = 1, , \ldots, \mu$, and all small oscillations are thus massive.

Note that the equations (4.2) are equivalent to $\frac{\partial}{\partial \Phi_A} W_0 = 0, A \neq 1$, plus $\frac{\partial}{\partial \Phi_1} W_0 = \lambda$. To solve these equations it is actually far simpler to revert to the Cartan subalgebra variables, ξ. Indeed the foregoing equations are equivalent to solving

$$\begin{aligned} p_{m_i+1}(\xi) &= 0 \qquad i = 1, \ldots, \ell - 1 \\ p_{m_\ell+1}(\xi) &= \lambda \ , \end{aligned} \qquad (4.3)$$

where the $p_{m_i+1}(\xi)$ are the Casimirs of G and $p_{m_\ell+1}(\xi) \equiv p_g(\xi)$ is the Casimir of highest degree. Observe if $\xi \equiv \xi_{(1)}$ is a solution of (4.3) and $w \in W(G)$, then $w(\xi_{(1)})$ is also a solution of (4.3). Conversely, it was established by Kostant [12] that if $p_{m_i+1}(\xi) = p_{m_i+1}(\tilde{\xi})$, $i = 1, \ldots, \ell$, then $\xi = w(\tilde{\xi})$ for some $w \in W(G)$. Thus, $W(G)$ acts simply transitively on all solutions of (4.3), and thus plays the role of a (subgroup of the) Galois group for these equations.

It actually turns out that equations (4.3) have already been solved in [12]. The solutions are related to the "cyclic elements" that are naturally associated with the principal three dimensional subalgebra (which itself is related to the exponents of G). It is however possible to characterize the solutions to (4.3) on a more mundane level, and for this we need some simple facts about Weyl groups.

Let $r_i : \lambda \to \lambda - \frac{2\langle\alpha_i,\lambda\rangle}{\langle\alpha_i,\alpha_i\rangle}\alpha_i$ be the fundamental Weyl reflections corresponding to the simple roots α_i. The product of the reflections:

$$S \equiv r_1 r_2 \cdots\cdot r_\ell \ ,$$

defines the Coxeter element of $W(G)$. One could take the product of the $r_i's$ in any order, or choose a different system of simple roots, and the result is still called a Coxeter element. It turns out that all Coxeter elements are conjugate to each other, and have all the same order, which is g, the dual Coxeter number. When acting on the Cartan subalgebra, the Coxeter element acts as a rotation whose diagonal form is:

$$S = \begin{pmatrix} e^{2\pi i m_1/g} & & & \\ & e^{2\pi i m_2/g} & & \\ & & \ddots & \\ & & & e^{2\pi i m_\ell/g} \end{pmatrix} , \tag{4.4}$$

where m_i are the exponents of G. Note that the ith eigenspace is the complex conjugate of the $(\ell+1-i)$th eigenspace. Now recall that $m_1 = 1$ and suppose that $\xi_{(1)}$ lies in this eigenspace. As the polynomials $p_{m_i+1}(\xi)$ are of homogenous degree m_i+1, it follows

$$p_{m_i+1}(e^{2\pi i/g}\xi_{(1)}) \ = \ e^{2\pi i(m_i+1)/g} p_{m_i+1}(\xi_{(1)})$$

However, these polynomials are Weyl invariant, whence

$$e^{2\pi i(m_i+1)/g} p_{m_i+1}(\xi_{(1)}) = p_{m_i+1}(\xi_{(1)})$$

and therefore (some appropriate multiple of) $\xi_{(1)}$ must satisfy (4.3). The complete set of solutions is thus the complete orbit of $\xi_{(1)}$ under the Weyl group. It turns out that $\xi_{(1)}$ is always a regular element of the Cartan subalgebra, and so there is no element of $W(G)$ that leaves $\xi_{(1)}$ invariant. That is, all the ground states occur at distinct values of $\xi_{(1)}$.

In practice it is very simple to construct $\xi_{(1)}$. Take any root, α, of G and let $\eta \equiv e^{2\pi i/g}$, then

$$\xi_{(1)} \ = \ k \sum_{j=0}^{g-1} \eta^j \, s^j(\alpha) \tag{4.5}$$

where k is some constant. In order to obtain the soliton masses we will need to evaluate the superpotential, W, on all Weyl images of $\xi_{(1)}$. However, from the quasihomogenity of the unperturbed superpotential, W_0, it follows that if $\tilde{\xi}_{(1)}$ is some Weyl group image of $\xi_{(1)}$, we have

$$\begin{aligned} W(\Phi_A(\tilde{\xi}_{(1)})) \ &= \ -\lambda \, \Phi_1(\tilde{\xi}_{(1)}) + \sum_B \omega_B \Phi_B \frac{\partial}{\partial \Phi_B} W_0(\Phi_A(\tilde{\xi}_{(1)})) \\ &= \ \lambda(\omega_1 - 1) \, \Phi_1(\tilde{\xi}_{(1)}) \ , \end{aligned} \tag{4.6}$$

where we have used $\frac{\partial}{\partial\Phi_B} W_0(\Phi_A(\tilde{\xi}_{(1)})) \equiv 0$ $(B \neq 1)$ as well as $\frac{\partial}{\partial\Phi_1} W_0(\Phi_A(\tilde{\xi}_{(1)})) \equiv \lambda$. The weight $\omega_1 = \frac{1}{g+1}$ is the $N=2$ $U(1)$ charge of the most relevant chiral primary field. The consequence that will be important for us later is that the value of $W(\Phi_A(\tilde{\xi}_{(1)}))$ is a simple (uniform) multiple of $\Phi_1(\tilde{\xi}_{(1)}) \equiv (\rho_G - \rho_H) \cdot \tilde{\xi}_{(1)}$.

As a simple example, consider $G = SU(n+1)$ and $H = SU(n) \times U(1)$ (corresponding to the minimal series A_{n+1}). Let $e_1, \ldots, e_{n+1}$ be the usual orthonormal basis for the Cartan subalgebra of $U(n+1)$ (see, for example, [13]). Taking $\alpha_j = e_j - e_{j+1}, j = 1, \ldots, n$ as the simple roots of G, the Coxeter element $S \in W(G)$ is the cyclic permutation $S : e_1 \to e_2 \to \ldots \to e_n \to e_{n+1} \to e_1$. The vector $\xi_{(1)}$ is given by

$$\xi_{(1)} \;=\; \sum_{j=0}^{n} \eta^j \, e_{j+1} \;, \tag{4.7}$$

where $\eta = e^{2\pi i/(n+1)}$, and the vector that defines the $U(1)$ factor of H is given by $(\rho_G - \rho_H) = (\sum_{j=1}^{n} e_j) - n e_{n+1}$. As will be explained momentarily, the physically distinct solutions to (4.3) are obtained by taking one representative $w \in W(G)$ of each class in the quotient $\frac{W(G)}{W(H)}$, and using $\tilde{\xi}_{(1)} = w(\xi_{(1)})$. It is elementary to see that there are $n+1$ possible such choices for w, corresponding to the possible coefficients of e_{n+1} in (4.6). Let w_j be such that $w_j(\xi_{(1)}) \cdot e_{n+1} = \eta^j$. It then follows that $w_j(\xi_{(1)}) \cdot (\rho_G - \rho_H) = -(n+1)\eta^j$. Consequently, we have shown that there are $\mu = n+1$ distinct ground states, and that the value of the superpotential at these ground states is $\kappa\eta^j$, $j = 1, \ldots, n$, where κ is some irrelevant constant. Of course, for this simple example one would have arrived at the same result more easily by directly solving (4.3) for $W(\Phi_1) = {\Phi_1}^{n+2} - \lambda\Phi_1$.

In general the Landau-Ginzburg fields are represented not by ξ but rather by Φ_A which are given by polynomials $p'_{m'_A+1}(\xi)$ that are invariant under $W(H)$. Applying the above-mentioned theorem of Kostant to H we see that all the Landau-Ginzburg fields will have the same expectation values for ξ and $\tilde{\xi}$ if and only if $\xi = w'(\tilde{\xi})$ for some $w' \in W(H)$. Thus the Landau-Ginzburg vacua are enumerated by the $W(H)$ equivalence classes of solutions of (4.3) and are given by

$$\{\, \Phi_{A,i} \,\} \;\equiv\; \{ p'_{m'_A+1}(w(\xi_{(1)})), \; w \in W(G)/W(H) \} \;. \tag{4.8}$$

The number of such classes is consistent with the vacuum state degeneracy

at the conformal point, given in (2.9). Thus, W is a fully resolved superpotential; that is, we have always $\mu \equiv |\frac{W(G)}{W(H)}|$ distinct solutions $\Phi_{A,i}$ to the equations of motion (4.2), and the superpotential evaluated on these is

$$W(\Phi_{A,i}) \;=\; \lambda(\omega_1 - 1)\, \Phi_{1,i} \;. \tag{4.9}$$

This result, which will be important for us later, is independent from the particular choice of quasihomogenous coordinates Φ_A.

In order to get at the soliton structure of these theories it is convenient to change our perspective. We have seen that the ground states of both the unperturbed and the perturbed theories are characterized by cosets of $\frac{W(G)}{W(H)}$. In the former instance, the Ramond states are characterized by branching functions with labels Λ and λ where $\lambda = w(\Lambda + \rho_G) - \rho_H$ with w chosen from a specific set of coset representatives of $\frac{W(G)}{W(H)}$, while in the latter instance one simply considers the coset representatives acting on $\xi_{(1)}$. In making the perturbation, we know that the Ramond ground states of the conformal theory must smoothly deform to ground states of the off-critical theory. Because the Weyl action on ground states off criticality is directly induced via the Weyl action on chiral, primary fields on criticality, it follows that if two off-critical ground states are related by some coset representative $w \in \frac{W(G)}{W(H)}$, then the corresponding Ramond ground states in the conformal theory must be related to each other by the same Weyl transformation, w. Consequently, it is the Weyl cosets that are the more fundamental characterization of the ground states. This has two important consequences.

The first consequence is that we can label the ground states in a different manner. Observe that $w(\rho_G - \rho_H) \equiv \rho_G - \rho_H$ if and only if $w \in W(H)$, thus there is a one-to-one correspondence between the Weyl images of $\rho_G - \rho_H$ and the cosets of $\frac{W(G)}{W(H)}$, and hence with the vacuum states of the perturbed theory. The vector $2(\rho_G - \rho_H)/g$ is a fundamental, miniscule* weight of G,

$\star$ A miniscule representation is one for which *all* the weights of the representation have the same length, and thus are Weyl images of the highest weight.

and indeed equal to the highest weight of the representation Ξ introduced in section 2. Therefore, its Weyl images are the complete set of weights of the representation Ξ ($dim\Xi \equiv \frac{|W(G)|}{|W(H)|}$). Thus the correspondence between the weights of Ξ and the ground states is preserved off criticality. *We can therefore use the weight diagram of* Ξ *to represent graphically the ground states of the theory.*†

Every pair of ground states must be linked by a minimum energy, 'solitonic' configuration. However, some of these configurations will be combinations of two or more fundamental solitons. More precisely, if σ is a spatial coordinate in two dimensional Minkowski space, and we impose boundary conditions that at $\sigma = -\infty$ and $\sigma = +\infty$ the Landau-Ginzburg fields approach there expectation values in ground state A and ground state B respectively, then it is possible that the minimum energy configuration will be a multi-soliton state running from ground state A to ground state B via other intermediate ground states. The problem now is to determine the fundamental solitons and which pairs of vacua they connect.

This leads us to the second consequence of our earlier discussion. Consider the Ramond vacuum state s_0 that is labelled by $\lambda = \rho_G - \rho_H$ or by the $SO(2d)$ highest weight $v_{\overline{\alpha}} = +\frac{1}{2}, \overline{\alpha} \in t_+$. The Ramond vacuum state $w(s_0)$, corresponding to a Weyl element w, can be obtained from s_0 by acting on it with the operator Φ_w in (2.7) (indeed it is only the fermionic zero-modes in Φ_w that are needed to obtain $w(s_0)$). Imagine slowly turning on the perturbation: the foregoing Ramond vacua go to ground states $\tilde{s}_0$ and $w(\tilde{s}_0)$ that are still Weyl images of each other. Thus, the operator Φ_w must deform to the soliton operator linking the two perturbed ground states. This implies that the soliton operators are linked to products of free fermions in the coset model. This is consistent with ones experience with sine-Gordon theory. It also suggests that the soliton configurations that are the products of several

† We believe that there is always some particular choice of (partly redundant) variables in terms of which the set of vacuum states is directly given by the weight diagram of Ξ. We will however make no use of this conjecture in the following.

fermions are multi-soliton states, whereas those that are associated with a single fermion operator should be viewed as fundamental. Putting this another way, all the fundamental solitons leaving $\tilde{s}_0$ should be associated with single Weyl reflections $r_{\overline{\alpha}}, \overline{\alpha} \in t_+$. More generally, there should be a fundamental soliton between $\tilde{s}$ and $\tilde{s}'$ if and only if there is a root $\alpha \in \Delta_+(G)$ such that the single Weyl reflection r_α takes $\tilde{s}$ to $\tilde{s}'$. Hence, to every fundamental soliton there is an associated root of G, and in our graphical representation of the set of ground states by the weight diagram of Ξ, the fundamental soliton will be represented by this root connecting the appropriate weights.

We will call the figure generated in the foregoing manner the *soliton polytope* of the theory. As we will see, it gives a complete characterization of the soliton structure of the theory. It turns out that, with the exception of the general grassmannian models, the polytopes in question are described in detail in the mathematical literature [14]; for a quick reference, see Table 1. The vertices are given by the weights of Ξ, and the 1-simplices, *i.e.*, the one dimensional edges lying on the surface of these figures, are given by the roots of G. Moreover, the figures are symmetric under the action of the Weyl group of G. Some elements of $W(G)$ may act trivially, but it is interesting to note that while $W(H)$ fixes an individual vacuum state, some elements of $W(H)$ can act non-trivially by permuting other vacua.

The foregoing conclusion about fundamental solitons is based upon some suppositions about the relationship between the conformal model and its perturbation. Therefore, we will adopt our conclusion more as a working hypothesis, but in the remainder of this paper we will amass a compelling amount of evidence for it.

To this end, we will first give a different, more geometric picture of the role of the fundamental solitons. Consider the Nicolai map [15] applied to the Landau-Ginzburg form of the perturbed theory and introduce new variables $u_A \equiv \frac{\partial \Phi_A}{\partial z} - (K''^{-1})_{A\overline{B}} \frac{\partial \overline{W}}{\partial \overline{\Phi}_{\overline{B}}}$. After integrating out the fermions and making this change of variables, the path integral becomes a trivial Gaussian. One of the useful properties of this map is that the Gaussian theory has a single ground state, and so the ground state degeneracy of the original Landau-Ginzburg theory is given by the number of times the variables Φ_A cover the configuration space of the u_A. This can be computed by ignoring the $\frac{\partial \Phi_A}{\partial z}$ term in the change of variables, and thus by computing the winding number of the map $\Phi_A \to \frac{\partial W}{\partial \Phi_A}$ at large values of Φ_A [16]. A single vacuum sector of the theory is covered by using $\frac{\partial W}{\partial \Phi_A} \sim p_{g-m'_A}(\xi)$ as coordinates. Thus, by making the Nicolai map, we are now using essentially the Casimirs of G to parametrize a single vacuum sector of the theory. In terms of the ξ variables, this means that a single vacuum sector coincides with the fundamental Weyl chamber of G. It is thus natural to expect that the fundamental solitons will be ones that connect immediately neighbouring chambers, or equivalently connect adjacent sheets in the multiple covering of the Nicolai variables. Neighbouring Weyl chambers are precisely those that are mapped onto each other by a single Weyl reflection, and their common wall is the hyperplane orthogonal to the root in question. While this argument is not by any means rigorous, we feel that there may be a way to make it so by relating the existence of fundamental solitons to intersection matrices of homology cycles associated with the resolved Landau-Ginzburg potential, and then relating these intersection forms to the structure of the roots that connect weights in Ξ.

5. Projections and Soliton Quantum Numbers

It was shown in [3] that the perturbed SLOHSS models based on G/H are integrable, massive field theories having conserved charges[⋆] $\mathcal{I}_s$, with spins s equal, modulo g, to the exponents m_i of G. Note that $\mathcal{I}_1$ is the energy operator, P. The eigenvalues $q^{(s)}$ of the integrals of motion $\mathcal{I}_s$ on all excitations of the theory are highly constrained by the bootstrap equations [4],

$$q_i^{(s)} e^{-is(\pi-\theta_{i\ k}^{\bar{j}})} + q_j^{(s)} e^{is(\pi-\theta_{j\ k}^{\bar{i}})} \ = \ q_{\bar{k}}^{(s)} \tag{5.1}$$

(here, i,j,k label the excitations and θ denote the fusion angles). It is of obvious interest to determine the $\mathcal{I}_s$ quantum numbers of the Landau-Ginzburg solitons.

It is relatively easy to obtain the masses (spin-1 quantum numbers) of the solitons, by employing [17][5] the central charge T of $N=2$ supersymmmetry algebra. The non-zero anti-commutators of this algebra are

$$\begin{aligned} \{Q_+,Q_-\} &= 2P \qquad & \{\overline{Q}_+,\overline{Q}_-\} &= 2\overline{P} \\ \{Q_+,\overline{Q}_+\} &= 2T \qquad & \{Q_-,\overline{Q}_-\} &= 2\overline{T}^* \ . \end{aligned} \tag{5.2}$$

By conformal perturbation theory, we can evaluate the topological charge T on the solitons explicitly. That is, in the perturbed theory, the conservation laws of the supercharges become:

$$\begin{aligned} \partial_{\overline{z}} G^+(z,\overline{z}) &= \lambda(1-\omega_1)\,\partial_z(\overline{G}^-_{-1/2}\Phi_1)(z,\overline{z}) \\ \partial_z \overline{G}^+(z,\overline{z}) &= \lambda(1-\omega_1)\,\partial_{\overline{z}}(G^-_{-1/2}\Phi_1)(z,\overline{z}) \end{aligned} \tag{5.3}$$

(similarly for G^-). Here, Φ_1 is the chiral superfield having $U(1)$ charge

⋆ We will consider in the following only $\mathcal{I}_s$ with spins $1 \leq s \leq g-1$, the eigenvalues of the higher integrals of motion on the soliton spectrum being only periodic repetitions.

$\omega_1 = \frac{1}{g+1}$, and whose top superpartner constitutes the perturbation. Thus

$$Q_+ \;=\; \int G^+ dz - \int \lambda(1-\omega_1)\, \overline{G}^-_{-1/2}\Phi_1\, d\overline{z}\;, \tag{5.4}$$

is a conserved charge (similar for the other supercharges). Therefore,

$$\begin{aligned} 2T \;\equiv\; \{Q_+,\overline{Q}_+\} \;&=\; \lambda(\omega_1-1)\int (dz\partial_z + d\overline{z}\partial_{\overline{z}})\Phi_1(z,\overline{z}) \\ &=\; 2\lambda(\omega_1-1)\big[\Phi_1(\infty)-\Phi_1(-\infty)\big] \\ &=\; 2\big[W(\Phi_A(\infty))-W(\Phi_A(-\infty))\big] \;\equiv\; 2\,\Delta W\;, \end{aligned} \tag{5.5}$$

where the last line follows from (4.9) and reproduces a well-known result [17]. From $\{\mathcal{Q},\mathcal{Q}^\dagger\}\geq 0$, where $\mathcal{Q} = Q_+ - \frac{\Delta W}{\langle \overline{P}\rangle}\overline{Q}_-$ one obtains:

$$m_{(i,j)} \;\equiv\; q^{(1)}_{(i,j)} \;\geq\; \big|W(\Phi^0_{A,i}) - W(\Phi^0_{A,j})\big|\;, \tag{5.6}$$

where $m_{(i,j)}$ is the mass of a soliton linking the ith with the jth vacuum. Equality in this equation corresponds to the Bogomolnyi bound, and it is saturated if and only if the supercharges $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ annihilate the soliton. We will make the highly plausible assumption that the fundamental solitons do indeed saturate this bound, as it is expected in an elastic scattering theory.

As we observed in the preceeding section, the value of the superpotential W at the solution $\Phi_A(\tilde{\xi}_{(1)})$ to (4.2) is given by

$$\lambda(\omega_1-1)\Phi_1(\tilde{\xi}_{(1)}) \;=\; \lambda(\omega_1-1)\,(\rho_G-\rho_H)\cdot\tilde{\xi}_{(1)}\;.$$

Also recall that $\tilde{\xi}_{(1)} = w(\xi_{(1)})$ for some $w\in\frac{W(G)}{W(H)}$, where $\xi_{(1)}$ is the $e^{2\pi i/g}$ eigenspace of the Coxeter element (4.4). Observing that $(\rho_G-\rho_H)\cdot\tilde{\xi}_{(1)} = \xi_{(1)}\cdot w^{-1}(\rho_G-\rho_H)$, we can conclude that the values of the superpotential at the ground states can be thought of as projection of the soliton polytope onto the eigenspace defined by $\xi_{(1)}$. Consequently, *the masses of the fundamental solitons are simply given by the lengths of the projections of the one-simplices, or roots of G.* We will call this projection on the $e^{2\pi i m_1/g}$ Coxeter eigenspace the "mass" or "spin-1 projection" of the soliton polytope, as it refers to the masses, or spin-1 integrals of motion ($s = m_1 = 1$).

It is elementary to generalize the arguments of [5] to more than one Landau-Ginzburg variable and to therefore conclude that solitons that saturate the Bogolmolny bound have semi-classical trajectories, $\Phi_A(\sigma)$, that project to straight lines in the W-plane, *i.e.*, $W(\Phi_A(\sigma))$ is a straight line in the complex W-plane. This does not imply that the soliton trajectories $\Phi_A(\sigma)$ are straight lines in $\mathbb{R}^M$, though it might be so for a particular choice of (partly redundant) coordinates.

From the foregoing observations, it is now elementary to determine the fundamental soliton masses for a general SLOHSS model. Up to an overall scale factor, the mass of a soliton corresponding to a root α is simply $|\alpha\cdot\xi_{(1)}|$. It is simple to compute this dot product in any example, but we will defer this and go into some more of the theory of roots and Coxeter elements so as to determine not only the soliton masses in general, but also their charges with respect to all other integrals of motion, $\mathcal{I}_s$.

First we need to fix a particular Coxeter element and the corresponding vector $\xi_{(1)}$. A natural choice was introduced by Carter [18,19]* . Given any system of simple roots $\alpha_i, i=1,\ldots,\ell$, the α_i can be decomposed into two disjoint sets, which we will denote by $\{\alpha_i : i\in I_1\}$ and $\{\alpha_i : i\in I_2\}$ with $I_1\cup I_2 = \{1,\ldots,\ell\}$, such that $\alpha_i\cdot\alpha_j = 0$ for $i,j\in I_1$ or $i,j\in I_2$. (This decomposition is equivalent to making a bicoloration of the Dynkin diagram.) Dual to this set of simple roots, introduce fundamental weights λ_i such that $\lambda_i\cdot\alpha_j = \delta_{ij}$. Let $C_{ij} = \alpha_i\cdot\alpha_j$ be the Cartan matrix of G. The eigenvalues of

$\star$ We are grateful to E. Corrigan and P. Dorey for poining this out to us.

the Cartan matrix can be parametrized in terms of the exponents of G,

$$\begin{aligned} \sum_j C_{ij}\, q_j^{(s)} \ &= \ \mu_s\, q_i^{(s)} \ , \qquad \text{where} \\ \mu_s \ &= \ 4\sin^2[s\tfrac{\pi}{2g}] \ , \qquad s\in\{m_1,\ldots,m_\ell\} \ , \ \ j=1,\ldots,\ell \ . \end{aligned} \tag{5.7}$$

Define vectors

$$a_\nu^{(s)} \ \equiv \ \sum_{j\in I_\nu} q_j^{(s)}\lambda_j \ , \qquad \nu = 1,2 \ . \tag{5.8}$$

Finally, introduce two Weyl elements w_1 and w_2 defined by $w_\nu \equiv \prod_{i\in I_\nu} r_{\alpha_i}$, $\nu = 1,2$. The order of the reflections r_{α_i} in these products does not matter because of the mutual orthogonality of the α_i within each of the disjoint subsets. The Coxeter element that we will use is given by

$$S \ \equiv \ w_1\, w_2 \ .$$

Observe that $w_1^2 = w_2^2 = 1$, and hence $S^{-1} = w_2\, w_1$. The following facts may then be proven [18][19]:

(i) The Coxeter element, S, acts as a rotation by $\frac{2\pi s}{g}$ on the space spanned by the two vectors $a_1^{(s)}$ and $a_2^{(s)}$. That is, these vectors define a natural, although non-orthogonal Coxeter eigenbasis (4.4) of the Cartan subalgebra.

(ii) For each exponent s the two vectors $a_1^{(s)}$ and $a_2^{(s)}$ have the same length.

(iii) The angle between $a_1^{(s)}$ and $a_2^{(s)}$ is $\frac{s\pi}{g}$.

The vectors $a_1^{(1)}$ and $a_2^{(1)}$ are thus real linear combinations of $\xi_{(1)}$ and its complex conjugate $\overline{\xi}_{(1)}$. Hence $a_1^{(1)} = z\xi_{(1)} + \overline{z}\overline{\xi}_{(1)}$ for some complex number z. From (ii) and (iii) it follows that we can chose $\xi_{(1)}$ so that $a_2^{(1)} = ze^{-\pi i/g}\xi_{(1)} + \overline{z}e^{\pi i/g}\overline{\xi}_{(1)}$. Inverting this we therefore have $\xi_{(1)} = [z(e^{-\pi i/g} - e^{\pi i/g})]^{-1}(a_2^{(1)} - e^{\pi i/g}a_1^{(1)})$. However the overall choice of scale in $\xi_{(1)}$ is arbitrary so we can

take

$$\xi_{(1)} = a_2^{(1)} - \omega\ a_1^{(1)}, \quad \omega = e^{\pi i/g}. \tag{5.9}$$

Now observe that for any simple root α_j, we either have $\alpha_j \cdot a_\nu^{(s)} = q_j^{(s)}$ or 0 depending on whether $j \in I_\nu$ or not. Also note that the components of $q_j^{(1)}$ are all positive (as it is the Perron-Frobenius eigenvector of C_{ij}). It follows that

$$|\alpha_j \cdot \xi_{(1)}| \ = \ q_j^{(1)} \ .$$

Therefore the the fundamental solitons corresponding to the simple roots have masses that are simply the components of the Perron-Frobenius eigenvector of C_{ij}.

To get the masses of the other solitons, we recall a theorem of Kostant [12]: a Coxeter element decomposes the entire system of roots into ℓ orbits of length g. There are also precisely ℓ roots, β_i, such that β_i is positive but $S(\beta_i)$ is negative, and these roots β_i all lie on distinct Coxeter orbits, and hence generate the Coxeter orbits [12]. Observe that for our choice of $S = w_1\, w_2$, $S(\alpha_i)$ is negative for $i \in I_2$. Moreover, since $S^{-1} = w_2\, w_1$, one similarly finds that $S^{-1}(\alpha_i)$ is negative for $i \in I_1$. The latter implies that for $i \in I_1$, $\beta_i \equiv -S^{-1}(\alpha_i)$ is positive but $S(\beta_i) = -\alpha_i$ is negative. It is easily seen that all the elements of $\{\alpha_i : i \in I_2\}$ and $\{-S^{-1}(\alpha_i) : i \in I_1\}$ are distinct, and therefore one can generate all the Coxeter orbits from $\{\gamma_i\} \equiv \{\alpha_i : i \in I_2\} \cup \{-\alpha_i : i \in I_1\}$.

Any root α has the form $S^n(\gamma_i)$ for some of the roots γ_i, and so

$$\begin{aligned} \xi_{(1)}\cdot\alpha \ &= \xi_{(1)}\cdot S^n(\gamma_i) = S^{-n}(\xi_{(1)})\cdot\gamma_i = e^{-2\pi i n/g}\xi_{(1)}\cdot(\pm\alpha_i) \\ &= \omega^{(1\pm1)/2}e^{-2\pi i n/g}\ q_i^{(1)} \end{aligned} \ , \tag{5.10}$$

where we have used (5.9) and the $\pm$ sign depends upon whether $i \in I_1$ or $i \in I_2$. Hence the mass of the soliton corresponding to any root α depends only upon the Coxeter orbit to which it belongs, and therefore again the mass is a component of the Perron-Frobenius eigenvector of the Cartan matrix, C_{ij}.

The foregoing explains the observation made in [3] that the lengths of lines in the W-plane diagram, corresponding to a perturbed SLOHSS model, relate to each other precisely as the masses of an affine-G Toda theory; indeed, it is well-known that these masses are given by the components $q_j^{(1)}$ of the Perron-Frobenius eigenvector. For instance, the soliton polytope of the $\frac{SU(n+1)}{SU(n)\times U(1)}$ theory is the n-simplex, and when it is projected into the $\xi_{(1)}$-plane the n-simplex is simply an $(n+1)$-gon with all the edges and diagonals drawn. The lengths reproduce the affine-$SU(n+1)$ Toda particle masses, $q_j^{(1)} = \sin(j\pi/n{+}1)$ [5]. More complicated examples, corresponding to the perturbed $\frac{E_6}{SO(10)\times U(1)}$ and $\frac{E_7}{E_6\times U(1)}$ models, are displayed in Fig.1 and Fig.2.

It is of course well-known that the components of the Perron-Frobenius eigenvector satisfy the bootstrap equations, (5.1), [20-24]. This can be also seen as follows* : consider three fundamental solitons whose endpoints form a triangle, and for which the corresponding roots are α_1, α_2 and α_3. Suppose that we direct the solitons so that $\alpha_3 = \alpha_1 + \alpha_2$. This means (and is born out by the results of [5]) that if we scatter soliton α_1 off soliton α_2 there should be a resonance pole corresponding to soliton α_3. Suppose that we project this triangle into the $\xi_{(1)}$ (spin-1) eigenspace, then the resulting triangle has side lengths equal to the masses of the solitons, and hence the angles of the triangle represent the imaginary rapidities at which the various resonances occur. We now take the dot product of both sides of $\alpha_3 = \alpha_1 + \alpha_2$ with $\xi_{(1)}$, and use the fact that for some choice of i, j, k, n_1, n_2 and n_3, we have $\alpha_1 = S^{n_1}(\gamma_i)$, $\alpha_2 = S^{n_2}(\gamma_j)$ and $\alpha_3 = S^{n_3}(\gamma_k)$ and obtain

$$e^{-2\pi i n_1/g}(\xi_{(1)}\cdot\gamma_i) \ + \ e^{-2\pi i n_2/g}(\xi_{(1)}\cdot\gamma_j) \ = \ e^{-2\pi i n_3/g}(\xi_{(1)}\cdot\gamma_k). \qquad (5.11)$$

If $i, j, k \in I_1$, this equation becomes:

$$q_i^{(1)} e^{2\pi i(n_3-n_1)/g} + q_j^{(1)} e^{2\pi i(n_3-n_2)/g} \ = \ q_{\bar k}^{(1)} \ ,$$

and this is precisely the bootstrap equation (5.1) for spin $s=1$ and $\theta_{1\ 3}^{\bar 2} = 2\pi(n_3-n_1)$, $\theta_{2\ 3}^{\bar 1} = 2\pi(n_2-n_3)$. The other choices of i, j, k can be dealt with in a similar manner.

* We are aware that P. Dorey [25] has recently, independently of us, given a similar proof of the bootstrap equations.

From the foregoing it is only a small step to evaluate the charges of the other integrals of motion acting on soliton states. The point is that in the proof above, it was just the closure of the soliton triangle in weight space that guaranteed the solution of the bootstrap equation. The bootstrap equation (5.1) for general spin, $s \in \{m_1, \ldots, m_\ell\}$, is very easily obtained from (5.11) by simply changing the projection vector (5.9) to

$$\xi_{(s)} \ = \ a_2^{(s)} - \omega^s a_1^{(s)} \ . \qquad (5.12)$$

However, this is just the $e^{2\pi i s/g}$ eigenvector of the Coxeter element (4.4). Consequently, the projection of the roots $\alpha_1, \alpha_2, \alpha_3$ onto this eigenspace yields precisely equation (5.1) for arbitrary s (with fusion angles defined by the mass projection onto the $\xi_{(1)}$ eigenspace). Thus, consistency requires that the charges to be associated with a soliton labelled by $\alpha \equiv S^k(\gamma_j)$ are precisely $q_j^{(s)}$. One of our main conclusions in this paper is, in other words, that *the quantum numbers of all solitons are just given by the projections of the soliton polytope on the various Coxeter eigenspaces.*

We note that the bootstrap equations (5.1) impose a vast number of non-trivial consistency conditions on the soliton charges. The fact that they are satisfied provides yet more support for our working hypothesis that associates solitons with roots. One can easily check that sums of roots that are not themselves roots do not appear to be consistent with (5.1).

That the components of the Cartan matrix eigenvectors solve the bootstrap equations was known empirically for some time in the Toda literature [20][21][22], but there was so far no geometric understanding of this fact† . It should also be noted that our Landau-Ginzburg soliton theory differs markedly from the usual Toda particle theory in that there are only ℓ Toda

† See however [25].

particles whereas there are many more solitons than this in the Landau-Ginzburg model.

As an example, consider the perturbed $\frac{E_6}{SO(10)\times U(1)}$ theory with Landau-Ginzburg superpotential (3.11). The eigenvalues that satisfy the bootstrap equations are given by [25]:

$$\{ q_j^{(s)} \} = \{\sin[s\tfrac{\pi}{12}], \sin[2s\tfrac{\pi}{12}], \sin[3s\tfrac{\pi}{12}], \sin[10s\tfrac{\pi}{12}], \sin[11s\tfrac{\pi}{12}], \sin[8s\tfrac{\pi}{12}] - \sin[2s\tfrac{\pi}{12}]\} \ . \tag{5.13}$$

In Fig.1, we have displayed the spin-1 projection of the soliton polytope, which gives the soliton masses. In Fig.3, we show the spin-4 projection which gives the $\mathcal{I}_4$ quantum numbers. These are simply given, in appropriate units, by $\{0, \pm 1\}$. Note that the diagram is highly degenerate and has only a rotational $\mathbb{Z}_3$ symmetry, and not a $\mathbb{Z}_g \equiv \mathbb{Z}_{12}$ symmetry. Such a reduction in symmetry happens always if s is not coprime to the Coxeter number, g, in which case $\mathbb{Z}_g$ gets reduced to $\mathbb{Z}_{g/(g,s)}$. In Fig.4, the spin-5 projection is shown; according to what we said above, it is fully $\mathbb{Z}_{12}$ symmetric and looks, a priori, the same as the spin-1 projection. However, the solitons are permuted so that the spin-1 and spin-5 quantum numbers are really independent. To make this more clear, we used the thickness of the lines in the figures to characterize the soliton masses, and the reader is invited to compare Fig.5 with Fig.1 in more detail. The diagrams corresponding to the remaining integrals of motion, $\mathcal{I}_7$,$\mathcal{I}_8$ and $\mathcal{I}_{11}$, are the complex conjugates (reflections) of Figs.4,3 and 1.

It would be, of course, much more satisfying if we could directly compute these higher spin quantum numbers in a manner similar to our computation of soliton masses, instead of obtaining the higher spin quantum numbers from consistency considerations. Unfortunately, have not succeeded, as yet, in generalizing the computation below (5.2) to higher spins.

We note, however, that it is possible to give an empirical generalization of (5.6) to all spins that is valid only for the perturbed, minimal A- and

D-series (based on cosets $\frac{SU(n+1)}{SU(n)\times U(1)}$ and $\frac{SO(n+2)}{SO(n)\times U(1)}$):

$$q_{(i,j)}^{(s)} = \left| W^s(\Phi^0_{A,i}) - W^s(\Phi^0_{A,j}) \right| \ .$$

This formula holds up to signs.

6. Final Comments, and the Relation to Toda Theory

As mentioned above, we expect from the results in [3] that a perturbed SLOHSS model (based on some G/H) should be quantum equivalent to an affine-G Toda theory,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{\lambda}{\beta^2}\sum_{i=1}^{\ell} e^{\beta \alpha_i \cdot \phi} - \frac{\lambda}{\beta^2} e^{-\beta\theta\cdot\phi} \ , \tag{6.1}$$

with coupling constant

$$\frac{\beta^2}{4\pi} = -\frac{g}{g+1} \tag{6.2}$$

and with non-canonical energy momentum tensor

$$T(z) = -\frac{1}{2}(\partial\phi(z))^2 + \left(\beta + \frac{1}{\beta}\right)\rho_H \cdot \partial^2\phi \ . \tag{6.3}$$

For the foregoing value of coupling constant and for this choice of energy-momentum tensor, an $N=2$ supersymmetry appears at the quantum level [3] [26] (though this is not manifest in (6.1)). It would certainly be interesting to understand the precise relationship between the Toda and Landau-Ginzburg descriptions. However, as the quantization of models (6.1) with imaginary coupling constants is not yet well understood, we will present in this section only some observations and speculations on this relationship.

An interesting observation is that the spin-1 projections, which give the masses of the solitons, contain building blocks of the Toda particle perturbation expansion in the form of dual diagrams. Therefore, by exploiting the soliton polytope, these dual diagrams can be related to each other by the Weyl group, $W(G)$. In particular, soliton triangles give just the three-point couplings in the Toda theory. This makes contact with the observations [22][24] [27] that the non-vanishing, three-point couplings are characterized by triangles whose side lengths are proportional to the corresponding particle masses. Remembering that soliton lines are just projections of roots, one easily finds that the Toda coupling triangles are just given by triples of roots that add up to zero[⋆] . More generally, n-point couplings correspond to n-gons, and the higher order poles in the S-matrix arising from loop corrections can be described [6] by all possible tilings of polygons in terms of fundamental coupling triangles (for an example, see Fig.5). In terms of solitons, a tree-level n-point amplitude would correspond to some skew n-gon on the polytope, and the quantum corrections to higher polytopes whose boundary is this n-gon. Poles in the S-matrix occur if the internal lines are on-shell, that is, if the spin-1 projection of the higher polytope lies completely within the projection of the skew n-gon. Note however that because of the finite extent of the soliton diagram, there is only a limited number of independent higher point couplings in the soliton theory. Note also that the dual diagrams do not, in general, correspond to physical, but rather to off-shell amplitudes in the Toda theory. In general, only a subset of all polygons can directly describe physical elastic scattering processes.

Actually, there does not seem to exist a very close relationship between Landau-Ginzburg solitons and Toda particle excitations. The S-matrices of both theories are different, though they must be composed out of the same building blocks (as any theory whose integrals of motion are related to the exponenents of some Lie algebra); this is required by the bootstrap equations.

⋆ This observation was also made in [25].

The main difference is the multiplicity of Landau-Ginzburg soliton states with the same quantum numbers. Consequently, as the sets of in- an out-going states need not coincide, the soliton S-matrices (associated to certain graphs or p-simplices on the polytope) must correspond to non-trivial solutions of the Yang-Baxter equation. There is also an added complication to the Landau-Ginzburg models. For the perturbed, minimal A-series it was shown [5] that the fundamental solitons indeed form a closed scattering theory. However, for more general coset theories, it is easy to see that on cannot always obtain a purely solitonic out-state. Therefore, in order to close the Landau-Ginzburg scattering matrix one has to include other particle, breather-like excitations as well.

It seems more likely that the Landau-Ginzburg solitons are closely related to Toda solitons. These exist only for imaginary coupling constants. It is well-known that the classical Toda vacua lie on weight lattices. In the full quantum theory, one expects a truncation of the allowed soliton sectors for rational coupling constants, $\frac{\beta^2}{4\pi} \in \mathbb{Q}_-$. For instance, it is known [28] that for the affine $SU(2)$-Toda theory with imaginary coupling, the sine-Gordon model, the soliton sectors that are allowed in the quantum theory correspond just to the weight diagram of some $SU(2)$ representation (which depends on the particular value of the coupling constant), and not to the whole weight space. Roughly speaking, the classical, infinite-well sine-Gordon potential gets effectively reduced to a potential with a finite number of wells [28].

It appears that our Landau-Ginzburg soliton models provide a very concrete, unitary realization of quantum affine-G Toda soliton theories, with truncation to weights of level one representations Ξ of affine-G. (Remember that for given G in our coset models, there are in general several different choices for H, and thus, Ξ. This corresponds to making different truncations of an affine-G Toda theory.) From the viewpoint of truncation, the Landau-Ginzburg models behave like Toda theories with quantum group[⋆] symmetry

⋆ Note that in usual Toda theories, where ρ_H in (6.3) is replaced by ρ_G, the relevant

$\mathcal{U}_q(G)$, $q = -e^{-i\pi/(g+1)}$. Though this quantum equivalence is valid only for the particular value of coupling (6.2), we believe that our Landau-Ginzburg description captures the relevant features of quantum Toda soliton theory. For other couplings, $N=2$ supersymmetry and the consequent very simple, well-defined Landau-Ginzburg description is lost.

For the integrable models considered here (and probably for all integrable models based on Lie algebras), it appears that it is most natural to describe the ground states and soliton structure in terms of weight spaces, Weyl groups and Cartan subalgebra variables. From any member of a hierachy of integrable systems, one can obtain the entire hierarchy by interpreting any integral of motion as a new Hamiltonian. We have seen that for SLOHSS models and Toda theories the integrals of motion are naturally associated with different projections of weight space. It is therefore tempting to suggest that the polytopes and their tesselations by p-simplices are more fundamental objects in the complete solution space of the entire hierarchy. Particular Hamiltonians would just amount to choosing projections onto particular Coxeter eigenspaces.

Acknowledgements. We like to thank Ed Corrigan for interesting discussions and in particular for raising the issue of polytopes. We would also like to thank Tohru Eguchi, Victor Kac and Dirk-Jan Smit for interesting discussions. We are grateful to the ITP at Santa Barbara for warm hospitality and for a very pleasant stay.

References

[1]. W. Lerche, C. Vafa and N.P. Warner,Nucl. Phys. B324 (1989) 427.

quantum groups $\mathcal{U}_q(G)$ have deformation parameter $q = -\exp[4\pi^2 i \frac{(1+\beta^2/4\pi)}{\beta^2}]$, and for the value of coupling (6.2), we have $q = -e^{-i\pi/g}$ and *all* soliton sectors become unphysical. This corresponds to the fact [29] that replacing ρ_H by ρ_G in (6.3)

[2]. Y. Kazama and H. Suzuki, Phys. Lett. 216B (1989) 112; Nucl. Phys. B321 (1989) 232.

[3]. P. Fendley, W. Lerche, S.D. Mathur and N.P. Warner, Nucl. Phys. B348 (1991) 66.

[4]. A.B. Zamolodchikov, JETP Letters 46 (1987) 161; "Integrable field theory from conformal field theory" in *Proceedings of the Taniguchi symposium* (Kyoto 1989), to appear in Adv. Studies in Pure Math; R.A.L. preprint 89-001; Int. J. Mod. Phys. A4 (1989) 4235.

[5]. P. Fendley, S. Mathur, C. Vafa and N.P. Warner, Phys. Lett. 243B (1990) 257;

[6]. H. Braden, E. Corrigan, P. Dorey and R. Sasaki, Nucl. Phys. B338 (1990) 689; *Multiple poles and other features of affine Toda Field Theory*, preprint NSF-ITP-90-174, DTP-90-57, YITP/U-90-25.

[7]. D. Gepner, Phys. Lett. 222B (1989) 207; *A comment on the chiral algebra of quotient superconformal field theory*, preprint PUPT 1130; *On the algebraic structure of N=2 string theory*, preprint WIS-90/47/Ph.

[8]. V.I. Arnold, *Singularity Theory*, Lond. Math. Soc. Lecture Series #53; V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, *"Singularities of differentiable maps"*, Birkhäuser (1985).

[9]. See, for example: C. Vafa and N. Warner, Phys. Lett. 218B (1989) 51; E. Martinec, Phys. Lett. 217B (1989) 431; D. Gepner, Phys. Lett. 222B (1989) 207; W. Lerche, C. Vafa and N.P. Warner,Nucl. Phys. B324 (1989) 427; E. Martinec, *"Criticality, catastrophes and compactifications"*, V.G. Knizhnik memorial volume; P. Howe and P. West, Phys. Lett. 223B (1989) 377; K. Ito, Phys. Lett. 231B (1989) 125; S. Cecotti and L. Girardello, Nucl. Phys. B328 (1989) 701.

[10]. R. Bott and L. Tu, *Differential forms in algebraic topology*, Springer 1982.

[11]. H. Hiller, *Geometry of Coxeter groups*, Pitman, London 1982.

[12]. B. Kostant, Am. J. Math. 81 (1959) 973.

[13]. B. Wybourne, *Classical groups for physicists*, Wiley 1974.

[14]. H.S.M. Coxeter, *Regular Polytopes*, Methuen & Co, London, 1948; *Regular Complex Polytopes*, Cambridge University Press, 1974; Am. J. Math. **62** (1940) 457.

[15]. H. Nicolai, Phys. Lett. 89*B* (1980) 341.

[16]. S. Cecotti and L. Girardello, Phys. Lett. 110*B* (1982) 39.

[17]. E. Witten and D. Olive, Phys. Lett. B78 (1978) 97.

[18]. R. Carter, *Simple groups of Lie type*, Wiley 1972.

[19]. J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Univ. Press 1990.

[20]. P. Dorey, Phd. thesis, unpublished.

[21]. T. Klassen and E. Melzer, Nucl. Phys. *B*338 (1990) 485.

[22]. H. Braden, E. Corrigan, P. Dorey and R. Sasaki, *Aspects of perturbed conformal field theory, affine Toda field theory and exact S-matrices*, to appear in the proceedings of the XVIII International Conference on Differential Geometric Methods in Theoretical Physics, Lake Tahoe, USA, July 1989.

[23]. P. Freund, T. Klassen and E. Melzer, Phys. Lett. 229*B* (1989) 243.

[24]. P. Christe and G. Mussardo, Nucl. Phys. *B*330 (1990) 465.

[25]. P. Dorey, *Root systems and purely elastic S-matrices*, preprint SPhT/90-169.

[26]. K. Kobayashi, *Quantum extension of symmetry in $N = 2$ supersymmetric theory*, Univ. of Rochester preprint, Dec. 1990.

[27]. C. Goebel, Progr. Theor. Phys. suppl. 86 (1986) 261.

[28]. See e.g.: A. LeClair, Phys. Lett. 230*B* (1989) 103; F. Smirnov, Int. Journ. Mod Phys. A4 (1989) 4213; T. Eguchi and S. Yang, Phys. Lett. 235*B* (1990) 282; D. Bernard and A. LeClair, Nucl. Phys. *B*340 (1990)721; N. Reshetikin and F. Smirnov, Comm. Math. Phys. 131 (1990)157; H. Itoyama and P. Moxhay, Phys. Rev. Lett. 65 (1990) 2102; T. Nakatsu, *Quantum group approach to affine Toda field theory*, preprint UT-567.

[29]. W. Lerche, Phys. Lett. 252*B* (1990) 349.

Table 1: Polytopes and solitons for various coset models. The notation is in accordance with ref. [14].

Polytope	Vertices (# of Vacua)	Edges (Solitons)	Triangles (Couplings)	Order of aut. Group	Perturbed Coset c.f.t.
α_n	$n+1$	$\frac{n}{2}(n+1)$	$\frac{n(n+1)(n-1)}{6}$	$(n+1)!$	$\frac{SU(n+1)}{U(n)}$
β_n	$2n$	$2n(n-1)$	$\frac{4n(n-1)(n-2)}{3}$	$2^n n!$	$\frac{SO(2n)}{SO(2n-2)\times U(1)}$
$h\gamma_n$	2^{n-1}	$n2^{n-2}$	$2^{n-4}n(n-1)$	$2^n n!$	$\frac{SO(2n)}{U(n)}$, $n \geq 4$
2_{21}	27	216	720	648	$\frac{E_6}{SO(10)\times U(1)}$
3_{21}	56	756	4032		$\frac{E_7}{E_6\times U(1)}$

Figure Captions

*Fig.*1 : W-plane diagram of the perturbed $E_6/SO(10)\times U(1)$ theory, as derived from the superpotential (3.11). The lengths of the soliton lines give precisely the affine-E_6 Toda masses (we use also the thickness of the lines to distinguish the masses). The diagram is the spin-1 projection of the “Hesse polytope” 2_{21} [14], which is the same as the weight diagram of $\Xi = \mathbf{27}$ of E_6. The dot in the center is three-fold degenerate, and all lines starting at the center are doubly degenerate. Note that there are no direct lines linking adjacent dots on the inner circle. The apparent connection is merely an artefact of the projection (similarly also for the other figures).

*Fig.*2 : W-plane diagram of the perturbed $E_7/E_6\times U(1)$ theory, as derived from the superpotential (3.12). The lengths of the soliton lines give precisely the affine-E_7 Toda masses. The diagram is the spin-1 projection of the polytope 3_{21} [14], which is the same as the weight diagram of $\Xi = \mathbf{56}$ of E_7. The dot in the center is doubly degenerate.

*Fig.*3 : Spin-4 projection (Coxeter eigenspace $m_2 = 4$) of the weight diagram of the **27** of E_6. The lengths of the lines (including zero) give, up to signs, the eigenvalues of $\mathcal{I}_4$ on the solitons of the perturbed $E_6/SO(10)\times U(1)$ theory.

*Fig.*4 : Spin-5 projection (Coxeter eigenspace $m_3 = 5$) of the weight diagram of the **27** of E_6. The lengths of the lines give, up to signs, the eigenvalues of $\mathcal{I}_5$ on the solitons of the perturbed $E_6/SO(10)\times U(1)$ theory. The thickness of the lines corresponds to Fig.1, that is, to the masses of the solitons.

*Fig.*5 : Spin-1 projection of a soliton scattering process in the perturbed $E_6/SO(10)\times U(1)$ theory, together with the corresponding dual diagram in Toda particle theory. The higher order poles in the S-matrix correspond to tilings of a polygon in terms of fundamental coupling triangles [6], as all lines are on-shell. The dots are projections of the vacuum states, and some of them misleadingly appear to lie on soliton lines; this is an artefact of the projection.