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On Topological W-Algebra Field Theories*

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Abstract

We show how certain conformal field theories with $\mathcal{W}$-algebraic structure and vanishing central charges can be interpreted as topological field theories, by mapping them to $N=2$ superconformal coset models. Appropriate identifications among vacuum states generate non-trivial ring structures for the topological observables.

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We investigate a class of topological field theories that is related to $N=2$ superconformal theories, and show that these correspond to certain Toda or $\mathcal{W}$-algebra "extended" conformal field theories with vanishing central charges. We thus provide a simple free field realization of these models, which may be useful for explicit computations after coupling to topological gravity [1,2]. We will speculate on further applications at the end of this letter.

1. Twisted N=2 Superconformal Theories. It is well-known [3] that the subset of chiral† primary fields $\{\Phi\}$, with

$$G^+_{-1/2}\Phi \;=\; G^-_{3/2}\Phi \;=\; 0\;, \qquad \longleftrightarrow \qquad h(\Phi)=\tfrac{1}{2}\,q(\Phi)\;,$$

defines a topological subsector of any two dimensional $N=2$ superconformal field theory. In this subsector, the operator algebra has a simple ring structure,

$$\Phi_\alpha(z)\cdot\Phi_\beta(w) \;=\; c_{\alpha\beta}^{\;\;\gamma}\Phi_\gamma(w)\;. \tag{1}$$

The set of the primary chiral fields together with the rules (1) is called the chiral Ring $\mathcal{R}$. Witten [1][2] has shown that any $N=2$ supersymmetric theory with a conserved R-current (with (half-)integral charge eigenvalues) can be "twisted", $T_{\mu\nu} \;\longrightarrow\; \widehat{T}_{\mu\nu} \;=\; T_{\mu\nu}+\epsilon_{\mu\sigma}\partial^\sigma R_\nu+\epsilon_{\nu\sigma}\partial^\sigma R^\mu$, to obtain a topological field theory. Under this twist, two supercharges combine into a BRST-charge such that the stress energy tensor becomes a BRST commutator. The observables $\mathcal{O}_\alpha$ are then precisely the non-trivial elements in the BRST cohomology. In particular, in a twisted $d=2$, $N=2$ superconformal theory, the basic observables are just the primary chiral fields. Specifically, under* [4]

$$T_{zz} \;\to\; \widehat{T}_{zz} \;=\; T_{zz}-\tfrac{1}{2}\partial_z J_z\;, \tag{2}$$

the operators acquire new dimensions $\hat{h}=h-\frac{1}{2}q$, so that $\hat{h}(\Phi_\alpha)=0$ and $\hat{h}(G^+)=1$, $\hat{h}(G^-)=2$. We thus have

† Chiral in the sense of supersymmetry.

* We will consider in the follwing only the holomorphic part of the theory. To obtain the complete theory, it must be combined with the anti-holomorphic part.

$$\mathcal{O}_\alpha(z) \;=\; \Phi_\alpha(z)\ ,\quad \{Q_{\rm BRST},\mathcal{O}_\alpha(z)\}=0\ ,\quad \mathcal{O}_\alpha \neq \{Q_{\rm BRST},\dots\}\ , \tag{3}$$

where

$$Q_{\rm BRST} \;=\; \oint \tfrac{dz}{2\pi i}\, G^+(z)\ ,$$

as well as

$$\widehat{T}_{zz}(z) \;=\; \tfrac{1}{2}\{Q_{\rm BRST},Q_{zz}(z)\}\ ,\qquad \text{with}\qquad Q_{zz}(z)=G^-(z)\ .$$

Further topological, non-local observables are determined from the ascent equation

$$d\,\mathcal{O}_\alpha \;=\; \tfrac{1}{2}\{Q_{\rm BRST},\mathcal{O}^{(1)}_\alpha\} \tag{4}$$

and are given by $\mathcal{O}^{(1)}_\alpha \;=\; G^-_{-1/2}\Phi_\alpha$. Having $\hat{h}=1$, they must be integrated and (combined with their anti-holomorphic counterparts) can be used to deform the topological theory,

$$S_0 \;\longrightarrow\; S_0+\sum t_\alpha \int \mathcal{O}^{(1,1)}_\alpha\ . \tag{5}$$

All other chiral fields of the $N\!=\!2$ theory, being descendants of some chiral algebra, are BRST trivial, and all non-chiral fields are BRST non-invariant. Thus, twisting effectively truncates the theory to its chiral ring $\mathcal{R}$. Note that from $J(z)J(w)\sim \frac{c}{3}(z-w)^{-2}$ follows that $\widehat{T}$ has vanishing central charge, $c=0$, a hallmark of any theory that is invariant under general coordinate transformations.

Eguchi and Yang [4] studied the twisting of members of the $N=2$ discrete series ($c=\frac{3k}{k+2}$) and found that in particular the $k=1,2$ models map to the coset theories $\frac{SU(2)_l\times SU(2)_k}{SU(2)_{k+l}}$ with $l=0$ and $c=0$. We will investigate a much larger class of theories, and link them formally with $\mathcal{W}$ algebras with vanishing central charges.

2. Twisted Coset Models. We study the map of certain $N=2$ coset models [5] based on $\frac{G_k\times SO(2D)_1}{H_{g-h+k}}$. Here, $2D\equiv dim G/H$, the subscripts denote the Kac-Moody levels and g and h are the (dual) Coxeter numbers of G and H, respectively. If $rank(G)=rank(H)$, G/H must be a Kähler manifold, and thus $H=H'\times U(1)$. We will restrict ourselves to the distinguished subclass of models with $k=1$, G simply laced and G/H being a hermitian symmetric space, in other words, to $G/H=\frac{SU(n+m)}{SU(n)\times SU(m)\times U(1)}$, $\frac{SO(n+2)}{SO(n)\times U(1)}$ (n even), $\frac{SO(2n)}{SU(n)\times U(1)}$, $\frac{E_6}{SO(10)\times U(1)}$ and $\frac{E_7}{E_6\times U(1)}$ ($\frac{SU(n+1)}{SU(n)\times U(1)}$ describes the minimal series of type A_{n+1}, whereas $\frac{SO(n+2)}{SO(n)\times U(1)}$ describes D_{n+2}). Note that the groups H can be characterized by deleting any dot of the G Dynkin diagram that corresponds to some fundamental weight λ_* with Kac label $a_*=1$. Note also that for the above models, the chiral ring $\mathcal{R}$ is isomorphic to the cohomology ring of G/H, that is, $\Phi_\alpha\sim H^{\alpha,\alpha}_{\bar\partial}(G/H,\mathbf{R})$ [3] (the index on Φ denotes the $U(1)$ charge in units of $\frac{1}{g+1}$).

As the embeddings $H_{g-h}\hookrightarrow SO(2D)_1$ and $H_1\hookrightarrow G_1$ are conformal and finitely reducible, one can represent these models also as* [6]

$$\left[\frac{H'_1\times H'_{g-h}}{H'_{g-h+1}}\right]\times U(1)\ \equiv\ \mathcal{W}_{H'_{g-h}}\Big|_{\min}\times U(1)\ ,\qquad \text{with}\quad c=\frac{3D}{g+1}\ . \tag{6}$$

(The $U(1)$ factor represents the $U(1)$ current of the $N=2$ algebra).

Free field realizations of $\mathcal{W}$-extended conformal field theories are well-known [7,8]. In order to give a convenient formulation of our models, let us set up the following conventions: $\partial\phi_i, i=1,...\ell\equiv rank(G)$ denote the currents that generate the CSA's of G and H, the free bosons satisfy $\phi_i(z)\cdot\phi_j(w)=-\delta_{ij}ln(z-w)$, and $\lambda_i\equiv\{\lambda_k,\lambda_*\}$ denote the fundamental weights of G, which are dual to the simple roots $e_i\equiv\{e_k,e_*\}$ (e_* is the simple root that extends the H' Dynkin diagram to the one of G).

$\star$ We use the same symbol $\mathcal{W}$ to denote either the algebra itself or a $\mathcal{W}$ extended conformal field theory.

The stress energy tensor of (6) is then

$$T(z) \ = \ -\tfrac{1}{2}\sum_{i=1}^{\ell}(\partial\phi_i)^2 + i\alpha_0\,\rho_H\cdot\partial^2\phi \ ,$$

where

$$\alpha_0 \ = \ \alpha_+ + \alpha_- \ , \qquad \alpha_+ = \sqrt{\frac{g+1}{g}} \ , \ \ \alpha_- = -\sqrt{\frac{g}{g+1}} \ ,$$

and $\rho_X = \frac{1}{2}\sum_{\mathbf{t}^+(X)} \alpha$ (positive roots of group X). The $U(1)$ current is given by

$$J(z) \ = \ -2i\alpha_0\,(\rho_G - \rho_H)\cdot\partial\phi \ , \tag{7}$$

and the supercharges can be represented as vertex operators [6] as follows:

$$G^+(z) \ =: e^{i\alpha_+ e_*\,\cdot\,\phi}:(z) \ , \qquad G^- \ =: e^{-i\alpha_+\psi\,\cdot\,\phi}:(z) \tag{8}$$

(ψ denotes the highest root of G). Furthermore, the unique fields with charges $\frac{1}{g+1}$, $-\frac{g}{g+1}$ and $\frac{D}{g+1}$ have representations

$$\begin{aligned} \Phi_1(z) \ &=: e^{i\alpha_0 e_*\,\cdot\,\phi}:(z) \ , \qquad G^-_{-1/2}\Phi_1(z) \ =: e^{i\alpha_- e_*\,\cdot\,\phi}:(z) \\ \Phi_D(z) \ &=: e^{-i\alpha_-\lambda_*\,\cdot\,\phi}:(z) \end{aligned}$$

More generally, the Hilbert space is generated by vertex operators $e^{i\alpha_{m_i,m'_i}\cdot\phi}$, with

$$\alpha_{m_i,m'_i} \ = \ -\left[(m_k-1)\alpha_- + (m'_k-1)\alpha_+\right]\lambda_k - \left[(m_*-1)\alpha_- + (m'_*-1)\alpha_+\right]\lambda_* \ , \tag{9}$$

where $m_i, m'_i \in \mathbf{N}$ and m_k, m'_k are restricted by the decompositions defined by $\frac{H'_l\times H'_1}{H'_{l+1}}$, $l = g-h$. More precisely, for each simple factor H'_r of H',

$$\sum^{rank\ H'_r} a_k^{(H'_r)} m_k \ \leq \ h_r + l_r \equiv g \ , \qquad \sum^{rank\ H'_r} a_k^{(H'_r)} m'_k \ \leq \ h_r + l_r - 1 \equiv g-1 \ . \tag{10}$$

Here, $a_k^{(X)}$ denote the Kac labels of the group X that satisfy $\sum a_k^{(X)} n_k \leq l$ for any highest weight $\sum n_k\lambda_k$ of the Kac-Moody algebra $\hat{X}$ at level l. The vertex operators with (9) generate completely degenerate representations of $\mathcal{W}_{H'_{g-h}}$. Null states are constructed from screening operators with unit conformal dimension

$$S_k^{\pm}(z) \;=\; :e^{i\alpha_{\pm} e_k \cdot \phi}:(z) \;. \tag{11}$$

We now twist (2) these models to obtain

$$\widehat{T}(z) \;=\; -\tfrac{1}{2}\sum_{i=1}^{\ell}(\partial\phi_i)^2 + i\alpha_0\,\rho_G\cdot\partial^2\phi \;, \tag{12}$$

and in particular

$$Q_{\rm BRST} \;=\; \oint \tfrac{dz}{2\pi i}\, S_*^{+}(z)\;, \qquad \mathcal{O}_1^{(1)}(z) \;=\; S_*^{-}(z)\;. \tag{13}$$

Together with (9) and (11), these are the ingredients that define free field realizations of what we like to call topological $\mathcal{W}_{G_0}$ theories, with $c=0$. These models are (presumably) equivalent to quantum, non-affine G-Toda theories [9][8] [10],

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial\phi_i)^2 - \frac{1}{\beta^2}\sum_i e^{\beta e_i\cdot\phi}\;,$$

with coupling constant $\beta^2 = -\frac{g}{g+1}$ ⋆ . One might be tempted to view these models as the first members of the principal minimal series $\mathcal{W}_{G_l}\Big|_{\rm min} \equiv \frac{G_l\times G_1}{G_{l+1}}$, $c = \ell\left[1-\frac{g(g+1)}{(g+l)(g+l+1)}\right]$, with $l=0$. This is however not quite correct, in general. By definition, these coset models would contain (up to identifications) only states created by

$$\Phi_{m_i,m_i'}(z) \;=\; :e^{i\alpha_{m_i,m_i'}\cdot\phi}:(z)\;, \tag{14}$$

with $\alpha_{m_i,m_i'}$ given in (9), with

$$\sum_{i=1}^{\ell} a_i^{(G)} m_i \;\le\; g+l\;, \qquad \sum_{i=1}^{\ell} a_i^{(G)} m_i' \;\le\; g+l-1\;, \quad (l=0)\;. \tag{15}$$

In general, states $|m_i,m_i'\rangle$ of the twisted models with (10) violate these conditions.

⋆ We understand that the proper quantization of Toda theories with imaginary coupling constant is not fully understood; the lagrangian is non-real and should be modified. This is of no relevance for our work.

This is obvious in the example we will present below. (The exception is for the minimal series with $G/H = \frac{SU(M+1)}{SU(M)\times U(1)}$). There is of course nothing wrong with $\mathcal{W}_{G_0}$ theories that do not obey (15). These bounds only provide a consistent truncation of the operator algebra, and just happen to match the group theory of $\frac{G_0\times G_1}{G_1}$. The distinction between "minimal" and "non-minimal" is of course irrelevant from the point of view of the $\mathcal{W}_{G_0}$ representation theory, as $c=0$. The topological observables belong just to multiple copies of the identity representation. The $\mathcal{W}$-algebraic structure of these models is manifest only (at least before coupling to topological gravity) in the free field formulation; in particular, there is no non-trivial chiral $\mathcal{W}$ algebra. It does, therefore, not seem to be particularly relevant at this point that the models posses conserved currents $W_n^{c=0}$ of spin n (the values of n are given by the orders of the independent Casimirs of G). This fact follows from the results of [11][6], where it was shown that any such $N=2$, G/H coset model under consideration has these currents (related to $\mathcal{W}_{G_0}$) in the Fock space. They are top components of $N=2$ superfields, and hence BRST trivial in the topological theory. Accordingly, all $\mathcal{W}_{G_0}$ descendants are null states.

3. Topological $\mathcal{W}_{G_0}$ Theories. We like to construct topological models from free field theories with $c=0$ stress energy tensor (12). The topological field theoretic interpretation of these models will in general not be unique: we have seen that for given G, all G/H coset models map to what we called a topological $\mathcal{W}_{G_0}$ theory. We thus like to understand, in particular, how the structure of the various chiral rings $\mathcal{R}$, corresponding to the various possible choices of H, is encoded in the set of $\mathcal{W}_{G_0}$ vacuum states.

Let us discuss the Feigin-Fuchs formulation of $\mathcal{W}_{G_0}$ theories in more detail, making use of results of refs. [7][8]. We require the spectra to be completely degenerate. Taking $S_i^{\pm}$ as screening operators, degenerate representations are created by vertex operators (14) with $\alpha_{m_i,m_i'} = -[(m_i-1)\alpha_- + (m_i'-1)\alpha_+]\,\lambda_i$, $m_i \neq 0\ mod\ g+1$, $m_i' \neq 0\ mod\ g$. As these are invariant under $m_i \longrightarrow m_i+g+1, m_i' \longrightarrow m_i'+g$, we can restrict ourselves to $m_i, m_i' \in \mathbf{N}$. The vertex operators create highest weight states of the algebra $\mathcal{W}_{G_0}$,

$$W_n^{c=0}(z)\cdot\Phi_{m_i,m_i'}(w) \;\sim\; w_n(\alpha_{m_i,m_i'})\,(z-w)^{-n}\,\Phi_{m_i,m_i'}(w)+\ldots \qquad (16)$$

In the special case $G = SU(M)$, these currents are generated* by the quantum Miura transformation

$$:\prod_{j=1}^{M}\Big[i\alpha_0\partial_z - h_j\cdot\partial_z\phi\Big]:(z) \;=\; \sum_{n=0}^{M} W_n^{c=0}(z)\,\big(i\alpha_0\partial_z\big)^{M-n}\;, \qquad (17)$$

where h_j are non-orthonormal basis vectors with $\sum h_j = 0$ (which can be taken to be the weights of the fundamental representation of $SU(M)$). Acting with (17) on (16), one has

$$\prod_j\Big[i\alpha_0\partial_z - \frac{i}{z-w}h_j\cdot\alpha_{m_i,m_i'}\Big] \;=\; \sum_n w_n(\alpha_{m_i,m_i'})\,(z-w)^{-n}\,\big(i\alpha_0\partial_z\big)^{M-n}\;,$$

and acting with this on $(z-w)^s$, $s = 0,...,M-2$, one obtains

$$i^M\prod_j\Big[\alpha_0(M-j+s) - h_j\cdot\alpha_{m_i,m_i'}\Big] \;=\; \sum_n \frac{s!}{(s-n)!}\,w_{M-n}(\alpha_{m_i,m_i'})\big(i\alpha_0\big)^n\;. \qquad (18)$$

Therefore, the w_n charges of the vertex operators are [7]

$$w_n(\alpha_{m_i,m_i'}) \;=\; (-i)^n \sum_{1\leq<j_1\ldots<j_n\leq M}\prod_{m=1}^{n}\Big[\alpha_0(n-m)+h_{j_m}\cdot\alpha_{m_i,m_i'}\Big]\;,$$

and specifically

$$w_2(\alpha_{m_i,m_i'}) \;\equiv\; \hat h(\Phi_{m_i,m_i'}) \;=\; \tfrac{1}{2}(\alpha_{m_i,m_i'})^2 - \alpha_0\,\rho_G\cdot\alpha_{m_i,m_i'}\;.$$

It is crucial to note that the l.h.s. of (18) (and thus the $w_n(\alpha_{m_i,m_i'})$) is invariant under an order $M!$ group of transformations

$$P_{SU(M)}\;:\quad h_j\cdot\alpha_{m_i,m_i'} \;\longrightarrow\; h_{\hat j}\cdot\hat\alpha_{m_i,m_i'} \;=\; h_j\cdot\alpha_{m_i,m_i'}+(j-\hat j)\alpha_0\;, \qquad (19)$$

where $\{\hat j\}$ is a permutation of $\{j\} = \{1,2,...,M\}$.

⋆ Up to total derivatives that are not relevant here.

More generally, one can check that P_G has the same order as the Weyl group of G, so that there are $|W(G)|$ vertex operators with the same $\mathcal{W}_{G_0}$ quantum numbers. In conventional, "minimal" $\mathcal{W}_{G_l}$ theories, the corresponding states are identified (and the labels m_i, m_i' are restricted (15) so as to match the coset structure $\frac{G_l\times G_1}{G_{l+1}}$). This is not what we want at this point, since all our basic topological observables will be vacuum states in the identity representation of $\mathcal{W}_{G_0}$, with $w_n = 0$. More specifically, the part of the spectrum that is important for us consists of $|W(G)|$ states $\mathcal{O}$ with $\hat{h} = 0$ (the orbit of $\alpha_{11..11,11..11} = 0$). In addition, there is an independent P_G orbit $\mathcal{S}^i$ for each pair of screening operators, $S_i^{\pm}$. Thus, we will consider $\ell|W(G)|$ operators with $\hat{h} = 1$.

So far, the operators in each $\mathcal{O}$ and $\mathcal{S}^i$ appear on equal footing. A non-trivial structure can be imposed on the theory by attaching labels to these operators. Essentially, these labels correspond to the $U(1)$ charges q the operators would have in an $N = 2$ G/H coset theory. There is in general an ambiguity in doing this, associated with the various choices of allowed subgroups H. That is, each dot of the G Dynkin diagram with Kac label $a_* = 1$ defines a valid choice of the $U(1)$ current, and determines a labelling system by

$$q(\Phi_{m_i,m_i'}) \ = \ -\alpha_-\,\lambda_*\cdot\alpha_{m_i,m_i'} \ .$$

More precisely, according to (6) one can decompose the P_G orbits into $P_{H'}$ orbits, which can be labelled by the $U(1)$ charge (up to accidental degeneracies). States in the same $P_{H'}$ orbit have then to be identified,

$$\mathcal{O}_\alpha \ = \ \frac{\{\Phi_{m_i,m_i'} \in \mathcal{O},\ \text{with}\ q = \frac{\alpha}{g+1}\}}{P_{H'}}\ , \quad \mathcal{S}^i_\alpha \ = \ \frac{\{\Phi_{m_i,m_i'} \in \mathcal{S}^i,\ \text{with}\ q = \frac{\alpha}{g+1}\}}{P_{H'}} \qquad (20)$$

We therefore find for the total number of $\mathcal{O}_\alpha$

$$dim\,\mathcal{R} \ = \ \frac{|W(G)|}{|W(H')|}\ ,$$

which is the correct result [3]. We also have that $S_k^{\pm} \in \mathcal{S}_0^k$ and that the BRST charge density belongs to $\mathcal{S}^*_{g+1}$; it does not seem to play any particular role in this formulation. Furthermore, the densities $\mathcal{O}_\alpha^{(1)}$ are given by $\mathcal{S}_{\alpha-g-1}$, and we see from (13) that $\mathcal{O}_1^{(1)}$ actually coincides with one of the screening operators. To have a well-defined, topological field theory interpretation, only $\mathcal{O}_\alpha$, $\oint \mathcal{O}_\alpha^{(1)}$ should be taken as physical operators.

Note that (20) identifies vertex operators acting on different Fock spaces. An operator product can be defined by restriction to the local part, ie., by taking only those products of vertex operators which are well-defined. In this way, we obtain a closed operator algebra for the $\mathcal{O}_\alpha$ of the form (1), and we expect from the connection to $N=2$ G/H coset models that this algebra should be isomorphic to the cohomology ring of G/H. We found this indeed be true for the examples we checked explicitly; one of these examples will be presented below. It is also easy to see that, using the above operator product rules, the $\mathcal{S}$ operators close into an algebra

$$[\oint \mathcal{S}_\alpha, \mathcal{O}_\beta] \; = \; a_{\alpha\beta}\mathcal{O}_{\alpha+\beta} \; , \qquad [\oint \mathcal{S}_\alpha, \mathcal{S}_\beta\} \; = \; b_{\alpha\beta}\partial\mathcal{O}_{\alpha+\beta} + c_{\alpha\beta}\mathcal{S}_{\alpha+b} \; ,$$

which contains and generalizes (4).

Correlation functions (on the sphere) can only be non-zero if the backgound charge, $(2g-2)\rho_G\alpha_0$, is saturated (we will consider only genus $g=0$ in the following). Specifically, the two-point function looks $\langle \Phi_{\alpha_{m_i,m_i'}} \Phi_{2\rho_G\alpha_0-\alpha_{m_i,m_i'}} \rangle = 1$. In fact, $\Phi_{2\rho_G\alpha_0-\alpha_{m_i,m_i'}} \equiv \Phi_{\alpha_{g+1-m_i,g-m_i'}}$ is just the Poincaré dual of $\Phi_{\alpha_{m_i,m_i'}}$ in $\mathcal{R}$, so that one can write $\langle\, \mathcal{O}_\alpha \, \mathcal{O}_{D-\alpha} \,\rangle = 1$ (up to normalization). By use of (1) and the fact that all topological correlators do not depend on the insertion points, one can relate all non-zero correlators of the $\mathcal{O}_\alpha$ to the one-point function of the top element of the chiral ring $\mathcal{R}$, $\langle\, \Phi_D \,\rangle = 1$.

4. An Example. We will consider $G = SU(4)$, with two choices for H, $H = SU(3)\times U(1)$ with $\lambda_* = \lambda_1$ (or equivalently, λ_3) and $H = SU(2)\times SU(2)\times U(1)$ $(\lambda_* = \lambda_2)$. For the first choice, the 4! vacuum state vertex operators of $\mathcal{W}_{SU(4)}$

decompose under $P_{SU(3)}$ as follows:

$$\begin{aligned}
\mathcal{O}_0 &= [\,\Phi_{344,333}\,,\,\Phi_{124,123}\,,\,\Phi_{213,212}\,,\,\Phi_{242,232}\,,\,\Phi_{331,321}\,,\,\Phi_{111,111}\,] \\
\mathcal{O}_1 &= [\,\Phi_{434,333}\,,\,\Phi_{133,123}\,,\,\Phi_{243,232}\,,\,\Phi_{222,212}\,,\,\Phi_{421,321}\,,\,\Phi_{112,111}\,] \\
\mathcal{O}_2 &= [\,\Phi_{134,123}\,,\,\Phi_{443,333}\,,\,\Phi_{333,232}\,,\,\Phi_{312,212}\,,\,\Phi_{422,321}\,,\,\Phi_{121,111}\,] \\
\mathcal{O}_3 &= [\,\Phi_{444,333}\,,\,\Phi_{224,123}\,,\,\Phi_{313,212}\,,\,\Phi_{342,232}\,,\,\Phi_{431,321}\,,\,\Phi_{211,111}\,]
\end{aligned}$$

One can easily verify that on the sphere (up to normalization)

$$\langle\,\mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\,\rangle \;=\; \langle\,\mathcal{O}_1\mathcal{O}_2\,\rangle \;=\; \langle\,\mathcal{O}_3\,\rangle \;=\; 1\;,$$

and find the chiral ring of the minimal model A_4, $\mathcal{R} = \{\mathcal{O}_0,\,\mathcal{O}_1,\,\mathcal{O}_2 = \mathcal{O}_1^2,\,\mathcal{O}_3 = \mathcal{O}_1^3$, with $\mathcal{O}_1^4 = 0\}$, which is isomorphic to the cohomology ring of CP_3. For the other choice of H, we have to make different identifications in the same set of vacuum states:

$$\begin{aligned}
\mathcal{O}_0 &= [\,\Phi_{434,333}\,,\,\Phi_{124,123}\,,\,\Phi_{421,321}\,,\,\Phi_{111,111}\,] \\
\mathcal{O}_1 &= [\,\Phi_{333,232}\,,\,\Phi_{213,212}\,,\,\Phi_{242,232}\,,\,\Phi_{312,212}\,] \\
\mathcal{O}_2 &= [\,\Phi_{211,111}\,,\,\Phi_{344,333}\,,\,\Phi_{443,333}\,,\,\Phi_{112,111}\,] \\
\mathcal{O}_2' &= [\,\Phi_{422,321}\,,\,\Phi_{133,123}\,,\,\Phi_{224,123}\,,\,\Phi_{331,321}\,] \\
\mathcal{O}_3 &= [\,\Phi_{243,232}\,,\,\Phi_{313,212}\,,\,\Phi_{342,232}\,,\,\Phi_{222,212}\,] \\
\mathcal{O}_4 &= [\,\Phi_{444,333}\,,\,\Phi_{134,123}\,,\,\Phi_{431,321}\,,\,\Phi_{121,111}\,]
\end{aligned}$$

A truncation to vertex operators satisfying (15) is obviously not possible here. One finds

$$\begin{aligned}
\langle\,\mathcal{O}_1^4\,\rangle = \langle\,\mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\,\rangle = \langle\,\mathcal{O}_1\mathcal{O}_3\,\rangle = \langle\,\mathcal{O}_2\mathcal{O}_2\,\rangle = \langle\,\mathcal{O}_2'\mathcal{O}_2'\,\rangle = \langle\,\mathcal{O}_4\,\rangle \;&=\; 1 \\
\langle\,\mathcal{O}_1\mathcal{O}_1\mathcal{O}_2'\,\rangle = \langle\,\mathcal{O}_2\mathcal{O}_2'\,\rangle \;&=\; 0\;,
\end{aligned}$$

and the operator algebra represents a ring of type D_6, $\mathcal{R} = \{\mathcal{O}_0,\,\mathcal{O}_1,\,\mathcal{O}_2 = \mathcal{O}_1^2,\,\mathcal{O}_3 = \mathcal{O}_1^3,\,\mathcal{O}_4 = \mathcal{O}_1^4 = \mathcal{O}_2'^2,\,\mathcal{O}_2'$, with $\mathcal{O}_1\mathcal{O}_2' = 0$ and $\mathcal{O}_1^5 = 0\} \cong H_{\bar\partial}^{*,*}(\frac{SU(4)}{SU(2)\times SU(2)\times U(1)},\mathbf{R})$.

5. Further Remarks. We have seen that certain conformal field theories with $c = 0$ allow for an interpretation as topological field theories, and appear to be equivalent to particular Toda theories. It might be possible to interpret these models in terms of the topology of the moduli spaces of the classical solutions. Since precisely the class of $N = 2$ coset models in question admits an $N = 2$ supersymmetric Landau-Ginzburg description [3], an explicit Lagrangian formulation of the topological models can also be given by twisting Landau-Ginzburg models. One obtains topological Landau-Ginzburg models [12] in this way, whose Lagrangian is explicitly a BRST variation. One is tempted to speculate that these models allow for an alternative interpretation in terms of the topology of Landau-Ginzburg soliton moduli spaces (note that solitons exist only in perturbed theories).

It also seems promising to study the coupling of these models to topological gravity [2]. Specifically, it appears [13] that the twisted $N = 2$ minimal models (topological $\mathcal{W}_{SU(n)}$ models in our language) coupled to topological gravity just describe n-matrix models [14]. The conjectured $\mathcal{W}_{n+1}$ structure of these models appears to be related to the $\mathcal{W}_n$ structure discussed above. Details will be discussed elsewhere.

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