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DUALITY SYMMETRIES IN N=2 LANDAU-GINZBURG MODELS

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ABSTRACT

We investigate the moduli spaces and their symmetries of the $c=3$, $N=2$ supersymmetric, Landau-Ginzburg models. We show that for the model that corresponds to the $\mathbf{Z}_3$ orbifold, the moduli space, $\mathcal{M}$, is isomorphic to the two-sphere, S_2, divided by the tetrahedral group $\mathcal{T}$. On this space, generalized duality transformations act like discrete rotations of a tetrahedron inscribed in S_2. We also find non-trivial monodromy when encircling certain points of $\mathcal{M}$.

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1. *Introduction.* It is now evident that $N = 2$ supersymmetric, two dimensional Landau-Ginzburg models can be used to describe σ-models on Kähler manifolds with vanishing first Chern class [1]. These Landau-Ginzburg models are characterized by superpotentials, $W(x_i)$, that are quasihomogenous functions with isolated singularities [2]. A σ-model on a topologically non-trivial target space must of course be invariant under the discrete automorphisms of this space. Moreover, there are discrete transformations of the target manifold (such as the duality transformation $R \to 1/R$ of the torus [3-5]), that are not symmetries of the manifold but are symmetries of the σ-model. For a Landau-Ginzburg theory that is quantum equivalent to such a σ-model, it is important to understand how these symmetries are encoded in the superpotential, as the superpotential is supposed to determine a given theory completely. Several aspects of such symmetries have recently been discussed in [6].

Our purpose here is to analyze completely the discrete symmetries of the moduli spaces of the simplest models, that is, we will consider the unimodal $c = 3$ Landau-Ginzburg theories that are equivalent to the $\mathbf{Z}_3, \mathbf{Z}_4$ and $\mathbf{Z}_6$ orbifolds of the ($N = 2$) torus. The corresponding superpotentials are given in table 1. We will focus on the type P_8 model, as it describes the well-known $\mathbf{Z}_3$ orbifold, and mention the other cases at the end. The Landau-Ginzburg fields x, y, z correspond to the twist fields of the orbifold, as will be discussed below. The states in the untwisted sector (the winding states in particular) correspond to the D-terms in the Landau-Ginzburg theory. In the orbifold model, the masses of the winding states depend continuously on the background fields B and G, and in the Landau-Ginzburg model, the conformal dimensions of the D-terms at the renormalization group fixed point depend smoothly on the background deformation parameter a. It is important to find the precise relation between the (complex) modulus a of the Landau-Ginzburg model and the background fields of the orbifold model; as we will see, different points in a-space can describe the same physical theory. Furthermore, we will show how the discrete duality transformations of the torus, which mix the various winding states, act on the chiral fields x, y, z and on the modulus a in the Landau-Ginzburg theory.

2. *$\mathbf{Z}_3$ orbifold model.* We first investigate the duality transformations and the structure of the moduli space for the compactification of two bosonic string coordinates $X^i(z, \bar{z})$ $(i = 1, 2)$ on a two-dimensional torus, $\mathbf{T}_2 = \mathbf{C}/\Lambda$. The lattice Λ is spanned by the basis vectors $\vec{e}_1$, $\vec{e}_2$. The σ-model action, $S \sim \int d^2z(G_{ij} + B_{ij})\partial X^i \bar{\partial} X^j$, depends on four background field parameters, the constant metric $G_{ij} = \vec{e}_i \cdot \vec{e}_j$ and the antisymmetric tensor $B_{ij} = B\epsilon_{ij}$. The second term in the action is topological, and the theory is invariant under shifts $B \to B + \frac{1}{2}$.

In addition to the oscillator excitations, the spectrum is generated by vertex operators $V_{p_L, p_R}(\bar{z}, z) = \exp i\vec{p}_L \cdot \vec{X}(\bar{z}) \exp i\vec{p}_R \cdot \vec{X}(z)$ with dimensions $(\bar{h}, h) = \vec{p}_L^2/2,\ \vec{p}_R^2/2$. These winding states depend on the background fields via

$$\vec{p}_{L,R} = \vec{e}_i(\pm n^i + \frac{1}{2}G^{ik}m_k - BG^{ij}\epsilon_{jk}n^k) . \tag{1}$$

Here, the n_i determine the winding vectors $\vec{w} = n_i\vec{e}_i$ and the m_i determine the momentum vectors $\vec{p} = m_i\vec{e}_i^*$. The vectors $(\vec{p}_L, \vec{p}_R)$ live on a Lorentzian even, self-dual lattice. Lorentz rotations of this lattice are parametrized by the moduli G_{ij} and B [7]. Thus, the local structure of the moduli space $\mathcal{M}_{T_2}$ is given by $\frac{SO(2,2)}{SO(2)\times SO(2)} \cong \frac{SU(1,1)}{U(1)} \times \frac{SU(1,1)}{U(1)}$. To describe its global structure, it is convenient to introduce the complex parameters

$$\lambda = 2(B + i\sqrt{G}) , \qquad \tau = \frac{G_{12}}{G_{22}} + i\frac{\sqrt{G}}{G_{22}} . \tag{2}$$

Here, τ represents the complex structure of $\mathbf{T}_2$ and thus takes values in the upper half-plane $\mathbf{H}_+ \cong \frac{SU(1,1)}{SU(1)}$, divided by the modular group $\Gamma = PSL(2, Z)$. Similarly, since the radius of the torus, $R^2 \propto \sqrt{G} > 0$, is a positive number, the parameter λ also takes values in $\mathbf{H}_+$. Periodicity in B and the duality transformations $\sqrt{G} \to \frac{\sqrt{G}}{4B^2+4G}$, $B \to -\frac{B}{4B^2+4G}$ can be expressed as $T: \ \lambda \to \lambda + 1$, $S: \ \lambda \to -\frac{1}{\lambda}$, respectively. This is just the action of the two generators of $PSL(2, Z)$. Therefore, the λ-moduli space is given by the fundamental domain, $\mathcal{F} = \mathbf{H}_+/PSL(2, Z)$ as well. The theory is also invariant under $\tau \leftrightarrow \lambda$ and $(\lambda, \tau) \to (-\bar{\lambda}, -\bar{\tau})$. Thus the global structure of the moduli space is given by $\mathcal{M}_{T_2} = \frac{\mathcal{F}\times\mathcal{F}}{Z_2\times Z_2}$ [3][4].

The fundamental region $\mathcal{F} \times \mathcal{F}$ contains orbifold points $\tau, \lambda = i\infty$, $e^{2\pi i/6}$, and i which are order ∞, 3, 2 fixed points of the transformations T, TS and S, respectively. When λ or τ is equal to one of these values, the theory acquires additional symmetries. We can use the freedom of interchanging λ and τ to fix one of them and study the effect of $PSL(2, Z)$ acting on the other. Specifically, let us choose $\tau = -\rho^2$ (where $\rho \equiv e^{2\pi i/3}$) which implies that the lattice Λ defining the torus has the same shape as the root lattice of $SU(3)$. That is, the basis vectors are $\vec{e}_1 = R(1, 0)$, $\vec{e}_2 = R(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then the vectors $\vec{p}_L$ and $\vec{p}_R$ are given by

$$\begin{aligned} p_{1L,R} &= \pm Rn_1 + (\pm\frac{1}{2}R - \frac{B}{R})n_2 + \frac{1}{2R}m_1 \\ p_{2L,R} &= \frac{1}{\sqrt{3}}\Big(\frac{2B}{R}n_1 + (\pm\frac{3}{2}R + \frac{B}{R})n_2 - \frac{1}{2R}m_1 + \frac{1}{R}m_2\Big) , \end{aligned} \tag{3}$$

and $\vec{p}_L^2 = 2\bar{h}$ and $\vec{p}_R^2 = 2h$ are expressed in terms of R and B as:

$$\begin{aligned}\vec{p}_{L,R}^2 = &(n_1^2+n_2^2)\frac{3R^4+4B^2}{3R^2} + (m_1^2+m_2^2)\frac{1}{3R^2} \\ &+ n_1 n_2 \frac{3R^4+4B^2}{3R^2} - m_1 m_2 \frac{1}{3R^2} \pm (n_1 m_1 + n_2 m_2) \\ &- (n_2 m_1 - n_1 m_2)\frac{4B}{3R^2} - (n_1 m_1 - n_2 m_2)\frac{2B}{3R^2} .\end{aligned} \tag{4}$$

Thus, by setting $\tau = -\rho^2$, the theory is invariant under a Z_3 symmetry generated by $2\pi/3$ rotations of $\vec{p}_L$ and $\vec{p}_R$.

It is now straightforward to determine how the duality transformations $\lambda \rightarrow \frac{a\lambda+b}{c\lambda+d}$ ($a,b,c,d \in \mathbf{Z}$, $ad-bc=1$), $\lambda = 2B + i\sqrt{3}R^2$, act on the theory. Consider first

$$S: \quad R^2 \rightarrow \frac{R^2}{4B^2+3R^4}\ , \quad B \rightarrow -\frac{B}{4B^2+3R^4}. \tag{5}$$

Clearly, performing this transformation on the background parameters transforms states of different conformal dimensions into each other. However, the conformal dimensions can be kept invariant if one simultaneously shifts the winding and momentum numbers as follows [3][5]:

$$\begin{aligned} n_1 &\rightarrow -m_1 + m_2 \\ n_2 &\rightarrow -m_2 \\ m_1 &\rightarrow -n_1 \\ m_2 &\rightarrow -n_1 - n_2 \end{aligned} \tag{6}$$

The invariance of (4) under T: $\lambda \rightarrow \lambda + 1$ ($R \rightarrow R$, $B \rightarrow B + \frac{1}{2}$) is manifest if one simultaneously acts on the momentum and winding numbers as follows:

$$\begin{aligned} n_1 &\rightarrow n_1 \\ n_2 &\rightarrow n_2 \\ m_1 &\rightarrow m_1 + n_2 \\ m_2 &\rightarrow m_2 - n_1 \end{aligned} \tag{7}$$

In contrast to S, the transformation T together with (7) maps each momentum and winding state onto itself.

We now consider the $\mathbf{Z}_3$ orbifold of this model. This requires us to fix one of either τ or λ to the value* $-\rho^2 = -e^{-2\pi i/3}$, and to mod out by the enhanced $\mathbf{Z}_3$ symmetry (we will fix τ and consider the dependence on λ). That is, we include the $\mathbf{Z}_3$ -twisted states, and project to $\mathbf{Z}_3$ invariant states. In the untwisted, winding sectors, this amounts to identifying vectors $(\vec{p}_L, \vec{p}_R)$ which differ by a $2\pi/3$ rotation. Thus the twist identifies $\vec{p}_L$ given in (3) with

$$\begin{aligned} p'_{1L} &= (-\frac{1}{2}R - \frac{B}{R})n_1 - Rn_2 - \frac{1}{2R}m_2 \\ p'_{2L} &= \frac{1}{\sqrt{3}}((\frac{3}{2}R - \frac{B}{R})n_1 - \frac{2B}{R}n_2 + \frac{1}{R}m_1 - \frac{1}{2R}m_2). \end{aligned} \tag{8}$$

The set of $\mathbf{Z}_3$ invariant combinations of winding states can be decomposed into equivalence classes, $V_{(M,N)}$, labelled by elements M, N of $\mathbf{Z}_3$, defined by [5]

$$\begin{aligned} M &= m_1 + m_2 \qquad mod\ 3 \\ N &= n_1 - n_2 \qquad mod\ 3\ . \end{aligned} \tag{9}$$

It is easy to see that the duality transformations act on these labels according to

$$\begin{pmatrix} M \\ N \end{pmatrix} \longrightarrow \begin{pmatrix} M' \\ N' \end{pmatrix} = \overline{\gamma} \begin{pmatrix} M \\ N \end{pmatrix} ; \qquad \overline{\gamma} \in \overline{\Gamma} \equiv SL(2,\mathbf{Z})\ . \tag{10}$$

More precisely, the generators of $\Gamma \equiv PSL(2,\mathbf{Z})$, $S: \lambda \rightarrow -\frac{1}{\lambda}$ and $T: \lambda \rightarrow \lambda + 1$, induce the $SL(2,\mathbf{Z})$ transformations

$$S\begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} M \\ N \end{pmatrix} ; \qquad T\begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} M \\ N \end{pmatrix} . \tag{11}$$

It is clear that the labels are invariant under the congruence subgroup

$$\overline{\Gamma}(3) = \left\{ \overline{\gamma} \in SL(2,\mathbf{Z}) : \overline{\gamma} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} mod\, 3 \right\} . \tag{12}$$

The quotient,

$$\overline{\Gamma}/\overline{\Gamma}(3) \cong SL(2,\mathbf{Z}_3)\ , \tag{13}$$

is a group that permutes the labels M and N. This quotient is, in fact, equal to the binary extension $\overline{T}$ of the tetrahedral group T, which is in turn equal to the group $\mathbf{A}_4$ of even

* Of course, we can take any other value that leads to an enhanced $\mathbf{Z}_3$ symmetry.

permutations of four elements. The tetrahedral group acts on the tetrahedron by rotating it, and indeed performs even permutations of the vertices. Thus T can be viewed as a subgroup of $SO(3)$, and $\overline{T}$ is defined to be the lift of T to $SU(2)$. That we have to consider binary extensions is a consequence of the fact that $S^2 \neq 1$ on the labels M and N (see (11)). The canonical presentation for $\overline{T}$ is

$$A^2 \;=\; B^3 \;=\; (AB)^3 \;=\; Z \qquad Z^2 \;=\; 1 \,, \tag{14}$$

where Z is a central element. We can identify $A = S$ and $B = S^2T = ZT$ (since $S^2 = (ST)^3 = -1$ in $SL(2,\mathbf{Z})$). Having obtained the action of $\overline{T}$ on the winding sectors, we now wish obtain the corresponding action on the twisted sectors. The twisted states are generated by the twist fields $\sigma_i(z,\overline{z})$ and $\overline{\sigma}_i(z,\overline{z})$, $i = 0,1,2$, of conformal dimensions $h_L = h_R = 1/9$, where the σ_i correspond to a twist by $\rho = e^{2\pi i/3}$ and the $\overline{\sigma}_i$ correspond to a twist by $\overline{\rho} = \rho^2$. The index i runs over the three fixed points of this twist. Because the conformal dimensions of the twist fields are independent of λ, they can only mix with each other under modular transformations of λ [4]. Moreover, the σ_i cannot mix with the $\overline{\sigma}_i$, and indeed, the σ_i transform in a representation R of $\overline{T}$, while the $\overline{\sigma}_i$ transform in the complex conjugate representation, $\overline{R}$. The explicit form of R can be derived from the following fusion rules [8]:

$$\begin{aligned} \sigma_j\,\overline{\sigma}_j \;&=\; V_{(0,0)} + \rho^{2j}V_{(1,0)} + \rho^{j}V_{(2,0)} \\ \sigma_{j+1}\,\overline{\sigma}_{j+2} \;&=\; V_{(0,2)} + \rho^{2j}V_{(1,2)} + \rho^{j}V_{(2,2)} \\ \sigma_{j+2}\,\overline{\sigma}_{j+1} \;&=\; V_{(0,1)} + \rho^{2j}V_{(1,1)} + \rho^{j}V_{(2,1)} \,, \end{aligned} \tag{15}$$

where the subscripts of σ are considered *mod* 3. From this it is clear that the twist fields can be taken to be invariant under $\overline{\Gamma}(3)$; however they must transform under $\overline{T} = \overline{\Gamma}/\overline{\Gamma}(3)$ since the $V_{(M,N)}$ are permuted under the action of this group. More precisley, it is elementary to deduce that the nine equivalence classes, $V_{(M,N)}$, decompose into the irreducible representations $1 \oplus 1 \oplus 2 \oplus 2 \oplus 3$ of $\overline{T}$. From (15) it follows that $R \otimes \overline{R}$ must yield precisely this list of representations, and this implies*

$$R \;=\; 1 \oplus 2 \,. \tag{16}$$

The generators S and T of $\overline{T}$ both act trivially on the singlet, while on the doublet they can be represented as: $S = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}; T = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}$. This singlet and doublet basis

* Since $\overline{T}$ has three distinct singlet and three distinct doublet representations, there is a minor ambiguity in this decomposition. It will be resolved below.

of R is linearly related to the σ-basis, and in the latter basis the generators read† :

$$S \;=\; -\frac{i}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho & \rho^2 \\ 1 & \rho^2 & \rho \end{pmatrix} \,, \quad T \;=\; \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{17}$$

The expression for S for vanishing B-field was discussed previously in [5]. The matrices correspond to the presentation (14) with

$$Z \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \,. \tag{18}$$

3. *Landau-Ginzburg theory.* It is well-known [2] that the $N = 2$ supersymmetric $\mathbf{Z}_3$ orbifold is equivalent to the Landau-Ginzburg model (at the renormalization group fixed point) with type P_8 superpotential

$$W(x,y,z) \;=\; x^3 + y^3 + z^3 + 6a\,xyz \,. \tag{19}$$

The (lowest components of) the superfields x,y,z are some linear combinations of the products of the three "bosonic" twist fields σ_i with the "fermionic" twist field, $e^{i/3(H(z)+\overline{H}(\overline{z}))}$. Here, $i\partial H(z)$ and $-i\overline{\partial H}(\overline{z})$ are the $U(1)$ currents of the left- and right-moving $N = 2$ superconformal algebras* . To determine the precise linear combinations, we observe that the equations of motion of the Landau-Ginzburg theory imply the following fusion rules,

$$\begin{aligned} 3x^2 + ayz \;&\sim\; \overline{x} \\ 3y^2 + axz \;&\sim\; \overline{y} \\ 3z^2 + axy \;&\sim\; \overline{z} \,. \end{aligned} \tag{20}$$

Comparing this with the space group selection rules for the σ_i, [8], we find that we can take

† A different choice of singlet and doublet representations in (16) would merely result in an overall, irrelevant power of ρ multiplying T.

* Note that with this choice of $U(1)$ currents and twist operators, the superfields x,y,z will be chiral-anti-chiral.

the lowest components of the superfields x, y, z to be:

$$x_0 \;=\; \sigma_0\, e^{i/3(H+\overline{H})}\ ,\quad y_0 \;=\; \sigma_1\, e^{i/3(H+\overline{H})}\ ,\quad z_0 \;=\; \sigma_2\, e^{i/3(H+\overline{H})}\ . \tag{21}$$

We now determine the exact relationship between the modal parameter a of the Landau-Ginzburg theory and the background fields of the orbifold model. Depending on which $\mathbf{Z}_3$ is modded out, the modal deformation parameter a can be viewed as a function of either τ or λ. We will take it to be a function of $\lambda \equiv 2(B + i\sqrt{G})$, and thus the modular transformations acting on λ will be generalized duality transformations. By a change of variables one can always bring the superpotential to the form

$$W' \;=\; z^2x - 4y^3 + Ayx^2 + Bx^3\ . \tag{22}$$

The equation $W = 0$ describes a singular curve in $\mathbf{C}^3$, but defines a smooth torus in $\mathbf{CP}_2$. From [1] we know that the conformal fixed point of the Landau-Ginzburg theory is equivalent to the $\mathbf{Z}_3$ orbifold of the $N = 2$ sigma model *on this torus*. Passing to projective coordinates, $\zeta = y/x$ and $\xi = z/x$, the curve is given by

$$W'(\zeta, \xi) \;=\; \xi^2 - 4\zeta^3 + A\zeta + B \;=\; 0\ . \tag{23}$$

To see that this defines a torus we first observe [9] that for all $A, B \in \mathbf{C}$, there exists some $\lambda \in \mathbf{H}_+/\Gamma$ such that

$$A \;=\; 60G_4(\lambda)\ ,\quad B \;=\; 140G_6(\lambda)\ , \tag{24}$$

where G_4 and G_6 are Eisenstein functions. Now let z be a coordinate on the torus, $z \sim z+1$, $z \sim z + \lambda$, and take

$$\zeta \;=\; \mathcal{P}(z;\lambda)\ ,\qquad \xi \;=\; \mathcal{P}'(z;\lambda)\ , \tag{25}$$

where $\mathcal{P}(z;\lambda)$ is the Weierstrass function. These substitutions identically satisfy (23), since it becomes the well-known differential equation‡ for $\mathcal{P}$. The map $z \to (\zeta, \xi)$ explicitly describes the smooth embedding of the torus $\mathbf{T}_2$ into $\mathbf{CP}_2$, and the image is isomorphic to the complete surface defined by $\frac{\mathbf{CP}_2}{W=0}$.

‡ Actually, the above is a manifestation of a deep relationship between singularity theory and the theory of automorphic forms on Riemann surfaces [10].

Because of the equivalence between the renormalization group fixed point of the Landau-Ginzburg model and the $\mathbf{Z}_3$ orbifold of the surface $W = 0$, we can thus identify λ in (24) with the parameter λ introduced earlier, and if we transform (22) back to the normal form (19), we obtain the modulus a in terms of the Eisenstein functions G_4 and G_6. More precisely, we find

$$j^{1/3}(\lambda) \;=\; 48\frac{a(1-a^3)}{(1+8a^3)}\ , \tag{26}$$

where j is the absolute modular invariant, and $j^{1/3}(\lambda) \equiv G_4(\lambda)/\eta^8(\lambda)$. From (26) we see that there are four linearly independent possible choices for a. Moreover, the twelfth order polynomial for a with coefficients linear in j that can be obtained by cubing both sides of (26) in fact defines modular functions of $\Gamma(3)$ (the index $[\Gamma\!:\!\Gamma(3)]$ is equal to 12). That is, all roots of the polynomial are modular functions of $\Gamma(3)$, and are permuted under the action of the Galois group of the polynomial. This Galois group is precisely the quotient $\Gamma/\Gamma(3) \cong \mathcal{T}$. Thus, up to the action of $\mathcal{T}$, we are free to choose any root of (26), for instance, we could take

$$a(\lambda) \;=\; -\frac{1}{2}\, s\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\,\middle|\, \frac{j(\lambda)}{1728}\right)\ . \tag{27}$$

The inverse triangle function $s(\alpha, \beta, \gamma | z)$ is defined in [11], and can be written, for instance, as proportional to the quotient of the hypergeometric functions $F(\frac{1}{2}(1-\alpha-\beta+\gamma), \frac{1}{2}(1-\alpha-\beta-\gamma), 1-\beta; z)$ and $F(\frac{1}{2}(1-\alpha+\beta+\gamma), \frac{1}{2}(1-\alpha+\beta-\gamma), 1+\beta; z)$. All choices for a map the fundamental region, $\mathcal{F}_3 = \mathbf{H}_+/\Gamma(3)$ of $\Gamma(3)$ one-to-one and onto $\mathbf{C} \cup \{\infty\}$. The region $\mathcal{F}_3$ is comprised of twelve copies of the fundamental region $\mathcal{F} = \mathbf{H}_+/\Gamma$. It can be taken to have cusps at $-1, 0, 1$ and $i\infty$. The region $\mathcal{F}_3$ also has four ρ-points and six i-points♮ . Our choice of (27) for a maps the cusps to $-\rho/2, -1/2, -\rho^2/2$ and ∞, respectively. The image of $\mathcal{F}_3$ in $\mathbf{C}$ is shown in the figure.

This diagram can be viewed as the stereographic projection of a spherical tetrahedron obtained by inscribing a tetrahedron inside a sphere of radius $\frac{1}{2\sqrt{2}}$ and then projecting from the center of the sphere. The vertices of the spherical tetrahedron are stereographically projected onto the images of the cusp points while the centers of the faces and edges are projected onto the images of the ρ- and i-points, respectively. The tetrahedral group acts in the obvious way on the spherical tetrahedron, and it is elementary to obtain the action

♮ The region $\mathcal{F}_3$ is a twelve-fold cover of $\mathcal{F}$ and thus by a "z-point" we mean any point of $\mathcal{F}_3$ above $z \in \mathcal{F}$.

of this group on the a-plane. Indeed the generators S and $\mathcal{T}$ act on a according to

$$S \;:\; a \longrightarrow \frac{1-a}{1+2a}\,, \qquad \mathcal{T} \;:\; a \longrightarrow \rho^2 a\,. \tag{28}$$

Observe that these transformations are precisely the action of $\Gamma/\Gamma(3)$ on (26) and that their action permutes the roots of (26).

We have deduced the action of $\mathcal{T}$ on the modulus a solely from the consideration of the toroidal surface in $\mathbf{CP}_2$. On the other hand, if one substitutes the duality transformations (17) of the twist fields into the superpotential (19) one also obtains precisely (28) as the transformation law for a (note that one must perform an irrelevant, overall rescaling of the superpotential to obtain this result). Thus, while the modulus, a, of (19) can take any value in $\mathbf{C}\cup\{\infty\}\cong S_2$, the moduli space of physically distinct Landau-Ginzburg models is given by $\mathcal{M}=\{a : a\in S_2/\mathcal{T}\}\cong S_2$, and can be represented by any of the twelve cells in the figure. This means, the "naive" a-moduli space $\mathbf{C}$ is one-to-one to the "naive" orbifold moduli space $\{\lambda : \lambda\in\mathcal{F}_3\}$, which is also a twelve-fold cover of the physical moduli space, $\mathcal{F}$.

Note that the non-degeneracy condition, $a^3\notin\{-1/8,\infty\}$, for the singularity of the superpotential corresponds to $\lambda\notin\{-1,0,1,i\infty\}$ in the orbifold theory. In other words, the points where the singularity ceases to be isolated are just the images of the decompactification limit $R\to\infty$. At these points, the local ring of the singularity becomes infinite dimensional, and the Landau-Ginzburg spectrum continuous.

For a generic, fixed choice of modulus, the Landau-Ginzburg model is not invariant under any of the duality transformations generated by (17). The theory is only invariant provided that one simultaneously readjusts the modulus a according to (28). This corresponds to rotating the spherical tetrahedron around one of its symmetry axes. However, if the particular value of $a=a_\gamma$ is a fixed point of some transformation $\gamma\in\mathcal{T}$, the Landau-Ginzburg model has a cyclic symmetry. It is important to remember that while it is only $\mathcal{T}$ that acts on a, it is the double cover, $\overline{\mathcal{T}}$, that acts on the Landau-Ginzburg fields. Thus, at a particular fixed point a_γ, the actual symmetry of the theory is generated by any $\overline{\gamma}\in\overline{\mathcal{T}}$, covering $\gamma\in\mathcal{T}$. We can always arrange $\overline{\gamma}$ to have twice the order of γ. Thus, depending on the particular fixed point, the symmetry is either $\mathbf{Z}_4$ or $\mathbf{Z}_6$. The complete list of fixed points and associated symmetries is given in table 2. It corresponds to a similar list in the orbifold model.

The doubling of the order of the symmetry implies that the theory has non-trivial

monodromy properties[◇]. That is, when one encircles some fixed point a_γ, the spectrum of the theory changes continuously but comes back to itself as a whole after a full cycle. However, individual states are not mapped back to themselves; rather, the theory spectrally flows back to itself only up to a symmetry transformation. For instance, moving around $a_S=(\sqrt{3}-1)/2$ induces $S^2=Z$, which is not the identity operation but a permutation of two Landau-Ginzburg fields (see (18)).

Finally, we comment on the structure of the other $c=3$ theories as given in table 1. The X_9 type theory corresponds to a $\mathbf{Z}_4$ orbifold of the torus, and all that changes compared to the above is that $\Gamma(3)$ is replaced by $\Gamma(2)$, the tetrahedral group $\mathcal{T}$ by the dihedral group $\mathcal{D}_3$, the tetrahedron inscribed in S_2 by a dihedron with three vertices and equation (27) by $a=4(s(\frac{1}{2},\frac{1}{2},\frac{1}{3}|\frac{j}{1728})-\frac{1}{2})$. The J_{10} theory, on the other hand, is a $\mathbf{Z}_2$ orbifold of the P_8 theory, and there appears to be no simple interpretation in terms of polyhedral groups; for the modulus we find $j=4\frac{a^3}{4a^3+27}$.

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◇ Note that the kind of monodromy we find here is different from the notion of monodromy as usually discussed in the context of singularity theory. This is because the deformation we are considering does not change the topological type of the singularity, and for all values of a we remain in what is called the level bifurcation set [12].

Singularity type	Superpotential $W(x_i)$	Non-degeneracy cond.	Orbifold
P_8	$x^3+y^3+z^3+6axyz$	$a^3 \neq -\frac{1}{8}$	$SU(3)/\mathbf{Z}_3$
X_9	$x^4+y^4+ax^2y^2$	$a^2 \neq 4$	$SO(4)/\mathbf{Z}_4$
J_{10}	$x^6+y^3+axy^4$	$4a^3 \neq -27$	$SU(3)/\mathbf{Z}_6$

Table 1: Landau-Ginzburg superpotentials leading to $N = 2$ superconformal theories that are equivalent to the orbifold models indicated in the last column.

Fixed point a_γ	Generator $\overline{\gamma} \in \overline{T}$	Symmetry
0	S^2T	$\mathbf{Z}_6$
1	STS	$\mathbf{Z}_6$
ρ	TS	$\mathbf{Z}_6$
ρ^2	ST	$\mathbf{Z}_6$
$-1/2$	STS	$\mathbf{Z}_6$
$-\rho/2$	TS	$\mathbf{Z}_6$
$-\rho^2/2$	ST	$\mathbf{Z}_6$
$(\sqrt{3}-1)/2$	S	$\mathbf{Z}_4$
$-(\sqrt{3}+1)/2$	S	$\mathbf{Z}_4$
$\rho(\sqrt{3}-1)/2$	$T^{-1}ST$	$\mathbf{Z}_4$
$-\rho(\sqrt{3}+1)/2$	$T^{-1}ST$	$\mathbf{Z}_4$
$\rho^2(\sqrt{3}-1)/2$	TST^{-1}	$\mathbf{Z}_4$
$-\rho^2(\sqrt{3}+1)/2$	TST^{-1}	$\mathbf{Z}_4$

Table 2: Points of enhanced symmetry in a-moduli space.

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Figure: The moduli space $\mathcal{M} = S_2/\mathcal{T}$ of the P_8 Landau-Ginzburg model. Each cell describes the same physical theory. Stars denote i-points, dots ρ-points and circles cusp points.