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Anomaly Cancelling Terms from the Elliptic Genus

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Abstract

We calculate the heterotic string one-loop diagram in $2n+2$ dimensions with one external $B_{\mu\nu}$ and n external gravitons and/or gauge bosons. The result is a modular integral over the weight zero terms of the character valued partition function (or elliptic genus) of the theory, and can be directly expressed in terms of the factor which multiplies $\mathrm{Tr}F^2$-$\mathrm{Tr}R^2$ in the field theory anomaly. The integrands have a non-trivial dependence on the modular parameter τ, reflecting contributions not only from the physical massless states but also from an infinity of "unphysical" modes. Some of them are identical to integrands which have been discussed recently in relation with Atkin-Lehner symmetry and the cosmological constant. As a corollary we find a method to compute these integrals without using Atkin-Lehner transformations.

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1. Introduction

In a recent paper [1] we have calculated the $BTrF^n$ terms in the effective field theory of $2n+2$ dimensional heterotic strings[1]. Such terms were expected to be present since they are needed to cancel part of the local gauge and gravitational anomalies, and indeed they turned out to appear with precisely the coefficient predicted in [3].

In addition to the $BTrF^n$ terms several other terms involving powers of lower traces of the gauge field two-form F and the curvature two-form R are expected to exist[2]. Although our previous calculation gives us no reason to doubt their existence there are several reasons for extending our previous calculations to include these terms.

First of all, the result of [1] gives little insight into how the anomaly cancelling terms are related to the 2n-form X_{2n} which multiplies TrF^2-TrR^2 in the anomaly. The structure of this factor can be derived from the character valued partition function A(q,F,R) of the chiral sector of the theory [4]. (Recently this function has also appeared in the mathematics literature under the name "elliptic genus" [5] [6]). In particular, it is the lack of modular invariance of this function which is responsible for the fact that the anomaly does not cancel completely. The modular invariance violating terms can be factored out of the character valued partition function, which then has the form

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[1] For related one-loop calculations in four dimensions see also [2].

[2] These terms have been considered in a recent paper [7]. None of the modular space integrals was however calculated, and so the result is inconclusive even with regard to the existence of these terms.

$$A(q,F,R) = e^{-\frac{1}{64\pi^4} G_2(\tau)(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2)} \tilde{A}(q,F,R) \ . \tag{1.1}$$

$\tilde{A}(q,F,R)$ is fully holomorphic and modular covariant (with weight $-n$), and can be expressed completely in terms of the Eisenstein functions G_4 and G_6 (in Appendix C this is illustrated for the ten-dimensional supersymmetric SO(32) heterotic string as well as for the O(16)×O(16) string). As explained in Appendix B, the Eisenstein functions $G_{2k}(\tau)$ are modular covariant quantities of weight 2k, except for the anomalous function G_2. It would certainly be satisfactory to find the relation between string loop diagrams and the "modular anomaly" of (1.1) (closely related to the holomorphic anomaly), which manifests itself through the appearance of G_2.

The second reason for considering the more general case is that it is far less straightforward, and hence more interesting than the calculation of $B\mathrm{Tr}F^n$. For the latter the integrand was independent of the modular parameters, and the τ-integral therefore trivial. In general however one has to calculate a non-trivial $n+1$ dimensional complex integral. For an arbitrary loop diagram there would be no hope of calculating such an integral analytically, and indeed in most calculations of loop diagrams which have appeared in the literature these integrals are left in the final result. The anomaly cancelling terms are among the very few cases were one expects to be able to calculate a string loop diagram completely[1], and get an answer which is non-trivial and physically meaningful.

[1]Assuming that there is a generalization of the Adler-Bardeen theorem to string theory, we expect that they are not even renormalized by higher loop corrections.

There are two kinds of integrals to be done, namely n integrals over the relative positions of vertex operators on the world-sheet, usually parametrized by integration variables ν_i, and a final integral over the modular parameter τ. Although the integrand has formally the correct modular transformation and periodicity behavior in all $n+1$ variables, it also has singularities which have to be dealt with carefully in order for the final τ integrand to be modular invariant. We have found that the existing expressions for correlation functions on the torus are ill-suited for this purpose, and were forced to find a new representation which makes all relevant modular and periodicity properties manifest. These expressions for the correlation functions are in fact quite natural, and are discussed in Appendix A.

As a result of the ν-integrals we obtain τ-integrands which are weight 12 products of G_4, G_6 and $\hat{G}_2 = G_2 - \pi/\mathrm{Im}\tau$, divided by Dedekind's η function to the power 24. Therefore, they have modular weight zero. In the absence of factors $\hat{G}_2$ the integrand is meromorphic (with poles only at $q=0$) and thus equal to a linear combination of the famous absolute modular invariant j (proportional to $(G_4)^3/\eta^{24}$) and a constant[1]. For diagrams involving only external gauge bosons, the coefficient of j has to vanish, so that the integral is simply proportional to the volume of the modular domain. This is the situation encountered in [1]. Diagrams with only external gravitons give integrands which depend non-trivially on j, but fortunately these integrals can also be performed analytically. An integral of this kind was recently discussed by Moore [8], who evaluated it using Atkin-Lehner symmetry. We will show that polynomials in j can in fact be integrated in a far more straightforward way, which then also has a simple extension to non-holomorphic zero weight functions

[1]See Appendix C for an explicit example.

containing factors of $\hat{G}_2$.

A noteworthy feature of these integrals, already pointed out by Moore, is that they receive contributions not just from "physical" states, but also from states that do not satisfy the left-right mass constraint, and which are therefore not part of the physical particle spectrum. This apparent violation of field theory intuition merely underscores the fact that string theory can not be regarded as a simple superposition of an infinite number of point field theories.

We want to stress that the purpose of this paper, as well as previous ones, is not to demonstrate the absence of anomalies in string theory, but to understand the cancellation of effective field theory anomalies from the fact that these field theories originate from strings. To prove that strings are anomaly free one has to calculate string loop diagrams with external gauge bosons and gravitons, and check their gauge invariance. This has been done in [9] [10]. Although certainly of great importance, these calculations miss the interesting structures that are present in the two effective field theory contributions which are contained in the loop diagram, namely the field theory polygon anomaly diagram, and the $B_{\mu\nu}$ exchange diagram involving the Green-Schwarz term. We are interested in the structure of these terms not to prove ***that*** the anomaly cancels, but to understand ***how*** it cancels. The first part of this understanding is provided by the relation between factorization of the field theory anomaly and modular invariance of the string theory from which it is derived [4]. (Because there has been some confusion about this we emphasize that the aim of [11] was not to prove that the string loop diagram is anomaly free, but to give a more natural interpretation of the character valued partition function in terms of the index of the Dirac-Ramond operator). The second part is the calculation of the anomaly cancelling term in the field theory limit, and to clarify its relation to the factorized anomaly. This task will be completed in the present paper.

In the next section we calculate the parity violating part of the one loop string amplitude involving N gauge bosons, M gravitons and a $B_{\mu\nu}$. We find that the result can be expressed as a modular integral over the zero weight part of the elliptic genus. In section 3 we perform this integral, and obtain as a result all anomaly cancelling terms.

2. Evaluation of the Anomaly-Cancelling Terms

Let us first consider contributions cancelling purely gravitational anomalies. In $D=2n+2$ dimensions, they correspond to one-loop diagrams with one external $B_{\mu\nu}$ and n external graviton lines, in the periodic-periodic (PP) sector of right-moving fermionic boundary conditions. To provide absolute normalization compared to field theory, we use the conventions given in [1]. The amplitude is then given by

$$i\Gamma = -\mathrm{Tr}\left[\tfrac{1}{2}\Gamma_{*}\,(-2iF_0)\,\Delta\, V_B^{R2} \prod_{i=1}^{n}(\Delta V_{g,i}^{R1})\right]. \tag{2.1}$$

Here, $V_B{}^{R2}$ and $V_g{}^{R1}$ are the vertex operators for the emission of $B_{\mu\nu}$ and gravitons (in the R2 and R1 pictures, respectively):

$$\begin{aligned} V_B^{R2} &= -g\,\zeta^B_{\mu\nu}\, P^\mu \Gamma^\nu e^{ik_0 X(0)} \\ V_{g,i}^{R1} &= 4ig\,\zeta^g_{\lambda\rho}\, P^\lambda\left(\tilde{P}^\rho - \tfrac{1}{4}k_{i,\sigma}\Gamma^\rho\Gamma^\sigma\right) e^{ik_i X(0)} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} P^\mu(z) &= \tfrac{1}{2}p^\mu + \sum_m{}' \alpha_m^\mu z^{-m} \\ \tilde{P}^\mu(\bar{z}) &= \tfrac{1}{2}p^\mu + \sum_m{}' \tilde{\alpha}_m^\mu \bar{z}^{-m} \\ \Gamma^\mu(\bar{z}) &= \gamma^\mu + \sqrt{2}\,\gamma_{*} \sum_m{}' d_m^\mu \bar{z}^{-m} \\ \Gamma_{*} &= \gamma_{*}\,(-1)^F \\ \gamma_{*} &= (-i)^{n+2}\gamma^0\cdots\gamma^{2n+1} \\ \Delta &= \frac{-i}{16\pi}\int \frac{d^2 z}{|z|^2}\, z^{L_0/2}\,\bar{z}^{\tilde{L}_0/2}. \end{aligned} \tag{2.3}$$

The operators P and Γ in (2.2) have arguments $z=\bar{z}=1$. $(-2iF_0)$ is the picture changing insertion with

$$F_0 = \oint \frac{d\bar{z}}{2\pi i \bar{z}} \Gamma^\tau \tilde{P}_\tau \, . \tag{2.4}$$

(In principle, for $D<10$, F_0 gets also a contribution from $T_F{}^{\rm int}$, the part of the supercurrent due to internal degrees of freedom. However, $T_F{}^{\rm int}$ cannot contribute to our calculation because it cannot saturate the zero modes of Γ^μ in the PP-sector.)

Performing the loop momentum integration, the lattice sum and the trace over the ghost modes, as well as the usual variable transformations from the plane to the torus, we can cast (2.1) into the form

$$\begin{aligned} i\Gamma = {} & \frac{i(-g)^{n+1}}{(16\pi)^{n+1}} \xi^B_{\mu\nu} \Big(\prod_{i=1}^{n} \xi^{g_i}_{\sigma_i\lambda_i} k_{g_i}\Big) \int_{\mathcal{F}} d^2\tau \Big(\frac{2}{{\rm Im}\,\tau}\Big)^{n+1} A(q,0,0) \\ & \times \int_{-1/2}^{1/2} d\,{\rm Re}\,\nu \int_{\mathcal{F}_\tau} \Big(\prod_{i=1}^{n} d^2\nu_i\Big) \,{\rm Tr}\Big\{\Gamma_* \Gamma^\tau(\bar{\nu}) \Gamma^\nu(0) \prod_{i=1}^{n} \big[\Gamma^{g_i} \Gamma^{\sigma_i}(\bar{\nu}_i)\big]\Big\} \\ & \times \Big\langle \tilde{P}_\tau(\bar{\nu}) P^\mu(0) e^{ik_0\cdot X(0)} \prod_{i=1}^{n} \big[P^{\lambda_i}(\nu_i) e^{ik_i\cdot X(\nu_i)}\big] \Big\rangle_\tau \, . \end{aligned} \tag{2.5}$$

Here, $d^2\tau = 2d({\rm Re}\tau)d({\rm Im}\tau)$ (similar for $d^2\nu$), the fundamental integration domains are $\mathcal{F} = \{\tau|\ {\rm Im}\tau>0,\ |\tau|\geq 1,\ -1/2 \leq {\rm Re}\tau \leq 1/2\}$ and $\mathcal{F}_\tau = \{\nu|,\ 0\leq{\rm Im}\nu\leq{\rm Im}\tau,\ -1/2 \leq {\rm Re}\nu \leq 1/2\}$, while $P^\mu(\nu) = -1/\pi\,\partial X(\nu)$, $\tilde{P}^\mu(\bar{\nu}) = 1/\pi\,\bar{\partial} X(\bar{\nu})$ and $q = e^{\pi i\tau}$.

A(q,0,0) is the light-cone partition function in the PP-sector, which could be written, for instance, in terms of theta functions. However, according to the covariant lattice approach developed in [12], we represent A(q,0,0) in the following way:

$$A(q,0,0) = \frac{1}{\eta^{24}(\tau)} \sum_{\substack{(\vec{w}_L,\vec{w}_R(s))\in\Gamma, \\ \vec{w}_R(s)^2=2}}{}' q^{\vec{w}_L^{\,2}} \quad , \tag{2.6}$$

where Γ is an even self-dual lorentzian lattice of signature $(24-2n;16-2n)$, and $w_R(s)=(u_R,v_R(s))$ is a $16-2n$ dimensional vector whose $n+4$ last entries $v_R(s)$ have fixed values of 1/2 (note that all $\bar{q}$ dependence is cancelled out in A(q,0,0); this is explained in more detail in [1] [12]). Although for convenience we have written A(q,0,0) for a specific class of models, the result is valid for any type of heterotic string construction [13].

As our notation suggests, A(q,0,0) is actually the character valued partition function for the case of vanishing external gravitational and gauge curvature two-forms R and F:

$$\begin{aligned} A(q,F,R) &= \frac{1}{\eta^{24}} \, e^{\sum_{k=1}^{\infty} \frac{1}{4k} \frac{1}{(2\pi i)^{2k}} G_{2k}(\tau) \, \mathrm{Tr}\left(\frac{iR}{2\pi}\right)^{2k}} \\ &\quad \times \sum{}' q^{\vec{w}_L^{\,2}} e^{\vec{w}_L\cdot\vec{s}} \quad , \end{aligned} \tag{2.7}$$

where $G_{2k}(q)$ are the Eisenstein series described in Appendix B, and s are the skew eigenvalues of $F/2\pi$.

To proceed we remark that in the Γ-trace only the zero modes can contribute:

$$\mathrm{Tr}\left\{ \Gamma_* \Gamma^\tau \Gamma^\nu \prod_{i=1}^{n} \Gamma^{\rho_i} \Gamma^{\sigma_i} \right\} = 2^{n+1} i^{n+2} \varepsilon^{\tau\nu\sigma_1\rho_1\cdots\sigma_n\rho_n} \quad . \tag{2.8}$$

It is easily checked that because of this ϵ-tensor the only terms which can contribute are those where $\tilde{P}^{\tau}(\bar{\nu})$ and $P^{\mu}(0)$ are contracted. This contribution is entirely due to zero modes:

$$\overbrace{\tilde{P}^{\tau}(\bar{v})\, P^{\mu}(0)} = \frac{1}{(2\pi)^2}\, \frac{\pi}{\mathrm{Im}\,\tau}\, g^{\tau\mu} \tag{2.9}$$

We thus can recast (2.5) into the form

$$\begin{aligned} i\Gamma &= \left(\frac{ig}{2\pi}\right)^{n+1} \frac{1}{8\pi}\, \varepsilon^{\mu\nu\sigma_1\rho_1\cdots\sigma_n\rho_n}\, \zeta^{B}_{\mu\nu}\, (k_{1,\sigma_1}\, \zeta^{g}_{\lambda_1\rho_1}) \cdots (k_{n,\sigma_n}\, \zeta^{g}_{\lambda_n\rho_n}) \\ &\times \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\, F^{\lambda_1\cdots\lambda_n}(\tau) \end{aligned} \tag{2.10}$$

with the modular weight zero function

$$\begin{aligned} F^{\lambda_1\cdots\lambda_n}(\tau) &= A(q,0,0)\, \frac{1}{(2\,\mathrm{Im}\,\tau)^n} \int_{\mathcal{F}_\tau} \Big(\prod_{i=1}^{n} d^2v_i\Big) \Big\langle e^{ik_0 X(0)} \prod_{i=1}^{n} P^{\lambda_i}(v_i)\, e^{ik_i X(v_i)} \Big\rangle_\tau \\ &= A(q,0,0)\, \frac{1}{(2\,\mathrm{Im}\,\tau)^n} \int_{\mathcal{F}_\tau} \Big(\prod_{i=1}^{n} d^2v_i\Big) \\ &\quad\times \partial_{s_1}\cdots\partial_{s_n} \Big\langle \prod_{i=0}^{n} e^{ik_i X + s_i P(v_i)} \Big\rangle_\tau \Big|_{s_i=0} \end{aligned} \tag{2.11}$$

where ∂_{s_i} denote derivatives with respect to the auxiliary variables $s^{i}_{\lambda_i}$ to be put to zero at the end of the computation. Using the Koba-Nielsen formula

$$\Big\langle \prod_{i=0}^{n} e^{ik_i X(v_i)} \Big\rangle_\tau = \prod_{0\le i<j}^{n} e^{\frac{1}{2} k_i k_j \ln \chi_{ji}} \tag{2.12}$$

involving the torus Green's function $\chi_{ji}=\chi(\nu_j-\nu_i,\tau)$ given in (A.3) and introducing the notation $\Delta_{ji}=\partial_j \ln\chi_{ji}$, $\Delta'_{ji}=(\partial_j)^2 \ln\chi_{ji}$ we can write

$$
\begin{aligned}
F^{\lambda_1\cdots\lambda_n}(\tau) &= A(q,0,0)\,\frac{1}{(2\,\mathrm{Im}\,\tau)^n}\int_{\mathcal{F}_\tau}\Big(\prod_{i=1}^{n} d^2\nu_i\,\partial_{s_i}\Big) \\
&\times \exp\Big[\frac{1}{2\pi^2}\sum_{0\le i<j}^{n}\vec{s}_i\,\vec{s}_j\,\Delta'_{ji}+\frac{1}{2\pi i}\sum_{0\le i<j}^{n}(\vec{k}_i\,\vec{s}_j-\vec{k}_j\,\vec{s}_i)\,\Delta_{ji}\Big]\Big|_{s_i=0} \qquad (2.13)\\
&\times \prod_{0\le i<j}^{n}(\chi_{ji})^{\frac{1}{2}\vec{k}_i\vec{k}_j}\,.
\end{aligned}
$$

(Note that the sums and products in the last two factors include $i=0$). Expressions of this type have already been considered by a number of authors [14] [9]. Since we are interested only in terms leading in the external momenta, we will drop $\chi^{k_i k_j/2}$ from now on. We will now, in this section, evaluate the ν-integrals, and in the next section the modular integrals.

In (2.13), the effect of the derivatives is equivalent to the following contraction rules in (2.11):

$$
\begin{aligned}
\overbracket{p^{\lambda_i}\,p^{\lambda_j}} &= \frac{1}{2\pi^2}\,g^{\lambda_i\lambda_j}\,\Delta'_{ij}\;, \\
\overbracket{p}^{\lambda_i} &= \frac{-1}{2\pi i}\sum_{i\neq j}^{n} k_j^{\lambda_i}\,\Delta_{ji}\;. \qquad (2.14)
\end{aligned}
$$

In order to deal with the ν-integrals in combination with these rules in a systematic way we introduce the following graphical notation for the world-sheet propagators:

$$
-\frac{1}{2\pi i}\,k_j^{\lambda_i}\,\Delta_{ji} \;=\; \overset{i}{\bullet}\!\longrightarrow\!\overset{j}{\bullet} \qquad (2.15)
$$

$$\frac{1}{2\pi^2}\, g^{\lambda_i \lambda_j}\, \Delta'_{ji} \quad = \quad$$ 

Then the various terms can be represented by graphs consisting of propagators connecting n closed dots and one open dot. The closed dots represent the momentum operators P^{λ_i} with indices $i=1,\ldots,n$. The open dot indicates the $B_{\mu\nu}$ vertex operator and differs from the other ones because there is no corresponding integration over $\nu_0=0$, nor a derivative with respect to s_0.

To saturate the derivatives with respect to s_i there must be precisely one solid line with incoming arrow, or one dashed line per closed dot. Furthermore the open dot cannot have an incoming solid line or a dashed line. In the representation for Δ_{ij} and Δ'_{ij} derived in appendix A, the ν_i-integrals simply enforce world-sheet momentum conservation at each of the n closed dots. Here "momentum" refers to the two integers m and k appearing in (A.12). Note that because of the restricted sum in (A.12) the momentum space propagators vanish at zero momentum. An immediate consequence is that at least two lines must end on each closed dot. If one such line is an incoming arrow, the others line(s) can only be outgoing solid lines. Hence we can follow the solid lines through the diagram along their arrows. Because the line cannot end on any open or closed dot, it must form a closed loop. If the line were to bifurcate, this would apply to all branches separately; however this would inevitably lead to a graph with more than one incoming line per vertex, which is not allowed. Therefore the solid lines must form disconnected closed loops. Furthermore there are no allowed graphs with dashed lines. This is due to formula (A.18) which implies that a ν-integral over a single Δ'_{ji} vanishes.

To summarize, only graphs involving only closed loops of solid lines (plus the isolated open dot) can contribute after the ν-integrations. As an example, the terms appearing in the loop expansion for $D=10$ ($n=4$) are shown in fig. 1.

Consider first the case where the entire diagram is a single loop with n vertices (plus the isolated point ν_0). Their are $(n-1)!$ such diagrams, and the resulting expressions differ only in their momentum factors. Denoting as $\pi(j)$, $j=1,\ldots,n$ a permutation of the indices $1,\ldots,n$ we can, using (A.1~~3~~9), immediately write down the result for the ν-integrals in (2.13):

$$\frac{1}{n}\sum_{\pi} k_{\pi(1)}^{\lambda_{\pi(2)}}\, k_{\pi(2)}^{\lambda_{\pi(3)}} \cdots\cdots k_{\pi(n)}^{\lambda_{\pi(1)}} \left(\frac{1}{4\pi i}\right)^{n} \hat{G}_n(\tau) , \tag{2.16}$$

where the sum is over all n! permutations ($\hat{G}_{2k} \equiv G_{2k}$ for $k>1$). Since this overcounts the actual number of graphs by the cyclic permutations, we have included a correction factor $1/n$. It is easy to see that the same result is obtained from

$$\partial_{s_1} \cdots \partial_{s_n} \left[\frac{1}{2n} \frac{1}{(2\pi i)^n} \mathrm{Tr} \left(\frac{iR_s}{2\pi}\right)^{n} \hat{G}_n(\tau) \right] \Bigg|_{s_i=0} \tag{2.17}$$

where

$$(R_s)_{ij} = -2\pi i \cdot \frac{1}{2} \left(\vec{k}_i \vec{s}_j - \vec{k}_j \vec{s}_i \right) . \tag{2.18}$$

If the diagram consists of more than one loop, the result is easily seen to be generated by acting with s_i derivatives on products of the square brackets in (2.17), multiplied with a factor 1/N! for each loop of a given order which appears N times. These combinatorical factors can be taken into account by replacing (2.17) by

$$\partial_{s_1} \cdots \partial_{s_n} \exp\left[\sum_{k=1}^{\infty} \frac{1}{4k} \frac{1}{(2\pi i)^{2k}} \mathrm{Tr} \left(\frac{i R_s}{2\pi}\right)^{2k} \hat{G}_{2k}(\tau)\right] \Bigg|_{s_i=0} . \tag{2.19}$$

Therefore the complete result for the gravitational contribution is

$$F^{\lambda_1 \ldots \lambda_n}(\tau) = \partial_{s_1} \cdots \partial_{s_n} e^{-\frac{1}{64\pi^4} \mathrm{Tr}\, R_s^2 \frac{\pi}{\mathrm{Im}\,\tau}} A(q, 0, R_s) \Big|_{s_i=0} , \tag{2.20}$$

where we have factored out the holomorphic anomaly term of $\hat{G}_2$, using (2.7) and (B.3).

Now let us consider the case of cancelling pure gauge anomalies. For the leading part (highest trace) the computation has been done already in [1]. It is sufficient to consider n external gauge bosons with quantum numbers in the Cartan subalgebra. Then the vertex operators for the gauge bosons are

$$V_{A,i}^{R1} = 4ig\, \xi_{I;\rho}^{A}\, P^{I} \left(\tilde{P}^{\rho} - \tfrac{i}{4} k_{i,\rho} \Gamma^{\rho} \Gamma^{\sigma}\right) e^{i k_i X(0)} , \tag{2.21}$$

where P^I are the internal momentum operators. It is clear that in the previous calculation one can just replace $\xi_{\lambda\rho}P^{\lambda}$ by $\xi_{I\rho}P^{I}$. Accordingly one gets from (2.10):

$$i\Gamma = \left(\frac{ig}{2\pi}\right)^{n+1} \frac{1}{8\pi} \varepsilon^{\mu\nu\sigma_1\rho_1 \cdots \sigma_n \rho_n} (k_{1,\sigma_1} \xi_{I_1\rho_1}^{A}) \cdots\cdots (k_{n,\sigma_n} \xi_{I_n\rho_n}^{A}) \cdot \xi_{\mu\nu}^{B}$$
$$\times \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} F^{I_1 \cdots I_n}(\tau) , \tag{2.22}$$

with $F^{I_1 \cdots I_n}(\tau) =$

$$\frac{1}{(2\,\mathrm{Im}\,\tau)^n}\,\frac{1}{\eta^{24}}\int_{\mathcal{F}_\tau}\Big(\prod_{i=1}^{n} d^2v_i\Big)\,\Big\langle e^{ik_0 X(0)}\prod_{i=1}^{n} e^{ik_i X(v_i)}\Big\rangle_\tau$$

$$\times\ \partial_{s_1}\cdots\partial_{s_n}\,\mathrm{Tr}'\Big[\,q^{L_0}\,e^{\sum_i s_i^I p^I}\Big]\Big|_{s_i=0}\ . \qquad (2.23)$$

Here, Tr′ runs over internal momentum zero and non-zero mode excitations, and ∂_{s_i} denotes $\partial/\partial s^i{}_{I_i}$. Evaluating this trace and disregarding again $\chi^{k_i k_j/2}$, we write, recalling the definition of A(q,F,0) in (2.7)

$$F^{I_1 \cdots I_n}(\tau) = \frac{1}{(2\,\mathrm{Im}\,\tau)^n}\int_{\mathcal{F}_\tau}\Big(\prod_{i=1}^{n} d^2v_i\,\partial_{s_i}\Big)\,e^{\frac{1}{2\pi^2}\sum_{i<j} s_i\cdot s_j\,\tilde{\Delta}'_{ji}}$$

$$\times\ \frac{1}{\eta^{24}}\sum_{\Gamma}{}'\,q^{\vec{W}_L^2}\,e^{\sum_i s_i\cdot W_L}\Big|_{s_i=0} \qquad (2.24)$$

$$= \frac{1}{(2\,\mathrm{Im}\,\tau)^n}\int_{\mathcal{F}_\tau}\Big(\prod_{i=1}^{n} d^2v_i\,\partial_{s_i}\Big)\,e^{\frac{1}{2\pi^2}\sum_{i<j} s_i\cdot s_j\,\tilde{\Delta}'_{ji}}\,A(q,F_s,0)\Big|_{s_i=0}\ ,$$

where $\tilde{\Delta}'_{ji}$ is the same as Δ'_{ji} except for the zero mode part,

$$\tilde{\Delta}'_{ji} = \Delta'_{ji} - \frac{\pi}{2\,\mathrm{Im}\,\tau} \qquad (2.25)$$

and

$$F_s = -2\pi i\sum_{j,I} s_j^I H^I\ . \qquad (2.26)$$

Here, H^I are charge operators of the Cartan subalgebra. We can easily compute the ν-integrals by using the fact that the integral over Δ'_{ji} vanishes (see A.18). Therefore,

$$F^{I_1\cdots I_n}(\tau) = \partial_{s_1}\cdots\partial_{s_n} e^{-\sum_{i<j} s_i\cdot s_j \frac{1}{(2\pi)^2}\frac{\pi}{\mathrm{Im}\tau}} A(q,F_s,0)\Big|_{s_i=0}$$

$$= \partial_{s_1}\cdots\partial_{s_n} e^{\frac{1}{64\pi^4}\frac{\pi}{\mathrm{Im}\tau}\mathrm{Tr}F_s^2} A(q,F_s,0)\Big|_{s_i=0}. \tag{2.27}$$

(Tr denotes a trace in the vector representation.) In [1] we calculated the leading trace contribution; it corresponds to the leading trace part of the zero mode part, i.e., to

$$F^{I_1\cdots I_n}(\tau) = \partial_{s_1}\cdots\partial_{s_n} A(q,F_s,0) = \partial_{s_1}\cdots\partial_{s_n}\frac{1}{n!}\mathrm{Tr}\left(\frac{iF_s}{2\pi}\right)^n + \cdots$$

$$= \frac{1}{n!}\mathrm{Tr}\,H^{I_1}\cdots H^{I_n}\ , \tag{2.28}$$

which does not depend on q, as explained in [1]. This means that only massless states contribute in the string loop. (The trace in (2.28) is over the representation of the massless states.) However, as it is clear from the above formula, subleading traces will in general depend on q (and on Imτ). As in the case of gravitational anomalies (2.20), these terms receive contributions from an infinite tower of "unphysical" states (i.e. states which do not satisfy the left-right mass constraint, and are therefore not part of the spectrum) and will involve nontrivial modular integrations.

Finally, it is now completely trivial to obtain a closed expression for the general counter term associated with mixed gravitational and gauge anomalies. An amplitude involving N external gauge bosons and M gravitons (M + N = n) is given by

$$i\Gamma = \left(\frac{ig}{2\pi}\right)^{n+1} \frac{1}{8\pi}\, \varepsilon^{\mu\nu\sigma_1\rho_1\cdots\sigma_n\rho_n}\, \zeta^{B}_{\mu\nu}\, (k_{1,\sigma_1}\zeta^{A}_{I_1\rho_1}) \cdots\cdots (k_{N,\sigma_N}\zeta^{A}_{I_N\rho_N})$$

$$\times\ (k_{N+1,\sigma_{N+1}}\zeta^{g}_{\lambda_{N+1}\rho_{N+1}}) \cdots\cdots (k_{n,\sigma_n}\zeta^{g}_{\lambda_n\rho_n}) \tag{2.29}$$

$$\times \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\, F^{I_1\cdots I_N\lambda_{N+1}\cdots\lambda_n}(\tau)$$

with

$$F^{I_1\cdots I_N\lambda_{N+1}\cdots\lambda_n}(\tau) = \prod_{i=1}^{N} \frac{\partial}{\partial s^i_{I_i}} \prod_{j=N+1}^{n} \frac{\partial}{\partial s^j_{\lambda_j}}\, \bar{A}(q, F_s, R_s)\Big|_{s_i=0}\,, \tag{2.30}$$

where we defined

$$\bar{A}(q,F,R) = e^{\frac{1}{64\pi^4}\frac{\pi}{\mathrm{Im}\,\tau}(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2)}\, A(q,F,R)\,. \tag{2.31}$$

$\bar{A}(q,F,R)$ can be regarded as generating functional for all anomaly cancelling terms; the s_i's may be considered as sources or background fields.

Note that $\bar{A}(q,F,R)$ is similar to the function

$$\tilde{A}(q,F,R) = e^{\frac{1}{64\pi^4}G_2(\tau)(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2)}\, A(q,F,R) \tag{2.32}$$

given in (1.1). Both $\bar{A}(q,F,R)$ and $\tilde{A}(q,F,R)$ are free of modular anomalies and have modular weight $-n$. In $\tilde{A}(q,F,R)$, the modular non-covariant terms due to the anomalous Eisenstein series G_2 are simply cancelled out. In $\bar{A}(q,F,R)$, all G_2's are replaced by $\hat{G}_2$'s, which are modular covariant, but not holomorphic, a reflection of Quillen's anomaly.

3. τ-Integral and Anomaly Cancellation

To calculate the amplitude (2.29) explicitly, one has still to perform the modular integration. The integrand is a sum of certain weight zero combinations of Eisenstein functions, and we show below how such functions can be integrated. For our problem it is not necessary to consider separate integrals, since we can solve it generically for all heterotic string theories in any dimension. In fact, we will show that we can calculate directly the effective action from the elliptic genus, and prove that it cancels all anomalies. We have collected some more explicit results for τ-integrals in Appendix B, and give some concrete examples of anomaly cancelling terms in Appendix C.

Recall that because of modular invariance, the anomaly has always a factorized form, i.e. [4],

$$I_{2n+4}(F,R) = \frac{1}{2\pi}\left(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2\right) X_{2n}(F,R)\ , \tag{3.1}$$

where X_{2n} is some 2n-form, and

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \quad , \quad R_{\alpha\beta} = \frac{1}{2} R_{\mu\nu\alpha\beta}\, dx^\mu \wedge dx^\nu \,. \tag{3.2}$$

The anomaly is given by the constant term of the 2n + 4-form in the elliptic genus:

$$I_{2n+4}(F,R) = A(q,F,R)\Big|_{\substack{2n+4\text{-forms}\\ \text{coefficient of } q^0}} \,. \tag{3.3}$$

It can be cancelled (up to irrelevant local counter terms) by the following term in the effective action [3] [1]

$$S = \int d^{2n+2}x \left(-4g\, B X_{2n}(F,R)\right) \tag{3.4}$$

($B = B^{\mu\nu} dx^\mu \wedge dx^\nu$).

Note that the s-derivatives in (2.30) automatically select the part of $\bar{A}(q,F_S,R_S)$ which is a 2n-form, if we replace F_S and R_S by F and R; this part has modular weight zero. Using the definitions (3.2) one can easily check[1] that our result (2.29) describing all possible amplitudes can be expressed in a closed form by the following effective action:

$$S = \int d^{2n+2}x \left(-4g\, B\right)\left(-\frac{1}{64\pi^2}\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\, \bar{A}(q,F,R)\Big|_{2n\text{-forms}}\right) . \tag{3.5}$$

This is our main result. Thus, to prove anomaly cancellation all we need is to show that

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[1]What has to be checked here is that the s-derivatives in (2.30), in combination with the ϵ tensor and the other factors in (2.29) give the same result as Feynman rules derived from (3.5). The conventions to be used in the latter derivation are those of [1].

$$-\frac{1}{(4\pi)^2}\int_{\mathcal{F}}\frac{d^2\tau}{(\mathrm{Im}\tau)^2}\,\bar{A}(q,F,R)\Big|_{2n\text{-forms}} = X_{2n}(F,R)\ . \tag{3.6}$$

To calculate the integral we use the following procedure. Consider the integral of a weight zero modular function F(τ) over the fundamental domain

$$I = \int_{\mathcal{F}}\frac{d^2\tau}{(\mathrm{Im}\tau)^2}\,F(\tau) \tag{3.7}$$

$$= 4i\int d\mathrm{Re}\tau\, d\mathrm{Im}\tau\left(\frac{d}{d\bar\tau}\frac{1}{\mathrm{Im}\tau}\right)F(\tau)\ . \tag{3.8}$$

The factor $1/\mathrm{Im}\tau$ can be expressed in terms of the difference between the holomorphic, but not modular invariant function $G_2(\tau)$ and the modular invariant non-holomorphic function $\hat{G}_2(\tau)$ defined in Appendix B. Using the fact that $\partial/\partial\bar\tau\, G_2(\tau)=0$ we get

$$I = \frac{-4i}{\pi}\int d\mathrm{Re}\tau\, d\mathrm{Im}\tau\left[\frac{d}{d\bar\tau}\hat{G}_2(\tau)\right]F(\tau)\ . \tag{3.9}$$

If F(τ) is holomorphic we can write the complete integrand as a total derivative, and the integral can be evaluated using Stokes' theorem

$$\int_S d\mathrm{Re}\tau\, d\mathrm{Im}\tau\,\frac{d}{d\bar\tau}f(\tau,\bar\tau) = \frac{i}{2}\int_L d\tau\, f(\tau,\bar\tau)\ . \tag{3.10}$$

Here f is any complex function, S a compact domain in the complex plane, and L the boundary of that domain. The line-integral on the right hand side goes clockwise along the boundary. In the case of interest to us the boundary is the one shown in fig. 2, in the limit $\Lambda \to \infty$. The one-form $f(\tau,\bar\tau)d\tau = \hat{G}_2(\tau)F(\tau)d\tau$ is modular invariant, so that the contribution of the edges C_1 and C_2 cancel those of C_3 and C_4 respectively. In other words, we are integrating over the boundary of modular space with the topology of a sphere from which a small disk around $q=0$ has been removed. This domain is obtained by glueing C_1 to C_3 and C_2 to C_4. The integral is over the boundary of this domain, which corresponds to the edge C_5. In the limit $\Lambda \to \infty$ the $(\mathrm{Im}\tau)^{-1}$ term in $\hat{G}_2$ vanishes, and only the constant term survives the integration over $\mathrm{Re}\tau$. Therefore we find

$$I = \frac{2}{\pi} G_2(\tau) F(\tau) \Big|_{\text{coefficient of } q^0} \tag{3.11}$$

For example if $F(\tau)=1$ the definition of G_2 gives $I=2\pi/3$, which is indeed the volume of a fundamental domain. In principle we could compute all the integrals relevant in our context for every string theory in this way. Since some of these integrals involving Eisenstein functions might appear in other contexts, we have collected some of them in Appendix B.

In this section the integrand $F(\tau)$ in question is

$$\begin{aligned} F(\tau) &= \bar{A}(q, F, R)\Big|_{2n\text{-forms}} \\ &= \exp\left[\frac{1}{64\pi^3} \frac{1}{\mathrm{Im}\tau} (\mathrm{Tr} F^2 - \mathrm{Tr} R^2)\right] \times A(q, F, R)\Big|_{2n\text{-forms}} \end{aligned} \tag{3.12}$$

Using (1.1) we can write

$$F(\tau) = \exp\left[-\frac{1}{64\pi^4} \hat{G}_2(\tau) (\mathrm{Tr} F^2 - \mathrm{Tr} R^2)\right] \times \tilde{A}(q, F, R)\Big|_{2n\text{-forms}} \tag{3.13}$$

Substituting this into (3.9) we obtain (using $\partial/\partial\bar{\tau}\,\tilde{A} = 0$)

$$I = -\frac{4i}{\pi}\int d\mathrm{Re}\,\tau\, d\,\mathrm{Im}\,\tau \left\{ \left[\frac{-64\pi^4}{\mathrm{Tr}F^2-\mathrm{Tr}R^2}\right] \frac{d}{d\bar{\tau}}\, F(\tau) \right\}\Big|_{2n-\text{forms}} \tag{3.14}$$

The integrand is a $\bar{\tau}$-derivative, and can be integrated using (3.10). The result is

$$I = \frac{-128\pi^3}{\mathrm{Tr}F^2-\mathrm{Tr}R^2}\; A(q,F,R)\Big|_{\substack{2n+4-\text{forms}\\ \text{coefficient of } q^0}} \tag{3.15}$$

By (3.3), the coefficient of q^0 and of the $2n+4$-form in A(q,F,R) is nothing but the field theory anomaly. In other words,

$$\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\; \bar{A}(q,F,R)\Big|_{2n-\text{forms}} = -64\pi^2 X_{2n}(F,R)\,, \tag{3.16}$$

proving that the effective action (3.5) we calculated from the string precisely cancels all anomalies.

4. Conclusions

There is much to be learned about anomalies in string-theory beyond the simple fact that they cancel. Previously, a study of the factorization of the field theory anomaly led to the character valued partition function or elliptic genus. In this paper we have shown that the same function, apart from a small modification, determines the terms in the effective action that cancel the anomaly.

The character valued partition function A(q,F,R) can be expressed completely in terms of the functions G_2, G_4 and G_6, multiplying traces of F and R. The presence of G_2 leads to anomalous behavior under modular transformations. There are two possible modifications which restore modular invariance: one can replace G2 by 0 to get $\tilde{A}$(q,F,R), or replace G_2 by $\hat{G}_2$ to get $\bar{A}$(q,F,R). By explicit calculation of the loop diagram we have shown that the latter function appears as the τ-integrand of this diagram. More precisely, while the anomaly is the coefficient of q^0 of the weight two terms of A(q,F,R), the anomaly cancelling terms are the τ-integrals of the weight zero terms of $\bar{A}$(q,F,R).

In general there is no known way of calculating τ-integrals of modular functions over the fundamental domain analytically. There are however so far two exceptions: integrals that vanish due to Atkin-Lehner symmetry [8] and integrals over weight zero functions in the extended ring of modular functions $\hat{G}_2(\tau)$, $G_4(\tau)$ and $G_6(\tau)$, divided by $\eta(\tau)$ to the appropriate power (the overlap between these two sets is formed by the holomorphic functions). The second case is the relevant one for anomaly cancellation.

Our results are valid for any chiral string theory in any dimension, provided that a character valued partition function with correct modular behavior can be defined (we expect that to be the case for any consistent string theory). As with the factorization of the field theory anomaly, the validity of the result extends formally even to dimensions higher than 10, because no part of the calculation depends critically on the dimension. Nevertheless we do not expect this extension to make sense unless one can avoid the conformal anomaly, since otherwise gauge and Lorentz-invariance would already be lost at string tree level, making a discussion of anomalies rather meaningless.

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APPENDIX A

GREEN'S FUNCTIONS ON THE TORUS

In this appendix we consider the following correlation functions

$$\Delta(v,\tau) = \partial_v \ln \chi(v,\tau) \tag{A.1}$$

$$\Delta'(v,\tau) = \partial_v^2 \ln \chi(v,\tau)\ , \tag{A.2}$$

where

$$\begin{aligned} \chi(v,\tau) &= \exp\left[\frac{(\ln|z|)^2}{2\ln|w|}\right] \times \left|\frac{\vartheta_1(v|\tau)}{\vartheta_1'(0|\tau)}\right| \\ w &= e^{2\pi i\tau} = q^2 \\ z &= e^{2\pi i v} \ . \end{aligned} \tag{A.3}$$

Other expressions for these functions are

$$\Delta(v,\tau) = i\pi\left[\frac{\ln|z|}{\ln|w|} - \frac{1}{2} - \sum_{m=1}^{\infty}\left(\frac{z w^{m-1}}{1-z w^{m-1}} - \frac{w^m/z}{1-w^m/z}\right)\right] \tag{A.4a}$$

$$= \frac{1}{2}\left[\frac{1}{v} - v\,\hat{G}_2(\tau) - \bar{v}\frac{\pi}{\mathrm{Im}\,\tau} - \sum_{k=1}^{\infty} v^{2k+1} G_{2k+2}(\tau)\right] \tag{A.4b}$$

$$\Delta'(v,\tau) = 2\pi^2\left[\frac{-1}{2\ln|w|} + \sum_{m=1}^{\infty}\left(\frac{z w^{m-1}}{(1-z w^{m-1})^2} + \frac{w^m/z}{(1-w^m/z)^2}\right)\right] \tag{A.5a}$$

$$= -\frac{1}{2}\hat{G}_2(\tau) - \frac{1}{2}\left(\frac{1}{v^2} + \sum_{k=1}^{\infty}(2k+1)\,v^{2k}\,G_{2k+2}(\tau)\right)\ . \tag{A.5b}$$

(A.4a) and (A.5a) are manifestly invariant under $\nu \to \nu+1$ and $\nu \to \nu+\tau$ (i.e. $z \to zw$). Furthermore they have the following modular transformation property

$$\Delta\left(\frac{\nu}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\,\Delta(\nu,\tau) \tag{A.6}$$

$$\Delta'\left(\frac{\nu}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\,\Delta'(\nu,\tau)\,. \tag{A.7}$$

This transformation property is essential for obtaining a modular invariant τ-integrand, but it is unfortunately not manifest in (A.4a) and (A.5a). On the other hand, (A.4b) and (A.5b) are manifestly modular covariant, but clearly not manifestly doubly periodic. If the propagators were well-behaved everywhere in the integration region ***manifest*** modular invariance and periodicity would only be convenient, but not essential. But the propagators have singularities at $z=1$ (i.e. $\nu=0$) and $z=w$ which tend to spoil modular invariance after ν-integration unless they are properly regulated. This can be circumvented by writing the propagators in yet another way, which makes their modular invariance and periodicity manifest.

We can achieve this as follows. Define new variables

$$x = \mathrm{Re}\,\nu - \mathrm{Re}\,\tau\,\frac{\mathrm{Im}\,\nu}{\mathrm{Im}\,\tau} \tag{A.8}$$

$$y = \mathrm{Im}\,\nu\,.$$

Both propagators are periodic in x with period 1 and in y with period $\mathrm{Im}\tau$, and can be Fourier expanded in both variables. They are in fact already a Fourier series in x, so that we only have to calculate their Fourier modes in y. Consider first Δ. Its Fourier modes are

$$\Delta_m(x,\tau) = \frac{1}{\mathrm{Im}\,\tau}\int_0^{\mathrm{Im}\,\tau} dy\; e^{-2\pi i m\frac{y}{\mathrm{Im}\,\tau}}\,\Delta(x,y,\tau)\,. \tag{A.9}$$

To calculate these integrals it is convenient to expand the denominators in (A.4a) and to perform a resummation on the result. We get

$$\Delta(v,\tau) = i\pi \left[\frac{\ln|z|}{\ln|w|} - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{z^k}{1-w^k} + \sum_{k=1}^{\infty} \frac{(w/z)^k}{(1-(w/z)^k} \right] . \tag{A.10}$$

(If one starts from bosonic oscillators this expression is in fact obtained as an intermediate step towards (A.4a); see e.g. [15]) The y integrals are now straightforward and the result is

$$\Delta_m(x,\tau) = \frac{i}{4\pi m^2} \left[\pi i m \left(e^{-2\pi i m} + 1 \right) + \left(e^{-2\pi i m} - 1 \right) \right] + \frac{1}{2} \sum_{k \neq 0} \frac{e^{2\pi i k x}}{k\tau - m} . \tag{A.11}$$

The first term is potentially singular for $m=0$. This singularity can be traced back to the aforementioned singularities of (A.4). A natural (and, as we will see, modular invariant) regularization which suggests itself is to define Δ_0 as the limit of the right hand side for $m \rightarrow 0$. This limit is indeed well-defined

$$\Delta_0(x,\tau) = \frac{1}{2} \sum_{k \neq 0} \frac{e^{2\pi i k x}}{k\tau} . \tag{A.12}$$

For $m \neq 0$ we get

$$\Delta_m(x,\tau) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \frac{e^{2\pi i k x}}{k\tau - m} . \tag{A.13}$$

We find then the following expression for Δ

$$\Delta(v,\tau) = \frac{1}{2} \sum_{k,m}{}' \frac{1}{k\tau - m}\, e^{2\pi i k \left(\mathrm{Re}\, v - \mathrm{Re}\, \tau \frac{\mathrm{Im}\, v}{\mathrm{Im}\, \tau} \right)} e^{2\pi i m \frac{\mathrm{Im}\, v}{\mathrm{Im}\, \tau}} , \tag{A.14}$$

where the prime indicates that $k=m=0$ is not included. The expression (A.14) has seemingly a more complicated dependence on the real and imaginary parts of ν and τ than (A.4a)[1], but in return we get explicit double periodicity in ν, and (almost) manifest modular properties. Indeed, the transformation of (A.14) under the generating modular transformations $\tau \to \tau+1$ and $\tau \to -1/\tau$ is easily verified, but the proof requires shifts and interchanges on m and k, an operation which is not allowed because the sum is not absolutely convergent. This is a well-known problem in the case of the function G_2, and is discussed for example in [16] (see also Appendix B). We can use the same solution here. To obtain a function with correct modular transformation properties, one can replace the sum in (A.14) by a ζ-function regulated expression

$$\Delta_s(v,\tau) = \frac{1}{2} \sum_{k,m}{}' \frac{1}{k\tau - m} \frac{1}{|k\tau - m|^s} e^{2\pi i k \left(\mathrm{Re}\, v - \mathrm{Re}\,\tau \frac{\mathrm{Im}\, v}{\mathrm{Im}\,\tau}\right)} e^{2\pi i m \frac{\mathrm{Im}\, v}{\mathrm{Im}\,\tau}} . \tag{A.15}$$

This function is manifestly modular invariant for $s>1$, and we define Δ by analytic continuation to $s=0$.

From (A.14) (or (A.15)) one can derive the corresponding expression for Δ' by differentiation

$$\Delta'(v,\tau) = \frac{-\pi}{2\,\mathrm{Im}\,\tau} \sum_{k,m}{}' \frac{m - k\bar{\tau}}{m - k\tau} e^{2\pi i k \left(\mathrm{Re}\, v - \mathrm{Re}\,\tau \frac{\mathrm{Im}\, v}{\mathrm{Im}\,\tau}\right)} e^{2\pi i m \frac{\mathrm{Im}\, v}{\mathrm{Im}\,\tau}} , \tag{A.16}$$

from which one easily derives

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[1]Anyway, holomorphicity is no issue here since Δ, Δ' are not holomorphic due to the zero modes.

$$\int_{-1/2}^{1/2} d\,\mathrm{Re}\,\nu\ \Delta'(\nu,\tau) = -\frac{\pi}{2\,\mathrm{Im}\,\tau}\sum_m{}' e^{2\pi i m \frac{\mathrm{Im}\,\nu}{\mathrm{Im}\,\tau}}$$

$$= \frac{\pi}{2}\left[\frac{1}{\mathrm{Im}\,\tau} - \delta(\mathrm{Im}\,\nu)\right]. \tag{A.17}$$

This agrees with a result given in [9].

The denominator $(k\tau - m)$ appearing in (A.14) is of course nothing but the Fourier transform of the derivative operator on a skewed torus. This allows us in section 2 to think of a set of contractions of string momentum operators under ν-integrals in terms of two-dimensional Feynman diagrams, with (k,m) playing the role of momenta, and the ν-integrals enforcing momentum conservation at each vertex. Because of the restricted sums there are no zero momentum singularities. An important consequence of this is that the ν-integrals of the propagators vanish

$$\int_{\mathcal{F}_\tau} d^2\nu\ \Delta(\nu,\tau) = 0$$

$$\int_{\mathcal{F}_\tau} d^2\nu\ \Delta'(\nu,\tau) = 0 \tag{A.18}$$

For a closed loop we find the following expression (using $d^2\nu = 2d(\mathrm{Re}\nu)d(\mathrm{Im}\nu)$)

$$\lim_{s\to 0}\int_{\hat{\mathcal{F}}_\tau} d^2\nu_1 \cdots d^2\nu_n\ \Delta_s(\nu_{n,n-1},\tau)\cdots\Delta_s(\nu_{1,n},\tau) = (\mathrm{Im}\,\tau)^n\,\hat{G}_n(\tau) \tag{A.19}$$

Here, $\hat{G}_{2k} \equiv G_{2k}$ for $k > 1$; because of the modular invariant regularization prescription (A.15) the function $\hat{G}_2$ appears rather than G_2 (see also Appendix B).

APPENDIX B

INTEGRALS OF MODULAR FUNCTIONS

Here we present some results for the integrals of weight zero modular functions over the fundamental domain. The integrands we consider are constructed out of Eisenstein functions. For the reader's convenience we list here some of their properties (more details may be found for example in [16] [17] [4]). Their definition is

$$G_{2k}(\tau) = \sum_{m,n\in\mathbb{Z}}{}' (m\tau+n)^{-2k} . \tag{B.1}$$

For all $k>1$ these are modular functions of weight 2k. For $k=1$ the proof of modular covariance does not hold because the sum is not absolutely convergent. This implies that G_2 is no modular function, but transforms in an anomalous way:

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - 2\pi i c(c\tau+d) \tag{B.2}$$

However, by means of ζ-function regularization one can define a function $\hat{G}_2$, which is a modular function of weight two, but is not holomorphic in τ:

$$\hat{G}_2(\tau) = \lim_{s\to 0} \sum_{m,n\in\mathbb{Z}}{}' (m\tau+n)^{-2} |m\tau+n|^{-s} \tag{B.3}$$

$$= G_2(\tau) - \frac{\pi}{\mathrm{Im}\,\tau} ,$$

$$\hat{G}_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \hat{G}_2(\tau) . \tag{B.4}$$

In order not to deal with irrelevant factors, it is convenient for the discussion of integrals to define also differently normalized Eisenstein functions E_{2k}

$$G_{2k}(\tau) = \alpha_{2k} E_{2k}(\tau) , \tag{B.5}$$

where the coefficients are given by (B_{2k} are Bernoulli numbers)

$$\alpha_{2k} = -\frac{(2\pi i)^{2k}}{(2k)!} B_{2k} \tag{B.6}$$

The modular functions G_{2k}, $k>1$, can all be written as products of G_4 and G_6, so that there is no need to consider functions with $k>3$. The first three Eisenstein functions have the following expansions

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^{2n}}{1-q^{2n}} \tag{B.7}$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}} \tag{B.8}$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^{2n}}{1-q^{2n}} \ . \tag{B.9}$$

Using the method described in section 3 it is straightforward to calculate the τ-integral of any weight zero function of the form ($\hat{E}_2 = E_2 - 3/(\pi \mathrm{Im}\tau)$)

$$\frac{\hat{E}_2^{m}(\tau) F_l(\tau)}{\eta^{24p}(\tau)} \ , \tag{B.10}$$

where F_l is a holomorphic function of weight $2l$, and $m+l=6p$. Using (3.10) we find

$$\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \frac{\hat{E}_2^{m}(\tau) F_l(\tau)}{\eta^{24p}(\tau)} = \frac{2\pi}{3(m+1)} \frac{\hat{E}_2^{m+1}(\tau) F_l(\tau)}{\eta^{24p}(\tau)} \Bigg|_{\text{coefficient of } q^0} \tag{B.11}$$

An interesting case is the integral of the absolute modular invariant j_{Leech}

$$j_{\mathrm{Leech}} = \frac{E_4^3}{\eta^{24}} - 720 = \frac{1}{q^2} + 24 + 196884 q^2 + \ldots.. \tag{B.12}$$

Here we have chosen the constant term in such a way that $j_{\rm Leech}$ is the left-mover partition function of a two-dimensional heterotic string compactified on the Leech lattice. Upon multiplication with G_2 we get a function without a constant term, so that we conclude that the integral of $j_{\rm Leech}$ vanishes. The same conclusion was reached by Moore [8] using Atkin-Lehner symmetry.

Other integrals, some of which are needed in Appendix C, are listed below. Define

$$I(m, \ell_1, \ell_2) = \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \; \frac{\hat{E}_2^{m} E_4^{\ell_1} E_6^{\ell_2}}{\eta^{24p}}(\tau) \tag{B.13}$$

with $m+2l_1+3l_2=6p$. Then

$$\begin{aligned}
I(0,0,0) &= \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} = \frac{2}{3}\pi \\
I(0,3,0) &= 480\pi \\
I(0,0,2) &= -672\pi \\
I(1,1,1) &= -96\pi \\
I(2,2,0) &= 96\pi \\
I(3,0,1) &= -96\pi \\
I(4,1,0) &= \frac{96}{5}\pi \\
I(6,0,0) &= -\frac{96}{7}\pi
\end{aligned} \tag{B.14}$$

APPENDIX C

EXAMPLES

To illustrate the discussion of section 3, we give some examples to show how to obtain the anomaly cancelling effective action directly from the elliptic genus by performing modular integrals. Consider first the ten-dimensional supersymmetric heterotic string with gauge group SO(32).

The partition function in the left sector associated with the right-moving PP-sector is

$$\begin{aligned} A(q,0,0) &= \frac{1}{\eta^{24}(\tau)} \frac{1}{2} \sum_{\ell=1}^{4} \vartheta_\ell^{16}(0|\tau) \\ &= \frac{1}{\eta^{24}(\tau)} E_8(\tau) , \end{aligned} \tag{C.1}$$

where $E_8 = (E_4)^2$ is an Eisenstein function normalized as in (B.5). The elliptic genus is then obtained by "gauging" (C.1):

$$A(q,F,R) = \frac{1}{2}\left[e^{\sum_{k=1}^{\infty} \frac{1}{4k} \frac{1}{(2\pi i)^{2k}} \mathrm{Tr}\left(\frac{iR}{2\pi}\right)^{2k} G_{2k}(\tau)} \right] \times \left[\frac{\sum_{\ell=1}^{4} \prod_{j=1}^{16} \vartheta_\ell\left(\frac{s_j}{2\pi i}\Big|\tau\right)}{\eta^{24}(\tau)} \right] \tag{C.2}$$

Here, s_j are the skew eigenvalues of $F_j/2\pi$ in the Cartan subalgebra of SO(32). After some manipulations one can expand it into Eisenstein series as follows (all traces are over the vector representation):

$$A(q,F,R) = e^{\frac{1}{64\pi^4} G_2(\tau)(\mathrm{Tr} R^2 - \mathrm{Tr} F^2)} \, e^{\sum_{k=2}^{\infty} \frac{1}{4k} \frac{1}{(2\pi i)^{2k}} \mathrm{Tr}\left(\frac{iR}{2\pi}\right)^{2k} G_{2k}(\tau)} \times$$

$$\times\Bigg[A(q,0,0)-\frac{2^5\cdot3^4\cdot5^2\cdot7}{(2\pi)^{10}}\frac{G_4G_6}{\eta^{24}}\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^2+\left\{\frac{2^3\cdot3^4\cdot5^3}{(2\pi)^{12}}\frac{G_4^3}{\eta^{24}}-\frac{1}{4}\right\}\left(\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^2\right)^2 \tag{C.3}$$

$$+\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^4-\frac{3^3\cdot5^3\cdot7}{(2\pi)^{14}}\frac{G_4^2G_6}{\eta^{24}}\left(\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^2\right)^3+\mathcal{O}(F^8)\Bigg]$$

Note that in the bracket the coefficients of certain powers of F have poles. One may however check that in the complete expression the terms containing only F's do not have poles. This is to be expected since the (anyway unphysical) tachyon has no gauge charge.

The anomaly is given by the 12-form multiplying q^0 in (C.3):

$$I_{12}=A(q,F,R)\Big|_{\substack{12\text{-form}\\ \text{coefficient of } q^0}}=-\frac{1}{2\pi}\left(\mathrm{Tr}\,R^2-\mathrm{Tr}\,F^2\right)X_8(F,R)\,, \tag{C.4}$$

$$X_8(F,R)=-\frac{1}{3\cdot2^{10}\cdot\pi}\left[32\,\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^4-4\,\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^2\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2+4\,\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^4+\left(\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\right)^2\right]$$

On the other hand, the integrand of the anomaly cancelling term is given by the 8-form in $\bar{A}(q,F,R)$, which is given by (C.3) when G_2 is replaced by $\hat{G}_2$. If we switch to the differently normalized Eisenstein functions E_{2k} (B.5), it reads

$$\begin{aligned}
&\bar{A}(q,F,R)\Big|_{8\text{-form}}\\
&=\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^4+\frac{1}{2^7\cdot3^2\cdot5}\frac{E_4^3}{\eta^{24}}\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^4\\
&\quad+\frac{1}{2^9\cdot3^2}\left[\frac{E_4^3}{\eta^{24}}+\frac{\hat{E}_2^2E_4}{\eta^{24}}-2\frac{\hat{E}_2E_4E_6}{\eta^{24}}-2^7\cdot3^2\right]\left(\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^2\right)^2\\
&\quad+\frac{1}{2^8\cdot3^2}\left[\frac{\hat{E}_2E_4E_6}{\eta^{24}}-\frac{\hat{E}_2^2E_4^2}{\eta^{24}}\right]\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^2\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\\
&\quad+\frac{1}{2^9\cdot3^2}\frac{\hat{E}_2^2E_4^2}{\eta^{24}}\left(\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\right)^2\,.
\end{aligned} \tag{C.5}$$

Using formulas (B.14) for the integrals involving E_4, E_6 and $\hat{E}_2$, it is easy to verify anomaly cancellation, i.e.,

$$\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\, \bar{A}(q,F,R)\Big|_{8\text{-form}} = \frac{-128\pi^3}{\mathrm{Tr}F^2-\mathrm{Tr}R^2}\, I_{12} = -64\pi^2 X_8(F,R)\,, \tag{C.6}$$

where X_8 is defined above. The effective action is of course given by (3.4).

To obtain the analogous expressions for the O(16)×O(16) string, one must replace the square brackets in (C.3) by

$$\Big[\mathrm{Tr}\Big(\frac{iF_1}{2\pi}\Big)^4 + \mathrm{Tr}\Big(\frac{iF_2}{2\pi}\Big)^4 - \frac{1}{4}\Big\{\Big(\mathrm{Tr}\Big(\frac{iF_1}{2\pi}\Big)^2\Big)^2 + \Big(\mathrm{Tr}\Big(\frac{iF_2}{2\pi}\Big)^2\Big)^2 - \mathrm{Tr}\Big(\frac{iF_1}{2\pi}\Big)^2\mathrm{Tr}\Big(\frac{iF_2}{2\pi}\Big)^2\Big\} + \mathcal{O}(F^8)\Big] \tag{C.7}$$

The simplicity of this result in comparison with (C.3) is due to the absence of a gravitino. A chiral gravitino (which can exist only in 6 and 10 dimensions) leads always to $1/q^2$ pole in the character valued partition function. The presence or absence of this pole determines all coefficient functions with negative weight completely, because there is just one modular function for each negative weight (this is also true for functions of weight two). In particular they must vanish if they have no poles. Because the gravitino is a gauge singlet, terms involving traces of F cannot have a pole. Therefore the negative weight and weight two coefficient functions of pure gauge terms, not involving $\mathrm{Tr}F^2$, must vanish (hence the absence of $\mathrm{Tr}F^6$ in (C.3)). This is true for supersymmetric as well as non-supersymmetric theories. For terms involving $\mathrm{Tr}F^2$ this argument is not valid, because such terms can come from the exponential prefactor (yielding G_2 terms) and the square brackets, as in (C.3). If both kinds of contributions occur, their poles must cancel. In the absence of a chiral gravitino there are no G_2 terms contributing to negative weight functions, because there are no other negative weight factors to multiply them with. Therefore in non-supersymmetric string theories, as well as in all 4 and 8 dimensional strings there are no negative weight functions at all.

Weight zero terms cannot be ruled out by this argument, because there are two weight zero modular functions, j and the constant function. The latter is determined by the massless fermions. The constant term takes its simplest form if one writes the traces in the actual representation of the massless fermions rather than the vector representation. Because the character valued partition function is essentially a Chern-character it contains no mixed traces, and one simply gets, in 2n+2 dimensions

$$\frac{1}{n!}\,\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^n \quad . \tag{C.8}$$

Indeed, (C.7) is simply $1/24\ \mathrm{Tr}(iF/2\pi)^4$, where $F=F_1+F_2$ and the trace is over the $O(16)\times O(16)$ representation $(16,16)-(128,1)-(1,128)$. In all non-supersymmetric theories and all 4 and 8 dimensional ones (C.8) is the only weight zero term that appears in the square brackets (cf. (C.3)), and thus is also the only factor multiplying $\mathrm{Tr}F^2-\mathrm{Tr}R^2$ in the anomaly. These cases were already completely covered by the results of [1].

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Figures

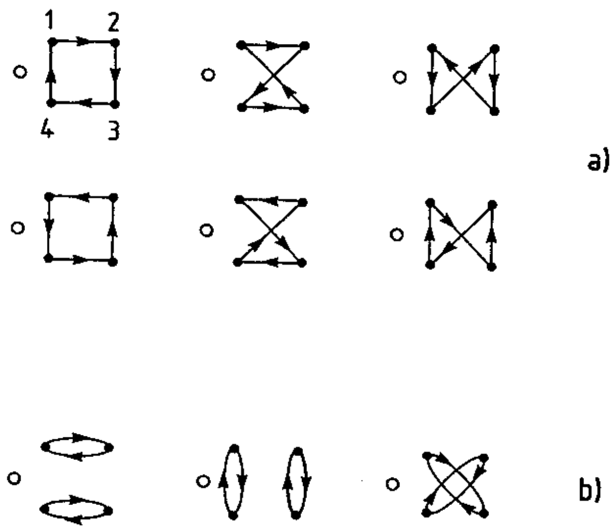

Fig 1: Contributions to the ν-integrals for the ten dimensional case. Each graph represents a particular momentum structure. The diagrams a) are proportional to G_4, while those of b) are proportional to $(\hat{G}_2)^2$.

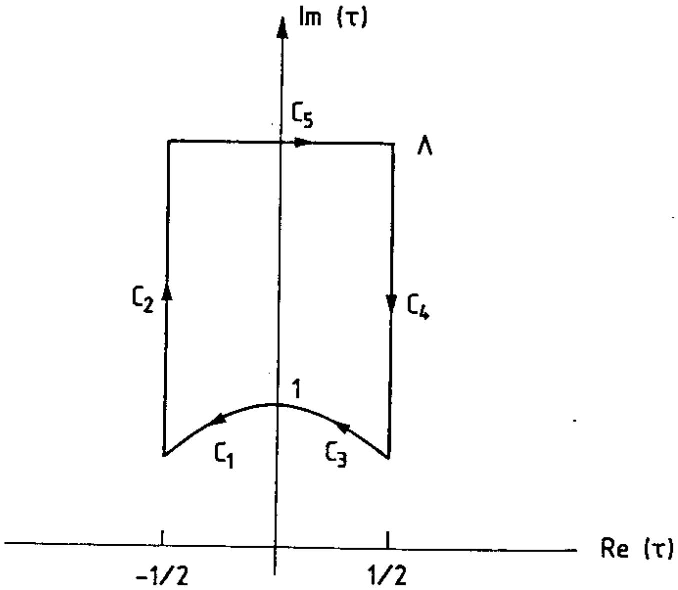

Fig 2: The path of integration along the boundary of modular space.