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Elliptic Index and Superstring Effective Actions

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ABSTRACT

Certain $N=1$ supersymmetric string one-loop effective actions can be obtained directly from the path integral. As the computation is essentially the same as the one leading to the index of the Dirac-Ramond operator, they are determined by the gauge and gravitational anomaly structure of the theory. Specifically, we calculate the four-point effective action in ten dimensions, the corrections to the kinetic terms in $d=6$ (including auxiliary fields) and the Fayet-Iliopoulos D-term in $d=4$. We also compute the β-function of four dimensional $N=2$ theories from the elliptic genus in $d=6$. Furthermore, we derive supersymmetry Ward type identities in terms of Kac-Moody characters, relating parity conserving with parity violating amplitudes.

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1. Introduction

1.1 General Remarks

Some progress has recently been made in explicit computations of certain one-loop amplitudes in various heterotic string theories [1-6]. Common to all of these calculations are (almost) holomorphic modular integrands in the low-energy limit, which allow for an *explicit* evaluation of the ν and τ integrals. Such a holomorphicity always arises if only massless zero modes contribute in the right-moving sector, in that the contributions of all massive states ($m_R^2 \neq 0$) cancel out. Then the right-moving partition function is a constant†. In particular this happens in the case of the Green-Schwarz anomaly cancelling terms [1][4], where only the zero modes ψ_0^μ contribute (giving the ϵ-tensor), and massive states cancel because of opposite helicity. This can directly be interpreted in terms of the index of the Dirac-Ramond operator [4].

A similar phenomenon occurs also for certain supersymmetric string amplitudes, where only the zero modes of the Green-Schwarz fields S_0^a contribute and massive states cancel because of supersymmetry. More specifically, it was noticed in [7][5] that various bosonic four-point one-loop amplitudes of supersymmetric ten dimensional heterotic strings have basically the same structure as the anomaly cancelling terms, except for the ϵ-tensor and the B-field. Furthermore, it is known that in four dimensional superstring theories, a certain part (the Fayet-Iliopoulos D-term [2][3]) of $S^{\text{eff}}_{1-\text{loop}}$ has a structure similar to the anomaly cancelling term, $\int B\,\text{Tr}F$.

Our objective is to clarify the relation between these and other observations, and to show that all holomorphic supersymmetric amplitudes can be directly related to and expressed in terms of the elliptic genus.

We will discuss in the following three generic cases, where holomorphic (bosonic) amplitudes can occur: four-point amplitudes in $d=10$, two-point amplitudes in $d=6$ and one-point amplitudes in $d=4$. In these dimensions, $N=1$ supersymmetric theories can be chiral, and amplitudes can be mapped to the chiral index*. For these numbers of external legs, the zero modes S_0^a are just saturated.

† Thus, infinitely many "unphysical" states with $m_L^2 \neq m_R^2 = 0$ may contribute to the modular integral.
* Therefore, three-point amplitudes in $d=8$ are missing in the list, as there is no chiral supersymmetric theory in $d=8$.

Amplitudes with less external legs vanish identically, and amplitudes with more external lines do not have, in general, holomorphic integrands, precluding model independent statements. Exceptions are amplitudes where all contributions come from the boundary of modular space, *i.e.*, from configurations where some vertex operators come close together, thus effectively producing holomorphic amplitudes with less external legs. Examples for this are the anomaly cancelling term which arises as a boundary in the anomaly diagram, and the four-dimensional D-term, which is a boundary in the scalar two-point amplitude [2][3]. We will also find that in $d=6$ holomorphic two-point amplitudes arise as boundary contributions in three gauge boson amplitudes, thus relating coupling constant renormalization to the elliptic chiral index.

Our results are relevant also for torus compactified lower dimensional theories with extended supersymmetry. As an example, we will show how the β-function in $d=4$, $N=2$ supersymmetric theories can be related to the chiral anomaly in six dimensions.

The above three cases are simply the only possibilities for supersymmetry one can have in heterotic string theories. According to the construction of ref. [8], they can be characterized by the exceptional groups E_6, E_7 and E_8 in a model and dimension independent way. We will use this in the appendix to derive supersymmetry Ward type identities between partition functions. They explain our results from the NSR point of view.

1.2 Anomaly Cancelling Terms Revisited

For later convenience, we recall that the anomaly cancelling effective action in $d=2n+2$ dimensions can be computed as [4][‡] (for heterotic strings)

$$S^{\rm eff}_{1-{\rm loop}}(B,F,R) = \int d^{2n+2}x \ \ \Big(\tfrac{1}{16\pi^2} B \int_{\mathcal{F}} \tfrac{d^2\tau}{(Im\,\tau)^2}\, \overline{\mathcal{A}}(q,{\rm Im}\tau,F,R)\Big)\Big|_{\rm 2n+2\ form} \quad , \tag{1.1}$$

where B is the antisymmetric tensor two-form, F and R are the gauge and gravitational curvature two-forms, and $\overline{\mathcal{A}}(q,{\rm Im}\tau,F,R)$ is related to the character valued index of the Dirac-Ramond operator [9-11]. The index is given by

$$\begin{aligned} {\rm ind}G_0(q,F,R) &= q^{-1}\int \widehat{A}(R)\,\mathcal{C}h(q,R)\,\mathcal{C}h(q,F)\Big|_{\rm top\ form} \\ &\equiv \int \mathcal{A}(q,F,R)\Big|_{\rm top\ form} \quad , \end{aligned} \tag{1.2}$$

‡ We set gauge coupling constants to unity, and $\alpha' = \frac{1}{2}$; $q \equiv e^{2\pi i\tau}$ refers to the left-moving sector.

where $\widehat{A}(R)$ is the Dirac genus and $Ch(q,R) = \sum_{n=0}^{\infty} q^n \mathrm{Tr}_{[n]} e^{\frac{iR}{2\pi}}$ the Chern character for every left-moving string level. More explicitly, the elliptic genus $\mathcal{A}(q,F,R)$ has the form [9]

$$\mathcal{A}(q,F,R) = e^{\sum_{k=1}^{\infty} \frac{1}{4k(2\pi i)^{2k}} \mathrm{Tr}\left(\frac{iR}{2\pi}\right)^{2k} G_{2k}(\tau)} \cdot \chi_{PP}(y_\beta|\tau) , \qquad (1.3)$$

where $G_{2k}(\tau) = \sum'(m\tau+n)^{-2k}$ are the Eisenstein series (modular forms of weight $2k$ for $k>1$), $\chi_{PP}(y_\beta|\tau)$ is the (gauge) character valued partition function associated with the periodic-periodic sector of the space-time fermions, and y_β are the skew eigenvalues of $\frac{i}{2\pi}F$. Due to anomalous modular properties of G_2 in the exponential (and χ_{PP}) above, $\mathcal{A}(q,F,R)$ does not have a nice modular behavior unless $\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2 = 0$. However, the function appearing in (1.1) is not (1.3) but

$$\begin{aligned} \overline{\mathcal{A}}(q,\mathrm{Im}\tau,F,R) &= e^{\left\{\frac{1}{64\pi^4}\frac{\pi}{\mathrm{Im}\tau}\left(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2\right)\right\}} \cdot \mathcal{A}(q,F,R) \\ &= e^{-\left\{\frac{1}{64\pi^4}\left(G_2(\tau)-\frac{\pi}{\mathrm{Im}\tau}\right)\left(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2\right)\right\}} \times [\dots] , \end{aligned} \qquad (1.4)$$

whose $2n$-form part is modular invariant. The appearance of $\frac{\pi}{\mathrm{Im}\tau}$ above is a manifestation of Quillen's holomorphic anomaly. It (partly) arises due to a particular regularization scheme [12], which maintains modular invariance but spoils holomorphicity: only

$$\widehat{G}_2(\tau) = \lim_{s\to 0} \sum{}' \frac{1}{(m\tau+n)^2} \frac{1}{|m\tau+n|^s} = G_2(\tau) - \frac{\pi}{\mathrm{Im}\tau} \qquad (1.5)$$

is modular covariant. Thus, the integrand in (1.1) as a sum of products of $\widehat{G}_2, G_4$ and G_6 does not factorize holomorphically, at least not in the most naive way♮ .

Using the formula

$$\int_{\mathcal{F}} \frac{d^2\tau}{(Im\,\tau)^2} f(q) = \frac{2}{\pi}\left[G_2(q) f(q)\right]\Big|_{\text{coeff. of } q^0} \qquad (1.6)$$

for weight zero functions $f(q)$, the modular integral (1.1) over the tower of infinitely many left-moving modes can explicitly be evaluated [4]:

$$\int_{\mathcal{F}} \frac{d^2\tau}{(Im\,\tau)^2} \overline{\mathcal{A}}(q,\mathrm{Im}\tau,F,R)\Big|_{2n-\mathrm{form}} = -64\pi^2 X_{2n}(F,R) . \qquad (1.7)$$

The $2n$-form X_{2n} is a certain polynomial in terms of traces of powers of F and R, and is

♮ Accordingly, with "holomorphic" we always mean holomorphic up to such $\mathrm{Im}\tau$'s and up to the pole at $q=0$.

defined by the chiral anomaly I_{2n+4} of the particular theory:

$$I_{2n+4}(F,R) \equiv \mathcal{A}(q,F,R)\Big|_{\substack{2n+4 \text{ form} \\ \text{coeff. of } q^0}} = \tfrac{1}{2\pi}\left(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2\right)\cdot X_{2n}(F,R)\ . \tag{1.8}$$

Because of modular invariance [9], (1.8) has always this factorized form. It follows

$$S^{\rm eff}_{1-\rm loop}(B,F,R) = -4\int d^{2n+2}x\ \ B\,X_{2n}(F,R)\ . \tag{1.9}$$

This is precisely the counter term anticipated by Green and Schwarz [13].

2. Effective Actions from Loop Space Index Theorems

2.1 The Ten Dimensional Case

We compute first the four-point one-loop effective action for gravitons and gauge bosons, for the well-known ten dimensional supersymmetric heterotic string theories. For reasons to become clear later, it is easiest to employ light-cone Green-Schwarz formalism. As there are eight zero modes S_0^a $(a = 1...8)$ and since bosonic vertex operators contain two S^a, saturation of the zero modes implies that the corresponding four-point amplitudes are holomorphic.

The effective action is given by

$$S^{\rm eff}_{1-\rm loop}(F,R) = \int \mathcal{D}X\,\mathcal{D}S\,\mathcal{D}\lambda\,\mathcal{D}h\ \ e^{-S_E(F,R)}\ , \tag{2.1}$$

where the path integral is to be evaluated on the torus. The fields X^i $(i = 1...8)$ and S^a have periodic boundary conditions along both cycles as required by world-sheet supersymmetry, $\mathcal{D}\lambda$ implicitly contains an appropriate sum over the spin structures of the left-moving fermions, and the integral over the world-sheet metric h can be traded for a modular integral. The two dimensional euclidean action is given by a σ-model on a torus characterized by the modular parameter τ:

$$S_E = \tfrac{1}{2\pi}\int_0^1 dt\int_0^\pi d\sigma\ \left[\frac{1}{\mathrm{Im}\tau}g_{ij}(X)\partial X^i\overline{\partial}X^j - \lambda^A\overline{\mathcal{D}}\lambda_A - S^aDS_a - \tfrac{1}{8}\mathrm{Im}\tau F^{AB}_{ij}S^a\gamma^{ij}_{ab}S^b\lambda_A\lambda_B\right]\ . \tag{2.2}$$

This action can be obtained via hamiltonian operator formalism along the lines of [10]. In (2.2), $\partial = 2i\mathrm{Im}\tau\partial_z$, $\overline{\partial} = -2i\mathrm{Im}\tau\partial_{\overline{z}}$, $z = \sigma + \tau t$, multiplication with appropriate two dimensional gamma matrices is understood, and

$$\begin{aligned} \overline{D}\lambda_A &= \overline{\partial}\lambda_A - iA_i^\alpha(T_\alpha)^{AB}\overline{\partial}X^i\lambda^B \\ DS_a &= \partial S_a + \tfrac{1}{4}\omega_{lk}^j(X)\partial X^l\delta_{ij}\gamma_{ab}^{ik}S^b \ . \end{aligned} \tag{2.3}$$

In the limit $\mathrm{Im}\tau \to 0$, the path integral is dominated by small fluctuations around the classical solutions. These are given by the zero modes of the periodic fields, and we can write in the usual way

$$\begin{aligned} X^i &= x_0^i + \sqrt{\mathrm{Im}\tau}X'^i \ , \\ S^a &= \sqrt{\tfrac{i}{2\pi\mathrm{Im}\tau}}S_0^a + S'^a \ . \end{aligned} \tag{2.4}$$

Since our result will be holomorphic, it follows by analytic continuation that the lowest order approximation is exact. Upon normal coordinate expansion around x_0,

$$\begin{aligned} g_{ij}(X) &= \delta_{ij} - \tfrac{1}{3}\mathrm{Im}\tau R_{ikjl}(x_0)X'^kX'^l + \dots \\ \tfrac{1}{4}\omega_{lk}^j(X)\partial X^l\delta_{ij}S^a\gamma_{ab}^{ik}S^b &= \tfrac{1}{8}\mathrm{Im}\tau R_{ijkl}(x_0)X'^k\partial X'^lS^a\gamma_{ab}^{ij}S^b + \dots \end{aligned} \tag{2.5}$$

and choosing the gauge $A_i^\alpha = 0$, the terms leading as $\mathrm{Im}\tau \to 0$ are:

$$\begin{aligned} S_E^{(2)} &= \tfrac{1}{2\pi}\int dt\, d\sigma \Big[\partial X_i'\overline{\partial}X'^i - S'^a\partial S_a' - \lambda^A\overline{\partial}\lambda_A \\ &\qquad - \tfrac{i}{16\pi}R_{ijkl}X'^i\partial X'^j\, S_0^a\gamma_{ab}^{kl}S_0^b - \tfrac{i}{16\pi}F_{ij}^{AB}S_0^a\gamma_{ab}^{ij}S_0^b\,\lambda_A\lambda_B\Big] \\ &= \tfrac{1}{2\pi}\int dt\, d\sigma \Big[\partial X'^i(\overline{\partial}\delta_{ij} + \tfrac{i}{2\pi}\widehat{\mathcal{R}}_{ij})X'^j - S'^a\partial S_a' - \lambda_A(\overline{\partial}\delta^{AB} + \tfrac{i}{2\pi}\widehat{\mathcal{F}}^{AB})\lambda_B\Big] \ . \end{aligned} \tag{2.6}$$

Here,

$$\begin{aligned} \widehat{\mathcal{R}}_{ij} &= \tfrac{1}{2}\widehat{R}_{ijab}\, S_0^a \wedge S_0^b \\ \widehat{\mathcal{F}}^{AB} &= \tfrac{1}{2}\widehat{F}_{ab}^{AB}\, S_0^a \wedge S_0^b \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \widehat{R}_{ijab} &= \tfrac{1}{4}\gamma_{ab}^{kl}R_{ijkl} \\ \widehat{F}_{ab}^{AB} &= -\tfrac{i}{4}\gamma_{ab}^{ij}F_{ij}^\alpha(T_\alpha)^{AB} \ . \end{aligned} \tag{2.8}$$

As $\widehat{\mathcal{R}}_{ij}$ and $\widehat{\mathcal{F}}^{AB}$ are valued in the exterior algebra of the spinor bundle generated by S_0^a, we may regard them as spinor two-forms. Such objects have been discussed recently [14].

The leading part of S_E is quadratic, and one can easily evaluate the non-zero mode integrals:

$$\begin{aligned}\int \mathcal{D}X\mathcal{D}S\mathcal{D}\lambda \; e^{-S_E^{(2)}} &= \int dx_0^i\, dS_0^a \left[\frac{\det'(\partial)}{\det'(\partial)}\right]^{\frac{1}{2}} \left[\frac{\det(\overline{\partial}\delta^{AB} + \frac{i}{2\pi}\widehat{\mathcal{F}}^{AB})}{\det'(\overline{\partial}\delta_{ij} + \frac{i}{2\pi}\widehat{\mathcal{R}}_{ij})}\right]^{\frac{1}{2}} \\ &= \int dx_0^i\, dS_0^a \prod_\alpha \left(\frac{\frac{ix_\alpha}{2\pi}\eta(\tau)}{\vartheta_1(\frac{ix_\alpha}{2\pi}|\tau)}\right) \sum_{\substack{\text{spin}\\ \text{structures}}} \prod_\beta \left(\frac{\vartheta_*(\frac{iy_\beta}{2\pi}|\tau)}{\eta(\tau)}\right) .\end{aligned} \tag{2.9}$$

In the first equation, summation over the spin structures of λ^A is implicitly understood, and x_α, y_β in the second equation denote the skew eigenvalues of $\frac{i}{2\pi}\widehat{\mathcal{R}}_{ij}$ and $\frac{i}{2\pi}\widehat{\mathcal{F}}^{AB}$, respectively ($\alpha = 1...4$, $\beta = 1...16$). Note that all dependence on Imτ disappears. The first bracket above cancels due to the world-sheet supersymmetry between X'^i and S'^a, hence (2.9) is holomorphic in q. The crucial point is that the calculation we are performing is very similar to the one leading to the index of the Dirac-Ramond operator [10][11]. In fact, the index is given by (2.9) if $\int dS_0^a$ is replaced by $\int d\psi_0^i$ and the spinor forms $\widehat{\mathcal{R}}_{ij}, \widehat{\mathcal{F}}^{AB}$ by

$$\begin{aligned}\mathcal{R}_{ij} &= \tfrac{1}{2} R_{ijkl}\, \psi_0^k \wedge \psi_0^l \\ \mathcal{F}^{AB} &= -\tfrac{i}{2} F_{ij}^\alpha (T_\alpha)^{AB} \psi_0^i \wedge \psi_0^j ,\end{aligned} \tag{2.10}$$

which are two-forms in the exterior algebra generated by ψ_0^i. Thus, we can borrow from the index calculation to rewrite (2.9) as

$$\int \mathcal{D}X\mathcal{D}S\mathcal{D}\lambda \; e^{-S_E^{(2)}} = \int d^8x \; \mathcal{A}(q, \widehat{F}, \widehat{R})\Big|_{\text{spinor 8-form}} = \text{ind}G_0(q, \widehat{F}, \widehat{R}) . \tag{2.11}$$

Here, 'spinor 8-form' refers to terms contributing after the Berezin integration proportional to

$$\int dS_0^a \; S_0^{a_1} S_0^{a_2} \dots S_0^{a_8} = \epsilon^{a_1 a_2 \dots a_8} , \tag{2.12}$$

similar to

$$\int d\psi_0^i \; \psi_0^{i_1} \psi_0^{i_2} \dots \psi_0^{i_8} = \epsilon^{i_1 i_2 \dots i_8} \tag{2.13}$$

in the index computation.

To obtain the effective action, one still has to perform the modular integral, with the correct measure. Using the results of a covariant treatment [15], one can infer a factor of $\mathrm{Im}\tau^{-1}$ from the ghosts and the same factor from the longitudinal modes of X^μ. Altogether we get the modular invariant measure $\frac{d^2\tau}{\mathrm{Im}\tau^2}$. Of course, we cannot derive the absolute normalization in such an easy way; our results will be correct only up to overall factors.

The integrand (2.11) is not yet modular invariant due to the above-mentioned modular anomalies associated with $G_2(\tau)$ in $\mathcal{A}(q,F,R)$ (1.3). Therefore, one has to regulate the determinants in (2.9) in a different, modular invariance preserving way. The same problem occurs of course also in explicit loop amplitude computations [4][5][6]. Performing the ζ-function regularization (1.5) of $G_2(\tau)$ effectively amounts to replacing $\mathcal{A}(q,\widehat{F},\widehat{R})$ by $\overline{\mathcal{A}}(q,\mathrm{Im}\tau,\widehat{F},\widehat{R})$ (presumably, one can also add suitable local counter terms to the σ-model). Then, using the formula (1.7), (2.11) can explicitly be integrated:

$$\begin{aligned} S^{\mathrm{eff}}_{1-\mathrm{loop}}(F,R) &\propto \int_{\mathcal{F}} \frac{d^2\tau}{(Im\,\tau)^2} \int d^8x\ \overline{\mathcal{A}}(q,\mathrm{Im}\tau,\widehat{F},\widehat{R})\Big|_{\text{spinor 8-form}} \\ &= -64\pi^2 \int d^8x\ \epsilon_{a_1a_2\ldots a_8}\widehat{X}_8^{a_1a_2\ldots a_8}(\widehat{F},\widehat{R})\ . \end{aligned} \tag{2.14}$$

The spinor form $\widehat{X}_8^{a_1a_2\ldots a_8}(\widehat{F},\widehat{R})$ is obtained from the eight-form $X_8(F,R)$ appearing in the anomaly polynomial (1.8) by simply replacing vector by spinor indices and the two-forms F, R by the spinor two-forms $\widehat{F}$ and $\widehat{R}$. Note that even though (2.14) contains an ϵ-tensor, $S^{\mathrm{eff}}_{1-\mathrm{loop}}(F,R)$ does not necessarily violate parity. Rather, the contraction of the γ-matrices in $\widehat{F}$ and $\widehat{R}$ with the ϵ-tensor produces the well-known t-tensor:

$$\left(\tfrac{1}{4}\gamma^{ij}_{ab}\right)\left(\tfrac{1}{4}\gamma^{kl}_{cd}\right)\left(\tfrac{1}{4}\gamma^{mn}_{ef}\right)\left(\tfrac{1}{4}\gamma^{pq}_{gh}\right)\epsilon^{abcdefgh} = t^{ijklmnpq}\ . \tag{2.15}$$

An explicit expression for $t^{ijklmnpq}$ can be found *e.g.* in [16]. We thus obtain the results derived by explicit string loop computations [5][6]. For example, from the formula for $X_8(F,R)$ corresponding to the $SO(32)$ heterotic string [13][4] one immediately deduces

$$\begin{aligned} S^{\mathrm{eff}\,[SO(32)]}_{1-\mathrm{loop}}(F,R) \propto \int d^8x\ \frac{1}{3\cdot 2^{12}\pi^3}\, t^{ijklmnpq} \Big\{ &4\,\mathrm{Tr}\,R_{ij}R_{kl}R_{mn}R_{pq} + \mathrm{Tr}\,(R_{ij}R_{kl})\,\mathrm{Tr}\,(R_{mn}R_{pq}) \\ &+32\,\mathrm{Tr}\,F_{ij}F_{kl}F_{mn}F_{pq} - 4\,\mathrm{Tr}\,(F_{ij}F_{kl})\,\mathrm{Tr}\,(R_{mn}R_{pq}) \Big\} \end{aligned} \tag{2.16}$$

(the traces are over the vector representations). Due to its close connection to the chiral index of the theory, it is conceivable that this expression does not receive corrections in higher loop orders.

As four-point amplitudes involving external fermion lines have the same structure as those with purely bosonic legs (modulo kinematical factors) [17], we conjecture that also the fermionic (*i.e.*, the complete supersymmetric) effective action should be expressible in a way similar to (2.14). However, it seems to be difficult to implement fermionic couplings in the light-cone σ-model (2.2), and one should probably better adopt a covariant formulation in order to check that.

The result that the effective action (2.14) is closely related to the anomaly (1.8) can be understood in terms of triality rotations in the $SO(8)$ transverse Lorentz group. More precisely, the transition between Green-Schwarz and Neveu-Schwarz formalism is achieved by

$$\begin{aligned} \psi^i &= \overline{\xi}^{\dot{a}} \gamma^i_{\dot{a}b} S^b \\ S^b &= -\gamma^{ib}_{\dot{a}} \xi^{\dot{a}} \psi_i \ , \end{aligned} \tag{2.17}$$

where ξ is a real, commuting constant 8_c spinor satisfying $\overline{\xi}\xi = 1$ [18]. Clearly (2.17) can only make sense if ψ^i and S^a have the same boundary conditions. As S^a is periodic along both cycles, ψ^i is also periodic. Thus, (2.17) maps the GS-theory to the PP-sector of the NSR-theory. But this is precisely the sector where ψ^i has zero modes, and where the anomalies (as well as all parity violating amplitudes) reside. How this relation between parity violating and parity conserving sectors comes about in the NSR-formalism (where it is not so obvious), will be discussed in the appendix.

The same method works also for four and six dimensional $N=1$ supersymmetric theories. However, since the number of the supercharges (or S_0^a zero modes) is smaller in these cases, the structure of the effective theory is less fixed, and accordingly we will be able to compute only a few terms of the effective action.

2.2 The Four Dimensional Case

The situation is simplest for $N=1$, $d=4$: here we have two zero modes S_0^a, $a=1,2$. One can therefore obtain results only for one-point functions. The only term we can add to the σ-model which will give a non-trivial result is

$$\Delta S_E = -\tfrac{1}{2\pi}\int dt\, d\sigma\, \tfrac{1}{2}\mathrm{Im}\tau\, D^{AB}\epsilon_{ab}S^a S^b\lambda_A\lambda_B \ . \tag{2.18}$$

The field D^{AB} is not a physical field, but rather it is (essentially*) the auxiliary D-component of a $N=1$ vector superfield. There is no reason to exclude such an auxiliary field background coupling from the σ-model. By the same argumentation as above, in terms of

$$\begin{aligned}\widehat{\mathcal{D}}^{AB} &= \tfrac{1}{2}\widehat{D}_{ab}^{AB}S_0^a\wedge S_0^b\\ \widehat{D}_{ab}^{AB} &= -i\epsilon_{ab}D^\alpha(T_\alpha)^{AB}\ ,\end{aligned} \tag{2.19}$$

the effective action is

$$S^{\mathrm{eff}}_{1-\mathrm{loop}}(D)\propto\int_{\mathcal{F}}\frac{d^2\tau}{(Im\,\tau)^2}\int dx_0^i\, dS_0^a\left[\frac{\det(\overline{\partial}\delta^{AB}+\frac{i}{2\pi}\widehat{\mathcal{D}}^{AB})}{\det'(\overline{\partial})}\right]^{\frac{1}{2}}\ , \tag{2.20}$$

with an implicit sum over spin structures. Using the relation to the elliptic genus, one gets immediately

$$\begin{aligned}S^{\mathrm{eff}}_{1-\mathrm{loop}}(D) &\propto \int_{\mathcal{F}}\frac{d^2\tau}{(Im\,\tau)^2}\int d^2x\ \overline{\mathcal{A}}(q,\mathrm{Im}\tau,\widehat{D},0)\Big|_{\text{spinor two-form}}\\ &= -64\pi^2\int d^2x\ \epsilon_{ab}\widehat{X}_2^{ab}(\widehat{D})\\ &= -\frac{1}{3}\int d^2x\ D^\alpha\,\mathrm{Tr}(T_\alpha)\ ,\end{aligned} \tag{2.21}$$

since $X_2(F)=-\frac{1}{96\pi}\mathrm{Tr}\left(\frac{iF}{2\pi}\right)$. As the trace is over the $U(1)$ charges of the massless chiral fermions, (2.21) is precisely the Fayet-Iliopoulos term. It was computed already in [2][3], although with different methods.

* It is not completely clear how to interpret auxiliary fields in the GS-formalism; (2.18) differs from the corresponding NSR-model by a certain term. As discussed in the appendix, this term does however not contribute.

2.3 The Six Dimensional Case

For $N=1$, $d=6$, there are four Green-Schwarz fields, $S_0^{\underline{a}}$, $\underline{a} = 1...4$. One expects therefore results for two-point functions. Usually, two point amplitudes involving gravitons and gauge bosons vanish on-shell. However, we will use nowhere the equations of motion, and obtain the two-point effective action directly from the path integral. Since we are not using the field equations, we can also derive the effective action for the auxiliary fields $Z_I^{AB}, I = 1...3$ of the gauge supermultiplet† .

The in the limit $\mathrm{Im}\tau \to 0$ relevant background dependent terms are

$$\Delta S_E = -\tfrac{1}{2\pi} \int dt\, d\sigma \Big[\tfrac{1}{8}\mathrm{Im}\tau R_{ijkl}X'^i\partial X'^j\, S^a\gamma^{kl}_{ab}S^b \;+\tfrac{1}{8}\mathrm{Im}\tau F^{AB}_{ij}S^a\gamma^{ij}_{ab}S^b\lambda_A\lambda_B$$
$$+\tfrac{1}{2}\mathrm{Im}\tau Z_I^{AB}\widetilde{S}^{\dot m}(\sigma^I)_{\dot m n}\widetilde{S}^n\lambda_A\lambda_B\Big]\;. \qquad (2.22)$$

Here, $\widetilde{S}^{\dot m}, \widetilde{S}^n (n, \dot m = 1, 2)$ in the vertex operator for the auxiliary field are the Green-Schwarz fields in a complex basis, as defined in the appendix. Note that the spinor indices in the other terms run from 1 to 8. The point is that even though S^a, $a = 5\ldots8$ are not free fields, the composite object

$$R^{ij} \;\equiv\; \tfrac{1}{4}S^a\gamma^{ij}_{ab}S^b \quad , \qquad a,b = 1\ldots8\;, \quad i,j = 1\ldots4 \qquad (2.23)$$

is a well-defined conformal field; it is the current of the uncompactified, transverse Lorentz group. Of course, only $S^{\underline{a}}, \underline{a} = 1\ldots4$ have zero modes,

$$S^{\underline{a}} \;=\; \sqrt{\tfrac{i}{2\pi\mathrm{Im}\tau}}S_0^{\underline{a}} + S'^{\underline{a}}\;, \qquad (2.24)$$

so that in the $\mathrm{Im}\tau \to 0$ limit the other fields play no role. Hence

$$S^{\text{eff}}_{1-\text{loop}}(F,R,Z) \propto \int_{\mathcal{F}} \frac{d^2\tau}{(Im\,\tau)^2} \int dx_0^i\, dS_0^{\underline{a}} \left[\frac{\det(\overline{\partial}\delta^{AB} + \frac{i}{2\pi}\widehat{\mathcal{F}}^{AB} + \frac{i}{2\pi}\widehat{\mathcal{Z}}^{AB})}{\det'(\overline{\partial}\delta_{ij} + \frac{i}{2\pi}\widehat{\mathcal{R}}_{ij})}\right]^{\frac{1}{2}} \qquad (2.25)$$

† For the auxiliary fields Z_I^i of the supergravity sector, the σ-model coupling term is not quadratic in the left-moving bosons. Therefore we cannot deal with these fields in an easy way. The same applies also to the other supergravitational auxiliary fields.

(with implicit summation over spin structures), where

$$\begin{aligned}
\widehat{\mathcal{R}}_{ij} &= \tfrac{1}{2}\widehat{R}_{ij\underline{ab}}\, S_0^{\underline{a}} \wedge S_0^{\underline{b}} \\
\widehat{\mathcal{F}}^{AB} &= \tfrac{1}{2}\widehat{F}^{AB}_{\underline{ab}}\, S_0^{\underline{a}} \wedge S_0^{\underline{b}} \\
\widehat{\mathcal{Z}}^{AB} &= \tfrac{1}{2}Z_I^{AB}\,\widetilde{S}_0^{\dot{m}}(\sigma^I)_{\dot{m}n}\widetilde{S}_0^{n}
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
\widehat{R}_{ij\underline{ab}} &= \tfrac{1}{4}\gamma^{kl}_{\underline{ab}} R_{ijkl} \\
\widehat{F}^{AB}_{\underline{ab}} &= -\tfrac{i}{4}\gamma^{ij}_{\underline{ab}} F^{\alpha}_{ij}(T_\alpha)^{AB} \\
Z_I^{AB} &= -iZ_I^{\alpha}(T_\alpha)^{AB} \ .
\end{aligned} \tag{2.27}$$

Using again the relation to the elliptic chiral index, it follows

$$\begin{aligned}
S^{\text{eff}}_{1-\text{loop}}(F,R,Z) \ &\propto \ \int_{\mathcal{F}} \frac{d^2\tau}{(Im\,\tau)^2} \int dx_0^i\, dS_0^{\underline{a}}\ \overline{\mathcal{A}}(q, \text{Im}\tau, \widehat{\mathcal{F}} + \widehat{\mathcal{Z}}, \widehat{\mathcal{R}}) \\
&= \ -64\pi^2 \int d^4x\ \ X_4(\widehat{F} + \widehat{Z}, \widehat{R})\Big|_{\text{spinor 4-form}} \ .
\end{aligned} \tag{2.28}$$

From the definition of $\mathcal{A}(q,F,R)$ one can derive

$$X_4(F,R) \ = \ -\frac{1}{192\pi}\left\{ \text{Tr}\left(\frac{iF}{2\pi}\right)^2 + 6\,\text{Tr}\left(\frac{iR}{2\pi}\right)^2 \right\} \ , \tag{2.29}$$

where the traces run over the representations of the massless chiral fermions. Together with

$$\begin{aligned}
\left(\tfrac{1}{4}\gamma^{ij}_{\underline{ab}}\right)\left(\tfrac{1}{4}\gamma^{kl}_{\underline{cd}}\right)\epsilon^{\underline{abcd}} \ &= \ t^{ijkl} \\
&\equiv -\tfrac{1}{2}\epsilon^{ijkl} - \tfrac{1}{2}\left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}\right)
\end{aligned} \tag{2.30}$$

one finally arrives at*

$$S^{\text{eff}}_{1-\text{loop}}(F,R,Z) \ \propto \ \int d^4x\ \left(-\frac{1}{12\pi}\right)\left\{\tfrac{1}{2}\,\text{Tr}\,Z^2 - \tfrac{1}{4}\,\text{Tr}\,F_{ij}F^{ij} - \tfrac{3}{2}\,\text{Tr}\,R_{ij}R^{ij}\right\} \ . \tag{2.31}$$

The first two terms represent the one-loop correction to the bosonic kinetic terms of the $d=6$ gauge supermultiplet; they describe a finite gauge coupling constant renormalization. We expect that upon coupling fermion background to the σ-model, the full supersymmetric kinetic terms appear. Similarly, we expect that also all supersymmetric partners of the gravitational term above appear upon coupling the corresponding fields to the σ-model.

---

* Like in the ten dimensional case, the ϵ-tensor is dropped as it does not survive covariantization to six dimensions.

As (2.31) is obtained by an index computation, is is natural to assume that it is not renormalized in higher loop orders. In particular this would mean that the gauge coupling constant does not receive contributions beyond one loop. This is in striking contrast to $N=1$, $d=6$ point particle super Yang-Mills theory which is not even renormalizable. Like in the ten dimensional case, besides the physical massless fermions also an infinite tower of unphysical ($m_L^2 \neq 0$) excitations contributes in the modular integral (2.28). This has no description in terms of particle theory.

In (2.31), the traces run over the representations of the massless chiral fermions, counting right- (left-) handed spinors with plus (minus) signs. Since the gravitinos are right-handed and the matter fermions left-handed, the gauge part of (2.31) can be expressed as

$$\frac{1}{12\pi}\Big(C_{adj}-C_M\Big)\int d^4x \;\; \Big\{\tfrac{1}{2}\,Z_I^\alpha Z_I^\alpha - \tfrac{1}{4}\,F_{ij}^\alpha F^{ij\alpha}\Big\}\,, \tag{2.32}$$

where C_{adj} and C_M are the corresponding gauge and matter second order Casimir invariants. Thus, if $C_{adj} = C_M$ and the above assumption holds, there are no quantum corrections to the gauge coupling at all.

One could of course obtain the same result by explicit loop calculations. As two-point amplitudes vanish on-shell, one would have to consider three particle scattering. Such kind of computation has recently been performed in [19] (for $d=4$). It was shown that the only contributions to $\mathrm{Tr}\, F^2$ come from boundary configurations where two vertex operators collide. The corresponding poles cancel then zeros in kinematical prefactors. Thus, one effectively computes two-point amplitudes.

In our case, the two-point boundary contributions are holomorphic due to the saturation of $S_0^{\bar{a}}$ zero modes, and therefore related to the elliptic index. This is analogous to two-point scalar scattering in $d=4$, where the boundary contribution is given by the D-term [2][3].

2.4 β-Functions in Four Dimensional $N=2$ Supersymmetric Theories

It is interesting to check to what extent the above results are relevant also for lower dimensional, torus-compactified theories. One certainly expects that some of the special features of index-related amplitudes persist after compactification. As we will see, the situation is slightly different in the lower dimensional theories, because the structure of the bosonic zero modes changes upon compactification.

As an example, we consider gauge coupling constant renormalization in $d=4$, $N=2$ superstring theories, which are torus compactifications of $d=6$, $N=1$ theories. As there are four zero modes $S^{\underline{a}}$, the simplest method is to directly compute the two-point effective action in a way similar to the $d=6$ calculation above.

We restrict ourselves to those gauge groups which are present already in the six dimensional theory. Generalization to gauge fields generated in the compactification process is straightforward. The background couplings we consider are thus the same as in the $d=6$ computation in the foregoing section,

$$\Delta S_E = -\tfrac{1}{2\pi}\int dt\, d\sigma\, \Big[\tfrac{1}{8}\mathrm{Im}\tau\, F^{AB}_{ij} S^a \gamma^{ij}_{ab} S^b \lambda_A\lambda_B + \tfrac{1}{2}\mathrm{Im}\tau\, Z^{AB}_I \widetilde{S}^{\dot m}(\sigma^I)_{\dot m n}\widetilde{S}^n \lambda_A\lambda_B\Big]\ . \tag{2.33}$$

In (2.33), Z^{AB}_I denotes the auxiliary field triplet of $N=2$, $d=4$ super Yang-Mills theory. For convenience we chose a six dimensional notation: F_{ij}, $(i,j=1\ldots4)$ are six dimensional (light-cone) gauge field strengths, which will be decomposed later into four dimensional gauge field strengths plus two scalars A and B. Evaluating the path integral, we get

$$\begin{aligned}\int \mathcal{D}X\mathcal{D}S\mathcal{D}\lambda\ e^{-S^{(2)}_E} &= \mathrm{Im}\tau\, \mathcal{L}_{2;2}(\tau,\overline{\tau})\int dx^{\underline{i}}_0\, dS^{\underline{a}}_0 \left[\frac{\det(\overline{\partial}\delta^{AB} + \frac{i}{2\pi}\widehat{\mathcal{F}}^{AB} + \frac{i}{2\pi}\widehat{\mathcal{Z}}^{AB})}{\det'(\overline{\partial})}\right]^{\frac{1}{2}}\\ &= \mathrm{Im}\tau\, \mathcal{L}_{2;2}(\tau,\overline{\tau})\int dx^{\underline{i}}_0\, dS^{\underline{a}}_0\ \overline{\mathcal{A}}(q,\mathrm{Im}\tau,\widehat{\mathcal{F}}+\widehat{\mathcal{Z}},0)\ ,\end{aligned} \tag{2.34}$$

where $\underline{i}=1,2$, $\underline{a}=1\ldots4$ and where $\overline{\mathcal{A}}(q,\mathrm{Im}\tau,F,R)$ is the elliptic genus of the *six dimensional* theory. The lattice sum in the modular invariant correction factor

$$\mathrm{Im}\tau\ \mathcal{L}_{2;2}(\tau,\overline{\tau}) = \mathrm{Im}\tau \sum_{(p_L;p_R)\in\Gamma_{2;2}} q^{\frac{1}{2}p_L^2}\, \overline{q}^{\frac{1}{2}p_R^2} \tag{2.35}$$

arises from the zero mode integration of the compactified bosons; $\Gamma_{2;2}$ denotes the even self-dual lorentzian lattice on which the six dimensional theory is compactified. The factor of $\mathrm{Im}\tau$ appears because the $\mathrm{Im}\tau$'s arising from the analogue of (2.4) do not cancel. Obviously, (2.34) is not any more in our sense holomorphic. Nevertheless, our result will still be related to an index.

In (2.34), the $S_0^{\underline{a}}$ integration picks out the modular invariant spinor four-form part. Thus, the effective action is

$$S^{\text{eff}}_{1-\text{loop}}(F,Z) \ \propto \ \int \frac{d^2\tau}{\text{Im}\tau}\, \mathcal{L}_{2;2}(\tau,\overline{\tau}) \int d^2x \ \overline{\mathcal{A}}(q,\text{Im}\tau,\widehat{F}+\widehat{Z},0)\Big|_{\text{spinor 4-form}} \quad . \tag{2.36}$$

The presence of a single factor $\text{Im}\tau^{-1}$ signals a possible logarithmic divergence of the modular integral.

Since the left-moving ground state does not carry gauge charge, it follows that the four-form part of $\overline{\mathcal{A}}(q,\text{Im}\tau,F,0)$ does not have a pole q^{-1}. Therefore the leading, logarithmic divergent contribution is associated with the $\text{Im}\tau$ independent part of

$$\mathcal{L}_{2;2}(\tau,\overline{\tau}) \ = \sum_{(p_L;p_R)\in\Gamma_{2;2}} e^{-\pi\text{Im}\tau(p_L^2+p_R^2)}\ e^{i\pi\text{Re}\tau(p_L^2-p_R^2)} \ , \tag{2.37}$$

that is, with $p_L^2 = p_R^2 = 0$. In other words, the lattice partition function does not play any role for the leading contribution. Introducing an infrared regulator μ, we find that

$$\lim_{\mu\to 0} \int_1^{(\mu^2\alpha')^{-1}} \frac{d\text{Im}\tau}{\text{Im}\tau} \int_{-1/2}^{1/2} d\text{Re}\tau \ \int d^2x \ \overline{\mathcal{A}}(q,\text{Im}\tau,\widehat{F}+\widehat{Z},0)\Big|_{\text{spinor 4-form}} \tag{2.38}$$

is the same as the divergent part of (2.36) above* . As the integration region is rectangular, the $\text{Re}\tau$ integral projects onto the physical massless states satisfying $m_L^2 = m_R^2 = 0$, that is, on the q^0 part of the four-form part of $\mathcal{A}(q,F,0)$. It is given by

$$\mathcal{A}(q,F,0)\Big|_{\substack{\text{4-form}\\ \text{coeff. of } q^0}} \ = \ -96\pi\, X_4(F,0) \ , \tag{2.39}$$

with X_4 defined in (2.29). We now decompose the six dimensional gauge fields into four dimensional gauge fields plus scalars A and B. Using (2.39), the divergent correction to the

* The finite parts of (2.36) and (2.38) are of course not the same; they differ e.g. by the integral over the non-rectangular part of the fundamental region $\{\tau \mid |\tau| \geq 1, -\frac{1}{2} \leq \text{Re}\tau \leq \frac{1}{2}, 0 < \text{Im}\tau \leq 1\}$. From this region, there are contributions of infinitely many unphysical states with $m_L^2 \neq 0$. The finite parts also differ by contributions from $\mathcal{L}_{2;2}(\tau,\overline{\tau})$.

(bosonic) $d=4$ $N=2$ gauge supermultiplet kinetic terms looks then

$$S^{\text{eff}\,(div)}_{1-\text{loop}}(F,A,B,Z) \;=\; -a\frac{1}{8\pi^2}\ln(\mu^2\alpha') \times \int d^2x \;\; \text{Tr}\Big\{ -\tfrac{1}{4}(F_{ij})^2 - \tfrac{1}{2}(D_iA)^2 - \tfrac{1}{2}(D_iB)^2 - \tfrac{1}{2}([A,B])^2 + \tfrac{1}{2}(Z_I)^2 \Big\} \,. \tag{2.40}$$

We have included some constant factor a, as we did not work out the absolute normalization. The kind of the trace above is determined by $X_4(F,0)$ which itself is defined in terms of the six dimensional chiral anomaly. It is identical to the trace occurring in the six dimensional computation above, counting $d=6$ gauginos and matter fermions with opposite signs. Upon compactification, gauge and matter fermions in $d=6$ are mapped to gauge and matter fermions in $d=4$, respectively. It follows that the corrected gauge coupling constant in four dimensions is given by

$$\frac{1}{g^2_{\text{eff}}(\mu^2\alpha')} \;=\; \frac{1}{g^2}\Big\{1+g^2b+a\frac{g^2}{8\pi^2}\Big(C_{adj}-C_M\Big)\ln(\mu^2\alpha')\Big\} \;, \tag{2.41}$$

where b denotes the finite corrections. Viewing α' as an inverse squared ultraviolet cutoff, the β-function is easily inferred from the divergent part:

$$\beta(g) \;=\; -2\alpha'\frac{\partial g_{\text{eff}}(\mu^2\alpha')}{\partial\alpha'} \;=\; a\frac{g^3}{8\pi^2}\Big(C_{adj}-C_M\Big) \,. \tag{2.42}$$

It is identical to the result obtained in $d=4$, $N=2$ super Yang-Mills particle theory if $a=-1$. This is consistent with the belief that string theory should always give the same answer as particle theory, whenever one computes a quantity which is meaningful in particle theory. In particular, in our string calculation only the physical massless spectrum ($m_L^2 = m_R^2 = 0$) contributes† .

In particle theory, it was shown that (2.42) is not renormalized beyond one loop [20]. This perfectly agrees with our conjecture, that index-related superstring amplitudes should not get renormalized in higher orders. From our viewpoint, higher loop corrections would be zero due to the vanishing of the six dimensional index factor in (2.36).

† In contrast, in the previous chapters we computed quantities which are not meaningful (since stronger than logarithmic divergent) in particle theory. There is no reason why only massless, physical states should contribute, as the decoupling theorem is not valid in these cases. Indeed, we found that infinitely many unphysical states ($m_L^2 \neq 0$) contribute in the ten and six dimensional cases. The situation is similar for the finite correction b above, which is also not a meaningful quantity.

3. Conclusions

We have shown that a certain class of supersymmetric string scattering amplitudes or effective actions is intimately related to the chiral anomaly structure. Specifically, in a $d = 2n + 2$ (n=1,2,4) dimensional $N = 1$ supersymmetric theory, the (light-cone) bosonic n-point effective action in question has the form

$$S^{\text{eff}}_{1-\text{loop}} = \int d^{2n}x \ \widehat{X}_{2n} . \tag{3.1}$$

The spinor form $\widehat{X}_{2n}$ is closely related to the $2n$-form X_{2n} which multiplies $\operatorname{Tr} F^2 - \operatorname{Tr} R^2$ in the expression for the anomaly of the theory. The close connection to the anomaly is due to the fact that the computation leading to (3.1) is essentially a triality rotated version of the one leading to the index of the Dirac-Ramond operator. Assuming the Adler Bardeen theorem holds, it is conceivable that (3.1) is not renormalized in higher loop orders. Our approach is useful also for torus compactified theories; for example, we derived the well-known result for the β-function in $d=4$, $N=2$ theories from the chiral anomaly in $d=6$. We expect analogous results for four-point scattering in $d=4$, $N=4$ theories.

In the appendix, we investigated the map between parity violating and parity conserving sectors in supersymmetric theories, from the NSR point of view. It can be expressed by the following Ward identity:

$$\Big\langle \prod_{i=1}^{n} J_i \Big\rangle_{\substack{\text{PP}\\ \text{sector}}} = \tfrac{2}{3} \Big\langle \prod_{i=1}^{n} \tfrac{1}{2}\mathcal{J}_i \Big\rangle_{\substack{\text{even spin}\\ \text{structures}}} . \tag{3.2}$$

Here, J_i are the number currents of the world-sheet fermions with space-time indices, ψ^i ($i = 1...2n$). At the right hand side, $\mathcal{J}_i$ are $U(1)$ Kac-Moody currents, which belong to the various world-sheet $N = 2$ superconformal algebras defining the supercharges. We found that (3.2) implies a wealth of Kac-Moody character identities that can be related to the representation theory of exceptional groups.

We remark that it might be possible to prove absence of supersymmetry anomalies by use of the techniques developed in this paper, in a way similar to the treatment of chiral anomalies [9].

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APPENDIX

Supersymmetric Character Ward Identities

The relation between parity violating and parity conserving amplitudes can easily be understood in terms of the triality map (2.17): periodic Green-Schwarz fields S^a are mapped to periodic NSR fields ψ^i, that is, to the odd spin structure sector in the NSR formalism. It is interesting to investigate how this relation comes about in the NSR formalism. Here, the parity preserving amplitudes reside in the even spin structure sectors. It follows that in supersymmetric theories there has to be a relation between even and odd spin structure sectors:

$$\int_{\substack{\text{PP}-\\\text{sector}}} d\psi\, e^{-S_E(\psi)} \propto \int dS\, e^{-S_E(S)} \equiv \sum_{\substack{\text{even spin}\\\text{structures}}} \int d\psi\, e^{-S_E(\psi)}\,. \qquad \text{(A1)}$$

Here, even and odd spin structures refer to the boundary conditions of the (free) space-time fermions ψ^i on the torus, and integrations over all other fields as well as appropriate operator insertions are understood.

A1 The Ten Dimensional Case

We start with the ten dimensional case (this includes of course torus compactifications). Saturating the zero modes, one gets

$$\int_{\substack{\text{PP}-\\\text{sector}}} d\psi\, e^{-S_E(\psi)}\, \psi_0^1\psi_0^2\ldots\psi_0^8 \propto \sum_{\substack{\text{even spin}\\\text{structures}}} \int d\psi\, e^{-S_E(\psi)}\, S_0^1 S_0^2 \ldots S_0^8\,. \qquad \text{(A2)}$$

In order to exhibit the relation between ψ^i and S^a, it is convenient to bosonize these fields:

$$\begin{aligned} :i\psi^1\psi^2: &= i\partial\phi_1 \equiv J_1 \\ :i\psi^3\psi^4: &= i\partial\phi_2 \equiv J_2 \\ :i\psi^5\psi^6: &= i\partial\phi_3 \equiv J_3 \\ :i\psi^7\psi^8: &= i\partial\phi_4 \equiv J_4\,, \end{aligned} \qquad \text{(A3)}$$

where $\phi_1 \dots \phi_4$ are free bosons. Then

$$
\begin{aligned}
:iS^1S^2: &= \tfrac{1}{2}\Big(J_1 + J_2 + J_3 + J_4\Big) \equiv \tfrac{1}{2}\mathcal{J}_1 \\
:iS^3S^4: &= \tfrac{1}{2}\Big(J_1 + J_2 - J_3 - J_4\Big) \equiv \tfrac{1}{2}\mathcal{J}_2 \\
:iS^5S^6: &= \tfrac{1}{2}\Big(J_1 - J_2 + J_3 - J_4\Big) \equiv \tfrac{1}{2}\mathcal{J}_3 \\
:iS^7S^8: &= \tfrac{1}{2}\Big(J_1 - J_2 - J_3 + J_4\Big) \equiv \tfrac{1}{2}\mathcal{J}_4 \ ,
\end{aligned}
\tag{A4}
$$

such that $\mathcal{J}_i = \sum_{j=1}^{4}(v_i)_j J_j$. One can therefore rewrite the relation (A2) as

$$
\Big\langle J_1 J_2 J_3 J_4 \Big\rangle_{\substack{\text{PP}\\ \text{sector}}} = \tfrac{1}{16}c \, \Big\langle \mathcal{J}_1\mathcal{J}_2\mathcal{J}_3\mathcal{J}_4 \Big\rangle_{\substack{\text{even spin}\\ \text{structures}}} \ , \tag{A5}
$$

where $J_i, \mathcal{J}_i$ stand either for the currents themselves or only for their zero modes. The distinction is irrelevant, since because of the Berezin integrations, only ψ and S zero modes contribute. The constant c is to be determined below.

Eq. (A5) represents a Ward identity, because it relates different scattering amplitudes through a symmetry principle. It is almost trivial from the Green-Schwarz point of view, but non-trivial in the NSR formalism. To prove it in the NSR-formalism, we employ operator formalism to rewrite the LHS of (A5) as ($F_i \equiv (J_i)_0$)

$$
\begin{aligned}
\Big\langle J_1 J_2 J_3 J_4 \Big\rangle_{\substack{\text{PP}\\ \text{sector}}} &= \frac{\partial}{\partial \nu_1} \cdots \frac{\partial}{\partial \nu_4} \tfrac{1}{2} \operatorname*{Tr}_{R} \Big[(-1)^F \, e^{v \cdot F} \Big]_{\nu_i = 0} \\
&= \tfrac{1}{2} \frac{1}{\eta^4(\tau)} \prod_{i=1}^{4} \Big\{ \frac{\partial}{\partial \nu_i} \vartheta_1\big(\tfrac{\nu_i}{2\pi}|\tau\big) \Big\}_{\nu_i = 0} .
\end{aligned}
$$

By the Riemann type identity

$$
\Big(\prod_{i=1}^{4} \frac{\partial}{\partial \nu_i}\Big) \Big\{ \tfrac{3}{2} \prod_{i=1}^{4} \vartheta_1(\nu_i|\tau) - \prod_{i=1}^{4} \vartheta_3(\nu_i'|\tau) + \prod_{i=1}^{4} \vartheta_4(\nu_i'|\tau) + \prod_{i=1}^{4} \vartheta_2(\nu_i'|\tau) \Big\}_{\nu_i = 0} = 0
$$

with $\nu_i' = \frac{1}{2} \sum_{j=1}^{4} (v_i)_j \nu_j$, this is the same as

$$
\begin{aligned}
&\tfrac{1}{3} \Big(\prod_{i=1}^{4} \frac{\partial}{\partial \nu_i} \Big) \Big\{ \operatorname*{Tr}_{NS} \Big[(1 - (-1)^F) e^{\nu' \cdot F} \Big] - \operatorname*{Tr}_{R} \Big[e^{\nu' \cdot F} \Big] \Big\}_{\nu_i = 0} \\
&= \tfrac{1}{24} \Big\langle \mathcal{J}_1\mathcal{J}_2\mathcal{J}_3\mathcal{J}_4 \Big\rangle_{\substack{\text{even spin}\\ \text{structures}}} .
\end{aligned}
$$

Thus it is a theta function identity which is responsible for the Ward identity ($c = \frac{2}{3}$).

In lower dimensions analogous identities have to hold. To derive these identities, it is convenient to switch to a different, bosonic language, where sums over spin structures are replaced by lattice sums. This is because in this formulation, all information about supersymmetry is encoded in weight lattices of the exceptional groups E_6, E_7 and E_8 [8] [21] (this applies to any kind of supersymmetric string theory). We can thus expect supersymmetry Ward identities to emerge as relations between certain Kac-Moody characters associated with exceptional groups.

There are certain rules how to extract the physical (and also the auxiliary field) spectrum from the exceptional groups [8]. The rules are as follows: decompose $E_n \to \Gamma_{n-4} \times D_4$, where Γ_{n-4} is some $n-4$ dimensional lattice associated with the physical states, and D_4 describes unphysical degrees of freedom. Accordingly, decompose all vectors (or better, conjugacy classes) in the same way. The physical light-cone states are then precisely those which are associated with the (v) or (s) conjugacy classes of D_4 (auxiliary fields with (0)). The partition function is thus characterized by a sum of characters of Γ_{n-4}, with positive signs for states in the NS-sector ((v) of D_4) and negative signs for states in the R-sector ((s) of D_4).

To make this more clear, we reformulate the ten dimensional case in this language. This case with maximal supersymmetry corresponds to E_8. According to our rules, we write $E_8 \to D_4 \times D_4$, with conjugacy classes $(0) \to ((0),(0)) \oplus ((v),(v)) \oplus ((s),(s)) \oplus ((c),(c))$. The physical states are those in (v) and (s) of the first D_4 above. The partition function in terms of D_4 Kac-Moody characters is therefore

$$Ch_v^{\widehat{D}_4}(0\,|\tau) - Ch_s^{\widehat{D}_4}(0\,|\tau) = 0 \ .$$

The $\widehat{D}_n$ characters are given by

$$\begin{aligned}
Ch_0^{\widehat{D}_n}(\nu_i\,|\tau) &= \tfrac{1}{2}\frac{1}{\eta^n(\tau)}\Big\{\prod_{i=1}^{n}\vartheta_3(\nu_i|\tau) + \prod_{i=1}^{n}\vartheta_4(\nu_i|\tau)\Big\}\\
Ch_v^{\widehat{D}_n}(\nu_i\,|\tau) &= \tfrac{1}{2}\frac{1}{\eta^n(\tau)}\Big\{\prod_{i=1}^{n}\vartheta_3(\nu_i|\tau) - \prod_{i=1}^{n}\vartheta_4(\nu_i|\tau)\Big\}\\
Ch_s^{\widehat{D}_n}(\nu_i\,|\tau) &= \tfrac{1}{2}\frac{1}{\eta^n(\tau)}\Big\{\prod_{i=1}^{n}\vartheta_2(\nu_i|\tau) + \prod_{i=1}^{n}\tfrac{1}{i}\vartheta_1(\nu_i|\tau)\Big\}\\
Ch_c^{\widehat{D}_n}(\nu_i\,|\tau) &= \tfrac{1}{2}\frac{1}{\eta^n(\tau)}\Big\{\prod_{i=1}^{n}\vartheta_2(\nu_i|\tau) - \prod_{i=1}^{n}\tfrac{1}{i}\vartheta_1(\nu_i|\tau)\Big\}\ .
\end{aligned}$$

In this formulation, the Ward identity (A5) is equivalent to the following identity between $\widehat{D}_4$ Kac-Moody characters:

$$\Big(\prod_{i=1}^{4}\frac{\partial}{\partial\nu_i}\Big)\ Ch_s^{\widehat{D}_4}(\nu_i|\tau)_{\nu_i=0} \;=\; \tfrac{2}{3}\Big(\prod_{i=1}^{4}\frac{\partial}{\partial\nu_i}\Big)\Big\{Ch_v^{\widehat{D}_4}(\nu_i'|\tau)-Ch_s^{\widehat{D}_4}(\nu_i'|\tau)\Big\}_{\nu_i=0}. \qquad \text{(A6)}$$

For the lower dimensional cases, we can now proceed in a similar way.

A2 The Four Dimensional Case

For $N=1$, $d=1$, we have only two S^a. The two space-time fermions ψ^i are free and can be bosonized as in (A3). However, as one can see from (A4), the S^a have also components in the internal, compact sector of the theory. The crucial point is that *this part* of the internal sector can always [22] be represented in terms of a free boson H, so that

$$:iS^1S^2: \;=\; \tfrac{1}{2}\Big(J_1+J^{(int)}\Big) \;\equiv\; \tfrac{1}{2}\mathcal{J}_1$$

$$J^{(int)} \;=\; i\sqrt{3}\,\partial H\ .$$

$\mathcal{J}_1$ is precisely the $\widehat{U(1)}$ current belonging to the global $N=2$ world-sheet supersymmetry [23] (similarly, the currents $\mathcal{J}_i=\sum(v_i)_jJ_j$ above and below belong to several copies of $N=2$ algebras). Saturating the zero modes in (A1), the supersymmetric identity then takes the form

$$\Big\langle\ J_1\ \Big\rangle_{\substack{\text{PP}\\ \text{sector}}} \;=\; \tfrac{1}{2}c\,\Big\langle\ \mathcal{J}_1\ \Big\rangle_{\substack{\text{even spin}\\ \text{structures}}}\ . \qquad \text{(A7)}$$

We like to identify the character identities that are responsible for that equation, and to determine c.

For the case at hand, the relevant lattice is E_6. Upon decomposition $E_6\to[U(1)D_1]\times D_4$, we see that besides the boson belonging to the transverse Lorentz group D_1, there exists another free boson associated with the $U(1)$ factor. It is of course identical to the boson H above. This observation is quite powerful, as it allows to investigate the relevant part of the partition function, no matter how complicated the full partition function (e.g., corresponding to a Calabi-Yau compactification) is. The relevant part is simply given by certain level one Kac-Moody characters of $[U(1)D_1]$.

We need of course to know what characters of $[U(1)D_1]$ actually do occur. This information is easily inferred from the weight lattice of E_6. There are three conjugacy classes of E_6, $(0),(1)$ and $(\bar{1})$ (we do not need to discuss $(\bar{1})$ separately); (1) is the class to which the $\underline{27}$ belongs. Applying our selection rules, we obtain the following conjugacy classes of $[U(1)D_1]$:

$$(0) \rightarrow \begin{cases} ([0],(v)) \\ ([1],(0)) \\ ([\frac{1}{2}],(s)) \\ ([-\frac{1}{2}],(c)) \end{cases}$$

$$(1) \rightarrow \begin{cases} ([\frac{1}{3}],(0)) \\ ([-\frac{1}{6}],(s)) \\ ([-\frac{2}{3}],(v)) \\ ([\frac{5}{6}],(c)) \ . \end{cases}$$

Here, $[q]$ means $U(1)$ charges $\sqrt{3}(q+2k), k \in \mathbf{Z}$. The classes (0) and (1) belong to different sectors of the theory; to (0) belong in particular all gauge particles, whereas (1) defines the matter sector. For example, graviton and gauge bosons belong to $(0,(v))$, the gravitino belongs to $(\frac{\sqrt{3}}{2},(s))$ and the (massive) holomorphic 3-form field to $(\sqrt{3},(0))$. The massless matter fields sit in $(\frac{\sqrt{3}}{3},(0))$ and $(-\frac{\sqrt{3}}{6},(s))$.

The characters associated with (0) and (1) do not factor out of the whole partition function, but rather multiply different (gauge and matter) sectors. That means, the Ward identity (A7) has to hold in the two sectors separately.

Defining now

$$Ch_q^{\widehat{U(1)}}(\nu\,|\tau) = \frac{1}{\eta(\tau)} \sum_{m\in\sqrt{3}(2\mathbf{Z}+q)} q^{\frac{1}{2}m^2} e^{2\pi i\sqrt{3}m\nu} \ ,$$

one can build $[U(1)D_1]$ characters in the following way:

$$Ch_{[q,*]}(\nu_1,\nu_2\,|\tau) \ = \ Ch_q^{\widehat{U(1)}}(\nu_1\,|\tau)\cdot Ch_*^{\widehat{D}_1}(\nu_2\,|\tau) \ .$$

In terms of these functions, the relevant factors of the partition function are

$$Ch_{[0,v]}(0,0\,|\tau) + Ch_{[1,0]}(0,0\,|\tau) - Ch_{[1/2,s]}(0,0\,|\tau) - Ch_{[-1/2,c]}(0,0\,|\tau) \ = \ 0 \qquad \text{(A8)}$$

in the gauge sector, and

$$Ch_{[1/3,0]}(0,0\,|\tau) + Ch_{[-2/3,v]}(0,0\,|\tau) - Ch_{[-1/6,s]}(0,0\,|\tau) - Ch_{[5/6,c]}(0,0\,|\tau) \;=\; 0 \qquad \text{(A9)}$$

in the matter sector. These identities are the essence of $N=1$ supersymmetry in four dimensional string theories. For instance, all cosmological constant loop calculations should boil down to (generalizations of) these equations. That these equations have to be true follows of course from the fact that the number of fermion matches the number of bosons at every string level. This can be proved by e.g. using triality properties of D_4 embedded in exceptional groups [21]. We checked these and all other equations below also explicitly by expanding them up to fifth order; in this way we determined the coefficients c.

Highly non-trivial are character-valued generalizations of (A8) and (A9), as these equations are not valid for $\nu_1, \nu_2 \neq 0$. For instance, the Ward identity (A7) is equivalent to the derivative character identities

$$\begin{aligned}&\frac{\partial}{\partial\nu_2}\Big\{Ch_{[-1/6,s]}(0,\nu_2\,|\tau) + Ch_{[5/6,c]}(0,\nu_2\,|\tau)\Big\}_{\nu_2=0}\\ &= \tfrac{1}{3}\frac{\partial}{\partial\nu_1}\Big\{Ch_{[1/3,0]}(\nu_1,0\,|\tau) + Ch_{[-2/3,v]}(\nu_1,0\,|\tau) - Ch_{[-1/6,s]}(\nu_1,0\,|\tau) - Ch_{[5/6,c]}(\nu_1,0\,|\tau)\Big\}_{\nu_1=0}\end{aligned} \qquad \text{(A10)}$$

in the matter sector, and

$$\begin{aligned}&\frac{\partial}{\partial\nu_2}\Big\{Ch_{[1/2,s]}(0,\nu_2\,|\tau) + Ch_{[-1/2,c]}(0,\nu_2\,|\tau)\Big\}_{\nu_2=0} \;=\; 0\\ &\frac{\partial}{\partial\nu_1}\Big\{Ch_{[0,v]}(\nu_1,0\,|\tau) + Ch_{[1,0]}(\nu_1,0\,|\tau) - Ch_{[1/2,s]}(\nu_1,0\,|\tau) - Ch_{[-1/2,c]}(\nu_1,0\,|\tau)\Big\}_{\nu_1=0} \;=\; 0\end{aligned} \qquad \text{(A11)}$$

in the gauge sector, for $c=\frac{2}{3}$. These equations are the $d=4$ analogues of (A6). (A11) tells that there are no contributions from the gauge sector; this is consistent with the fact that in $d=4$ gauge supermultiplets do not contribute to anomalies. Equation (A10) is not precisely the same as (A7), but rather an expression of

$$\Big\langle\, J_1 \,\Big\rangle_{\substack{\text{PP}\\ \text{sector}}} \;=\; \tfrac{1}{3}\Big\langle\, J^{(int)} \,\Big\rangle_{\substack{\text{even spin}\\ \text{structures}}} .$$

This equation has already has been discussed [3] in the context of the D-term calculation. In fact, $J^{(int)} \equiv i\sqrt{3}\,\partial H$ is the (right-moving part of the) vertex operator of the D-field. In our Green-Schwarz computation above, $\langle \mathcal{J}_1 = J_1 + J^{(int)} \rangle_{\substack{\text{even spin}\\ \text{structures}}}$ appears instead. This is however equivalent to the above expression, because

$$\Big\langle\, J_1 \,\Big\rangle_{\substack{\text{even spin}\\ \text{structures}}} \;\equiv\; 0 \;.$$

A3 The Six Dimensional Case

The $N=1$, $d=6$ case can be dealt with in an analogous way (this includes also $N=2$, $d=4$). The four free fermions ψ^i can be bosonized as in (A3). Like in the previous case, part of the internal compactified sector can be represented by a free boson H, so that

$$\begin{aligned}
:iS^1S^2: \;&=\; \tfrac{1}{2}\Big(J_1 + J_2 + J^{(int)}\Big) \;\equiv\; \tfrac{1}{2}\mathcal{J}_1 \\
:iS^3S^4: \;&=\; \tfrac{1}{2}\Big(J_1 + J_2 - J^{(int)}\Big) \;\equiv\; \tfrac{1}{2}\mathcal{J}_2 \\
J^{(int)} \;&=\; i\sqrt{2}\,\partial H \;,
\end{aligned}$$

and the supersymmetry Ward identity in question is

$$\Big\langle\, J_1 J_2 \,\Big\rangle_{\substack{\text{PP}\\ \text{sector}}} \;=\; \tfrac{1}{4}c\,\Big\langle\, \mathcal{J}_1\mathcal{J}_2 \,\Big\rangle_{\substack{\text{even spin}\\ \text{structures}}} \;. \tag{A12}$$

The spectrum is characterized by $E_7 \to [A_1D_2] \times D_4$, where D_2 is the transverse Lorentz algebra in six dimensions (for $d=4$, one decomposes of course $D_2 \to D_1D_1$). The boson H is associated with the internal $\widehat{A}_1$ Kac-Moody algebra (which is part of a $N=4$ superconformal algebra [24]). According to our physical state selection rules, the two conjugacy classes of E_7 lead to the following list of $[A_1D_2]$ conjugacy classes:

$$(0) \;\to\; \begin{cases} ([0],(v)) \\ ([\tfrac{1}{2}],(s)) \end{cases}$$

$$(1) \;\to\; \begin{cases} ([\tfrac{1}{2}],(0)) \\ ([0],(c)) \end{cases}$$

($[0]$ and $[\frac{1}{2}]$ mean isospin I_3 even and odd, respectively). The $[A_1D_2]$ Kac-Moody characters are defined by

$$\mathcal{C}h_{[I_3,*]}(\nu_1,\nu_2,\nu_3\,|\tau) \;=\; \mathcal{C}h^{\widehat{A}_1}_{I_3}(\nu_1\,|\tau)\cdot \mathcal{C}h^{\widehat{D}_2}_{*}(\nu_2,\nu_3\,|\tau)$$

with

$$\mathcal{C}h^{\widehat{A}_1}_{I_3}(\nu\,|\tau) \;=\; \frac{1}{\eta(\tau)}\sum_{m\in\sqrt{2}(\mathbf{Z}+I_3)} q^{\frac{1}{2}m^2}e^{2\pi i\sqrt{2}m\nu}\;.$$

The vanishing of the partition function is then due to

$$\mathcal{C}h_{[0,v]}(0,0,0\,|\tau)-\mathcal{C}h_{[1/2,s]}(0,0,0\,|\tau) \;=\; 0$$

$$\mathcal{C}h_{[1/2,0]}(0,0,0\,|\tau)-\mathcal{C}h_{[0,c]}(0,0,0\,|\tau) \;=\; 0$$

in the gauge and matter sectors, respectively. These identities and their character valued generalizations are the essence of $N=1$ ($d=6$) and $N=2$ ($d=4$) supersymmetry in string theories. For (A12), the following derivative identities are relevant (they imply $c=\frac{2}{3}$):

$$\frac{\partial^2}{\partial\nu_1\partial\nu_2}\mathcal{C}h_{[1/2,s]}(0,\nu_1,\nu_2\,|\tau)_{\nu_1=0} \;=\; \tfrac{1}{4}\frac{\partial^2}{\partial\nu_1\partial\nu_2}\Big\{\mathcal{C}h_{[0,v]}(\nu_1-\nu_2,\nu_1+\nu_2,\nu_1+\nu_2\,|\tau) -\mathcal{C}h_{[1/2,s]}(\nu_1-\nu_2,\nu_1+\nu_2,\nu_1+\nu_2\,|\tau)\Big\}_{\nu_1=0}$$

$$\frac{\partial^2}{\partial\nu_1\partial\nu_2}\mathcal{C}h_{[0,c]}(0,\nu_1,\nu_2\,|\tau)_{\nu_1=0} \;=\; \tfrac{1}{4}\frac{\partial^2}{\partial\nu_1\partial\nu_2}\Big\{\mathcal{C}h_{[1/2,0]}(\nu_1-\nu_2,\nu_1+\nu_2,\nu_1+\nu_2\,|\tau) -\mathcal{C}h_{[0,c]}(\nu_1-\nu_2,\nu_1+\nu_2,\nu_1+\nu_2\,|\tau)\Big\}_{\nu_1=0}$$

In this case, both gauge and matter sectors contribute to the Ward identity. Furthermore, via similar identities one can derive

$$\Big\langle J_1J_2\Big\rangle_{\substack{\text{PP}\\ \text{sector}}} \;=\; -\tfrac{1}{4}\Big\langle\,(J^{(int)})^2\,\Big\rangle_{\substack{\text{even spin}\\ \text{structures}}} \;=\; \Big\langle\,(J_1)^2\,\Big\rangle_{\substack{\text{even spin}\\ \text{structures}}} \;=\; \Big\langle\,(J_2)^2\,\Big\rangle_{\substack{\text{even spin}\\ \text{structures}}}$$

for the current zero modes.

Since relevant in sections 2.3 and 2.4, we need to discuss some aspects of auxiliary fields in $N=1$, $d=6$ (and $N=2$, $d=4$) supersymmetry. Auxiliary fields can also be inferred from the decomposition $E_7\to[A_1D_2]\times D_4$: they are associated with the (0) conjugacy class of

D_4 [25] (we consider here only the right-moving parts of their vertex operators; they still have to be combined with the appropriate left-moving, gauge or lorentz part). Thus, the scalar auxiliary fields Z_I, $I = 1 \ldots 3$ which we consider in our computation above are simply given by the roots of A_1, and their vertex operators are*

$$V^{Z_{\pm,0}} \;=\; e^{\pm i\sqrt{2}H},\, i\sqrt{2}\,\partial H \equiv J^{(int)} \;.$$

Adopting a complex basis for the Green-Schwarz fields, $\widetilde{S}^m = \frac{1}{\sqrt{2}}(S^{2m-1} + iS^{2m})$, $\widetilde{S}^{\dot{m}} = \frac{1}{\sqrt{2}}(S^{2m-1} - iS^{2m})$, $m, \dot{m} = 1, 2$ one can also write

$$V^{Z_I} \;= : \widetilde{S}^{\dot{m}}(\sigma^I)_{\dot{m}n}\widetilde{S}^n :$$

using the Pauli matrices.

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