LATTICES AND STRINGS

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Abstract:

Self-dual lattices can be used to construct simple conformal field theories for "compactified" degrees of freedom in string theory. While these are by no means the most general conformal field theories that can be used, one does obtain in this way a large number of bosonic, heterotic and type-II theories that display nearly all interesting features of more general string theories. We present a pedagogical introduction to the construction and properties of these lattice theories, indicating generalizations whenever possible. While lattice constructions of bosonic strings are merely torus compactifications, heterotic and type-II strings provide more interesting possibilities, because one can include bosonized NSR-fermions and ghosts on the lattice. This covariant lattice construction of fermionic strings is explained in detail. We include a discussion of two subjects that are not limited to lattices, but originated from lattice constructions historically, namely the relation between exceptional algebras and space–time supersymmetry in string theory, and the relation between anomaly cancellation, modular invariance and elliptic genera. Three more mathematical topics are discussed in appendices, namely properties of self-dual lattices, Weyl groups and shift vectors, and modular functions such as multi-loop ϑ-functions for Lie algebra conjugacy classes as well as for spin structures.

1. Introduction

The main goal of string theory is an ambitious one: to obtain a consistent theory of all interactions, including gravity. Just a few years ago it seemed that gravity was all it could describe, but that changed drastically with the discovery of the heterotic string [1]. Although at first it seemed that the theory would fail its first confrontation with experiment by predicting the wrong space–time dimension, that too changed very quickly after its birth. Indeed, what once may have seemed an insurmountable obstacle is now a gaping hole: not only *can* we construct string theories in four dimensions, but it is also so easy that we can make them in embarrassing abundance. Meanwhile the miracles that started the excitement have survived, although we are now so used to them that we tend to forget them: chiral string theories in four dimensions exist and their spectra have the same general features (gauge groups, representations and family replication) as the standard model. If all this is realized in a class of theories that holds the promise of offering a finite and consistent theory of quantum gravity, then there is certainly reason for excitement.

Lattices played an important role in the recent history of string theory. Although compactification on tori (flat spaces modulo lattices) was known for a long time, lattices started playing a more interesting role after Green and Schwarz [2] discovered an anomaly cancellation in ten-dimensional super-Yang–Mills theories with $E_8 \times E_8$ or SO(32) gauge groups. Several people realized that exactly these groups appeared in the classification of the 16-dimensional even self-dual lattices. These facts were put together in ref. [1], where it was shown that the self-duality of the lattice is required to satisfy a one-loop consistency condition (modular invariance).

Another key ingredient in the recent development of the subject is the use of lattice-related string operators as generators of various algebras, the most interesting of which (for our purposes at least) are Kac–Moody algebras.

This illustrates the two different aspects of the role of lattices in string theory, namely their use in the construction of modular invariant string theories, and their use in the realization of certain string symmetry algebras. Although we will discuss the latter aspect in some detail, the emphasis of this review is more on modular invariant string construction. For reviews emphasizing the algebraic aspect we refer to ref. [3].

Large numbers of bosonic and fermionic*) string theories can easily be constructed by means of lattices, but nevertheless these lattice theories form by no means the most general class of consistent string theories. Indeed, especially in the most intensively studied case of chiral fermionic strings, they give us only a set of measure zero in the space of all solutions. Therefore our main interest will not, and should not be to find the elusive "theory of everything", since the probability for finding it within the class of lattice theories (and indeed within any class that is accessible with presently available methods) is discouragingly small. It appears therefore to make more sense to aim at more mundane goals, and to try to understand general properties of string theory, rather than exhaustively study special cases.

Our present understanding of string theory suggests that general bosonic or fermionic string theories correspond to certain *conformal field theories*, with properties which we will describe in a moment. We believe that lattice theories serve as a useful model for this general case. Nearly all relevant features of string theory have their simplest realization in lattice theories, and by studying them within that subclass one can learn a lot, and often everything, about the generalization to conformal field theory. This aspect of lattice theories, which has been emphasized recently in ref. [4], will be a recurring theme in this report.

Sections 2–7 provide a fairly pedagogical explanation of lattice constructions of bosonic and fermionic strings. It is of course not possible to review all of the concepts involved, and some basic knowledge of string theory is assumed (and can be acquired by studying for example ref. [5]). Some of the principles of string construction are very briefly explained in the remainder of this introduction, which serves also as a guide to sections 2–7.

Sections 8 and 9 are respectively devoted to space–time supersymmetry and anomaly cancellation. Both these topics are good examples of how knowledge first gained from lattice theories applies to the general case. In the case of space–time supersymmetry there is not even a need for generalization: the complete supersymmetric structure of any superstring theory can be understood by studying lattice theories. In the case of anomalies the relevant expressions (called "elliptic genera" and "Kac–Moody characters" by mathematicians) were simply most easy to write down for lattice theories, and although the generalization is now understood, this is indeed where the subject started historically.

We have included three appendices containing various mathematical results, namely properties of self-dual lattices (appendix A), Weyl groups (appendix B), and modular functions (appendix C).

1.1. *Consistency conditions for string construction*

The most fundamental quantity in any string theory is a function $X^\mu(\sigma^0, \sigma^1)$ that specifies the space–time position of every point of the string (denoted by σ^1) for any value of the proper time, σ^0. Here μ labels the space–time coordinates, $\mu = 1, \ldots, d$, where d is the space–time dimension. Throughout this paper we will only consider *closed* strings, which means that $X^\mu(\sigma^0, 0) = X^\mu(\sigma^0, \pi)$ (the choice of the σ_1-interval is a matter of convention). When it propagates through space–time the string sweeps out a two-dimensional surface parametrized by σ^0 and σ^1. This surface is called the *world sheet*, in analogy to world lines for point particles. For non-interacting closed strings it has the shape of

*) By fermionic string theories we mean string theories with *space–time* fermions in their spectrum.

a tube. If strings interact they do so by splitting and joining, and as a result they sweep out world sheets that look like complicated pieces of plumbing. We will restrict ourselves to *oriented* strings, for which the world sheet has a well-defined orientation (i.e. it looks like a real piece of plumbing, and not for example like a Klein bottle.)

The modern way of describing strings is to regard them as two-dimensional field theories. One regards $X^{\mu}(\sigma_0, \sigma_1)$ as a set of d scalar fields, which are functions of a two-dimensional coordinate σ^{α}, $\alpha = 0, 1$. Once one realizes that one is dealing with two-dimensional field theories, one is more or less naturally led to the questions: Which conditions should be imposed on such a field theory for it to describe a consistent string theory? And which two-dimensional field theories satisfy these conditions?

One has to be careful in formulating these conditions, because many claims about uniqueness of string theory were based on implicitly or explicitly imposed conditions which were unnecessarily strong. So we should begin by carefully listing and examining the conditions. Our list of what we assume to be necessary conditions for d-dimensional strings is:
(1) the presence of d unconstrained scalar fields X^{μ},
(2) reparametrization invariance,
(3) conformal invariance,
(4) modular invariance,
(5) world sheet supersymmetry and superconformal invariance (for fermionic strings),
(6) unitarity for all matter fields (i.e. all fields except ghosts).

Reparametrization invariance means that the description of string propagation does not depend on how one chooses the coordinates σ_0 and σ_1 on the world sheet. This has a simple interpretation in two-dimensional field theory language: it is just two-dimensional general coordinate invariance. Conformal invariance is an additional symmetry related to rescalings of the world sheet metric. This will be discussed in more detail in the following subsection. Modular invariance means invariance under global diffeomorphisms, i.e. coordinate transformations that cannot be continuously connected to the identity. Such transformations exist as soon as the world sheet has non-trivial topology, for example when the world sheet is a torus. This will be discussed in more detail in section 3. By world sheet supersymmetry we mean local two-dimensional $N = 1$ supersymmetry. By unitarity we mean that the two-dimensional theory describing all fields except ghosts must be unitary, have positive norm and have a spectrum that is bounded from below.

We do not really know whether these are necessary and sufficient conditions for string theory. They are likely to be sufficient for consistency of superstring perturbation theory, but very little is known about non-perturbative effects. We are also not claiming that all these conditions are necessary. If one violates them one usually gets into serious trouble, but we will not make the mistake of trying to prove no-go theorems. However, we can give some examples of problems that are avoided as a consequence of some of these conditions.

Conformal invariance guarantees Lorentz invariance and general coordinate invariance in space–time as well as gauge invariance (if the theory has massless vector bosons, as is usually the case), and it also guarantees absence of space–time ghosts.

One of the nice consequences of *modular invariance* is that it ensures that chiral fermion representations coming from string theory are anomaly free. This is highly non-trivial, especially in higher dimensions, and the discovery of a rather spectacular cancellation in 10 dimensions [2] was the starting point of the current wave of interest in the subject. The precise relation with modular invariance was only understood much later, and this will be discussed in section 9. Anomalies are certainly not the only problems one can expect in the absence of modular invariance, but it is the easiest problem to identify.

World sheet supersymmetry is essential for fermionic strings, and one of the problems one has in its absence is a violation of space–time Lorentz invariance. We will discuss this in section 6.

Note that by imposing conditions on the world sheet theory, we obtain desirable properties of the resulting space–time theory.

1.2. Conformal invariance

The two-dimensional action for the bosons X^μ has the form

$$S \propto \int d^2\sigma \sqrt{(-g)} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu ,$$

where $g_{\alpha\beta}$ is the world sheet metric and g its determinant. This action is not only general coordinate invariant, but has an additional invariance under $g_{\alpha\beta} \rightarrow \Lambda(\sigma) g_{\alpha\beta}$.

Using general coordinate transformations, one can bring the world sheet metric to the form $e^{\phi(\sigma_0,\sigma_1)}\delta_{\alpha\beta}$. It is instructive to write this in terms of complex coordinates $z = \sigma_0 + i\sigma_1$, in which the metric may be written as

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 ; \qquad g_{\bar{z}z} = g_{z\bar{z}} = \tfrac{1}{2} e^{\phi} .$$

This form is preserved by *conformal transformations*

$$z \rightarrow f(z) , \qquad \bar{z} \rightarrow f(\bar{z}) ,$$

where f is an analytic function. Such a transformation does change the function ϕ, but ϕ can always be brought back to its original form by a Weyl rescaling. Hence the theory has a symmetry, which is called *conformal invariance*.

As is usually the case with symmetries in physics, conformal invariance provides us with a powerful tool for solving the theory. The technology used to exploit this symmetry is called *conformal field theory*, and some of its basic principles and applications are reviewed in section 4.

One way of using this invariance is to choose a gauge (light-cone gauge) in which one gets an action which depends only on the $d-2$ "transverse" (i.e. transforming in the vector representation of the little group $SO(d-2)$ of a massless particle) degrees of freedom X^i, and which is non-interacting and manifestly ghost-free. In the absence of conformal invariance the extra mode ϕ of the metric (the Liouville mode) becomes a dynamical field, and the world sheet theory is then no longer free. This makes life considerably more complicated. Therefore we would like to maintain conformal invariance even if other terms are added to the action.

Although conformal invariance is a symmetry of the classical action, it is afflicted by an anomaly in the quantized theory. This anomaly receives contributions from all the two-dimensional fields. In units in which a scalar contributes 1, one gets a contribution $\frac{1}{2}$ for a Majorana–Weyl fermion, -26 for the ghosts b and c of general coordinate invariance, and $+11$ for the ghosts β and γ of local supersymmetry (two-dimensional gravitons and gravitini have no physical degrees and contribute only via their ghosts). There are also other kinds of fields that one can have on the world sheet, as well as string building blocks that do not have a simple description in terms of two-dimensional actions involving bosons and fermions.

1.3. General features of oriented closed strings

String theories can be classified by their local world sheet symmetries. Since every local sypersymmetry requires ghosts for gauge-fixing, we can start with their contribution to the conformal anomaly, and then try to cancel it with matter fields. Since we always have reparametrization invariance, there is always a ghost-contribution of -26. If there is in addition one local world sheet supersymmetry, we get a total reparametrization plus superghost contribution of $-26+11=-15$. The d scalar fields X^μ add d to this, and in addition world sheet supersymmetry requires their supersymmetric partners ψ^μ to be present, contributing $d/2$. Since we assume that the remaining matter fields give only positive contributions to the conformal anomaly, it follows that $d \leq 10$. Extending the number of supersymmetries leads to maximal dimensions less than four, which is sufficient reason not to consider these cases*). It might be possible to construct string theories with other local world sheet symmetries, but we will not consider that possibility. Thus the two conformally invariant systems that we will use consist of the following fields

$$N=0: \quad b, c, X^\mu, \mathscr{C}_{26-d}\,,$$

$$N=1: \quad b, c, X^\mu, \beta, \gamma, \psi^\mu, \mathscr{C}_{15-3d/2}\,.$$

Here $\mathscr{C}_c$ denotes any unitary conformal field theory that contributes a value c to the conformal anomaly (c is also known as the central charge). Obvious candidates for these systems are c additional scalars (if d is even) or $2c$ additional Majorana–Weyl fermions, but there is no need to be specific yet.

At this point we should make a trivial but crucial remark. The traditional (pre-1985) approach to string theory was to saturate the conformal anomaly with scalars, or (for $N=1$) with boson–fermion pairs, and to relate all the scalars to space–time coordinates. In that case the $N=0$ system lives in 26 dimensions, and the $N=1$ system in 10 dimensions. This is not necessary however. One can equally well associate just a subset of the scalars with space–time coordinates, and regard the others as some sort of "internal" degrees of freedom. Then the numbers 26 and 10 are "maximal" rather than "critical" dimensions. In some cases, and in particular for bosonic strings, this means nothing more than that some dimensions are "compactified", but for fermionic strings there are more interesting possibilities, as we will see.

A special feature of conformally invariant oriented closed strings is that the modes of vibration of the string can be separated into *left-moving* ones and *right-moving* ones. Since the world sheet theory is non-interacting, and since there are no endpoints of the string that modes can bounce back from, a left-moving mode will move to the left forever. This left/right-mover separation goes so far that one can actually choose *different* conformal systems for them, as was first realized in ref. [1]. This leads in an obvious way to three kinds of closed string theory:

Bosonic	$N=0$ left-movers	$N=0$ right-movers	$d \leq 26$,
Heterotic	$N=0$ left-movers	$N=1$ right-movers	$d \leq 10$,
Type II	$N=1$ left-movers	$N=1$ right-movers	$d \leq 10$.

*) More precisely, for local $N=2$ supersymmetry one gets $d \leq 2$ and for $N=4$ $d \leq -2$, again under the assumption that the matter fields give a positive contribution. There may also be *global* extended world sheet supersymmetries, but since they do not have ghosts associated with them, they do not affect the space–time dimension.

This also specifies our convention for left- and right-movers in the heterotic string.

Note that condition (1) states that the scalars X^μ must be unconstrained. This means in particular that they must have right-moving as well as left-moving components, since one cannot associate an unpaired left-moving mode with a flat space–time dimension (this would be in conflict with modular invariance). Thus the maximal dimension for the heterotic string is 10.

The field content of d dimensional strings divides naturally into two groups, which play a rather different role in the problem: the set of "space–time" fields $\{X^\mu, \psi^\mu, b, c, \beta, \gamma\}$ and the "internal" conformal systems contributing the remainder of the conformal anomaly, $\mathscr{C}_{26-d;26-d}$ for bosonic strings, $\mathscr{C}_{26-d;15-3d/2}$ for heterotic strings and $\mathscr{C}_{15-3d/2;15-3d/2}$ for type-II strings*). Lattices can be used for the description of both of these systems, but with a rather different degree of generality.

The set of space–time fields is essentially universal. It depends of course on the general type of string theory (heterotic, type-II or bosonic) and in a trivial way on the space–time dimension, but apart from that it is the same for any string theory. In fermionic theories this system is responsible for one of the main technical problems in perturbation theory, namely the treatment of the $\beta\gamma$ ghost system. This ghost system has rather bizarre properties that have proved serious (but not insurmountable) obstacles in understanding the perturbation expansion and proving finiteness. Lattices enter into the discussion of this system if one bosonizes the fermions ψ^μ (an option one always has in two dimensions). One is then naturally led to bosonize also the $\beta\gamma$ system. The result of this procedure is a set of bosons that is periodic, and whose center-of-mass momenta are therefore quantized, i.e. lie on a lattice. This bosonization of the $\psi^\mu\beta\gamma$ system was first described in ref. [6], and will be reviewed in section 5. Clearly anything lattices teach us about this system has a completely general validity.

The internal system is by no means universal. It contains all the information about the spectrum of the theory, and determines the gauge group, the presence of chiral fermions and gravitini, etc. Only the graviton, the dilaton and the anti-symmetric tensor $B_{\mu\nu}$ owe their existence entirely to the space–time sector, and hence they are the only fields that are always present. The internal sector is responsible for another serious problem in string theory, namely that of its classification. There are several ways of constructing conformal field theories $\mathscr{C}_{N;M}$ that satisfy all the consistency conditions, and one of them uses lattices. In lattice constructions one builds up $\mathscr{C}_{N;M}$ by N left-moving and M right-moving bosons whose momenta lie on a lattice. This will be explained in much more detail in section 2, and in section 3 we will discuss the conditions for modular invariance for such theories.

Although the above may suggest that the two systems can be treated completely independently, this is not quite true. First of all, to discuss modular invariance one has to consider both systems simultaneously. This can be done quite elegantly if $\mathscr{C}_{N;M}$ is a conformal field theory described by a lattice. In that case we can combine this lattice with the one resulting from the bosonization of ψ^μ, β and γ. The condition for modular invariance translates then into a certain self-duality condition for the combined lattice. This *covariant lattice construction* is just one of many constructions of fermionic string theories (see e.g. [7–16]). It will be discussed in sections 5 and 7.

The other way in which the space–time and internal sector are linked is via the condition of world sheet supersymmetry for the right-movers of the heterotic string, and both left- and right-movers of the type-II string. Every approach to constructing $\mathscr{C}_{N;M}$ requires its own realization of world sheet supersymmetry, which is usually not a manifest symmetry between bosons and fermions in two dimensions. This should not be surprising, since fermions can be bosonized and vice versa. In section 6

*) We separate left-movers and right-movers by a semicolon in the following, with quantities belonging to left-movers appearing to the left of semicolon.

we discuss the purely bosonic world sheet supersymmetry realizations for covariant lattice constructions.

Section 7 contains a discussion of the spectrum of heterotic and type-II strings, as well as some examples, and a discussion of the relation between various constructions of $\mathscr{C}_{N:M}$.

2. Torus compactification of the bosonic string

In this section we review in some detail the closed bosonic string, compactified on a torus. The presentation given here is essentially the one of refs. [9, 17], although these authors compactified the heterotic string rather than the bosonic string. To warm up, and to introduce some notation, we will begin with a very brief reminder of the basics, the quantization of the uncompactified string.

2.1. The uncompactified closed bosonic string

The propagation of a string through flat d dimensional space–time is governed by the action

$$S = \frac{1}{4\pi\alpha'} \int d\sigma_0 \int d\sigma_1 \, \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu} \,, \tag{2.1}$$

where σ_0 and σ_1 are the world sheet coordinates, indicating respectively the time direction and the direction along the string (one often denotes these variables as τ and σ, but we will not do this to prevent confusion with the usual notation for the one-loop modular parameter). In eq. (2.1) $\eta_{\mu\nu}$ is a flat metric and $X^\mu(\sigma_0, \sigma_1)$ is a map from the world sheet to space–time, which specifies the position of every point of the string at any time. The "slope parameter" α' is set to the value $\frac{1}{2}$ in some of the string literature, but we will refrain from fixing a particular value for a while. We shall only consider closed strings, for which one has the boundary condition

$$X^\mu(\sigma_0, 0) = X^\mu(\sigma_0, \pi) \,. \tag{2.2}$$

From the action one can, in the usual way, derive the operators H and P which generate translations in the σ_0 and σ_1 directions on the world sheet:

$$H = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma_1 \, [(\partial_0 X^\mu)(\partial_0 X_\mu) + (\partial_1 X^\mu)(\partial_1 X_\mu)] \,, \tag{2.3}$$

$$P = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma_1 \, (\partial_0 X^\mu \partial_1 X_\mu) \,. \tag{2.4}$$

The quantization of this system is simplest in light-cone gauge, in which the dynamical variables are X^i, $i = 1, \ldots, d-2$ (for consistency of the theory we will later have to choose $d = 26$). The canonical momenta of X^i, Π_i, are easily obtained from the action

$$\Pi_i = (1/2\pi\alpha')\partial_0 X_i \,.$$

The solutions of the equations of motion have the form

$$X^i(\sigma_0, \sigma_1) = q^i + 2\alpha' p^i \sigma_0 + \frac{i}{2}\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} (\alpha^i_n \, e^{-2in(\sigma_0+\sigma_1)} + \tilde{\alpha}^i_n \, e^{-2in(\sigma_0-\sigma_1)}) . \tag{2.5}$$

Using the equal time commutators

$$\begin{aligned} &[X^i(\sigma_0, \sigma_1), \Pi^j(\sigma_0, \sigma_1')] = i\delta^{ij}\delta(\sigma_1 - \sigma_1') , \\ &[X^i(\sigma_0, \sigma_1), X^j(\sigma_0, \sigma_1')] = 0 , \\ &[\Pi^i(\sigma_0, \sigma_1), \Pi^j(\sigma_0, \sigma_1')] = 0 , \end{aligned} \tag{2.6}$$

one finds the commutation relations for the quantized modes

$$\begin{aligned} &[q^i, p^j] = i\delta^{ij} , \qquad &[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{n+m,0} , \\ &[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = m\delta^{ij}\delta_{n+m,0} , \qquad &[\alpha^i_m, \tilde{\alpha}^j_n] = 0 . \end{aligned} \tag{2.7}$$

The spectrum is obtained by requiring that both H and P vanish. The vanishing of H leads to the conditions*)

$$-p^2 = 2p_+p_- - p_i^2 = (2/\alpha')(\mathcal{N} + \tilde{\mathcal{N}} - 2) , \tag{2.8}$$

where the -2 is due to the zero-mode fluctuations of the oscillators, and we have defined the number operators

$$\mathcal{N} = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n ,$$

and similarly for $\tilde{\mathcal{N}}$. From the vanishing of P we get $\mathcal{N} = \tilde{\mathcal{N}}$. In the quantized theory this is imposed as a condition on the physical spectrum, while eq. (2.8) gives the mass-shell condition.

It is convenient to define an operator H', which is equal to H, but without the zero-mode terms (i.e. the operators p^i and q^i). It is also useful to define the linear combinations

$$H_L = \tfrac{1}{4}(H' - P) = (\tilde{\mathcal{N}} - 1) , \qquad H_R = \tfrac{1}{4}(H' + P) = (\mathcal{N} - 1) ,$$

which are the generators of world sheet translations in the $\sigma_0 - \sigma_1$ and $\sigma_0 + \sigma_1$ directions respectively (without the zero-point subtraction -1, these operators are equal to the Virasoro generators $\tilde{L}_0$ and L_0, which will be discussed in section 4). Their eigenstates are called left- and right-moving modes respectively. Denoting the eigenvalues of H_L and H_R on a state $|\psi\rangle$ as $\frac{1}{4}\alpha' m_L^2$ and $\frac{1}{4}\alpha' m_R^2$ we can write the result obtained above as

*) Our space–time metric has signature $(- + + +, \ldots, + +)$.

$$H_{\mathrm{L}}|\psi\rangle = \tfrac{1}{4}\alpha' m_{\mathrm{L}}^2|\psi\rangle\,, \qquad H_{\mathrm{R}}|\psi\rangle = \tfrac{1}{4}\alpha' m_{\mathrm{R}}^2|\psi\rangle\,,$$
$$-p^2 = m^2 = \tfrac{1}{2}(m_{\mathrm{L}}^2 + m_{\mathrm{R}}^2)\,,$$

with the condition $m_{\mathrm{L}}^2 = m_{\mathrm{R}}^2$.

2.2. The compactified closed bosonic string

Now we consider space–times that are compactified to a torus. A torus in N dimensions is simply $\mathbb{R}^N$ modulo a lattice Λ. This means that points in $\mathbb{R}^N$ which differ by a vector on Λ are identified. If a string propagates in a space–time with this topology it is not necessary to require, as in eq. (2.2), that it begins and ends in the same point in $\mathbb{R}^N$. We should allow for the possibility that the points $\sigma_1 = 0$ and $\sigma_1 = \pi$ of the string are identified points in $\mathbb{R}^N$:

$$X^I(\pi) = X^I(0) + 2\pi l^I\,, \tag{2.9}$$

where $l^I \in \Lambda$ and the factor 2π is put in for convenience. Here we use $I = 1, \dots, N$ to refer to the internal coordinates. Physically, eq. (2.9) corresponds to a string winding around the torus a certain number of times. To accommodate this possibility one modifies the string field X^I in the following way

$$X^I(\sigma_0, \sigma_1) = q^I + 2\alpha' p^I\sigma_0 + 2L^I\sigma_1 + \frac{\mathrm{i}}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^I\,\mathrm{e}^{-2\mathrm{i}n(\sigma_0+\sigma_1)} + \tilde{\alpha}_n^I\,\mathrm{e}^{-2\mathrm{i}n(\sigma_0-\sigma_1)}\right), \tag{2.10}$$

where L^I is an operator whose eigenvalues are the lattice vectors l^I. Of course there is no change in X^i, $i = 1, \dots, d-2-N$, the uncompactified components.

Having done this, we go now through the same exercise as for the uncompactified string. It is easy to verify that the equal time commutators (2.6) again imply (2.7) for both the compactified and uncompactified operators, with the additional requirement that L^I should commute with all other operators. The mass formula replacing (2.8) acquires on the right-hand side an extra term L^2/α'^2. Furthermore, to interpret the left-hand side as a mass in $d - N$ dimensions we have to move the internal components of p to the right, to get

$$m^2 = \sum_{I=1}^{N}\left(p_I^2 + \frac{1}{\alpha'^2}L_I^2\right) + \frac{2}{\alpha'}(\mathcal{N} + \tilde{\mathcal{N}} - 2)\,.$$

The momenta p_I are canonical momenta of a periodic coordinate, and therefore (for reasons familiar from solid state physics) they are quantized. Indeed, if we write their eigenfunctions of momentum k_I as $\mathrm{e}^{\mathrm{i}k_I x_I}$, then we see that these functions are periodic on the boundaries of the unit cell of Λ if and only if $\boldsymbol{k}\cdot\boldsymbol{l} \in \mathbb{Z}$ for every $\boldsymbol{l} \in \Lambda$. This means that the eigenvalues of k_I must lie on the dual lattice Λ^* of Λ.

The condition obtained from the vanishing of P now reads

$$\mathcal{N} - \tilde{\mathcal{N}} + p_I L_I = 0\,.$$

It is again useful to define left and right Hamiltonians H' exactly as before, but with the understanding that only the space–time momenta are removed from H to get H', not the internal ones. Then we get

$$H_{\mathrm{R}}=\tfrac{1}{2}p_{\mathrm{R}}^2+\mathcal{N}-1\,,\qquad H_{\mathrm{L}}=\tfrac{1}{2}p_{\mathrm{L}}^2+\tilde{\mathcal{N}}-1\,,\tag{2.11}$$

where as before left and right masses must be equal to each other and to m^2. The left and right momentum operators are defined as

$$p_{\mathrm{R}}^I=\frac{1}{\sqrt{2}}\left(\sqrt{\alpha'}p^I+\frac{1}{\sqrt{\alpha'}}L^I\right),\qquad p_{\mathrm{L}}^I=\frac{1}{\sqrt{2}}\left(\sqrt{\alpha'}p^I-\frac{1}{\sqrt{\alpha'}}L^I\right).\tag{2.12}$$

There is a further generalization of this, obtained by adding a constant background anti-symmetric tensor field. This background field is coupled to X^μ via an additional term in the action

$$S_B=\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma_0\int \mathrm{d}\sigma_1\ \varepsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\,.\tag{2.13}$$

We take $B_{\mu\nu}$ to be a constant background field, which has non-vanishing components only in the compactified dimensions. The extra term is then a total derivative, which does not affect H or P. So how can it make any difference? It can, because there is a change in the canonical momentum operator in the compactified dimensions. This operator becomes now

$$\begin{aligned}\Pi_I&=\frac{1}{2\pi\alpha'}\,(\partial_0\delta_{IJ}+B_{IJ}\partial_1)X^J\\&=\frac{1}{\pi}\left(p_I+\frac{1}{\alpha'}\,B_{IJ}L^J+\text{oscillators}\right).\end{aligned}$$

It is easy to check that this does not affect the commutation relations of the oscillators. The difference occurs for the zero-modes, where p_μ is replaced as the generator of translations by $\pi_I=p_I+(1/\alpha')B_{IJ}L^J$. So it is π, rather than p, that has its eigenvalues quantized on the dual of Λ. Although the mass formula remains the same as a function of p, it makes now more sense to express it in terms of π. The result is the same as eq. (2.11) above, but with

$$\begin{aligned}p_{\mathrm{R}}^I&=\frac{1}{\sqrt{2}}\left(\sqrt{\alpha'}\pi^I+\frac{1}{\sqrt{\alpha'}}\,(\delta^{IJ}-B^{IJ})L_J\right),\\p_{\mathrm{L}}^I&=\frac{1}{\sqrt{2}}\left(\sqrt{\alpha'}\pi^I-\frac{1}{\sqrt{\alpha'}}\,(\delta^{IJ}+B^{IJ})L_J\right).\end{aligned}\tag{2.14}$$

Here p_{R}, p_{L}, π^I and L^I are operators, whose eigenstates are $|r^I,l^I\rangle$ with

$$\pi^I|r^I,l^I\rangle=r^I|r^I,l^I\rangle\,,\qquad L^I|r^I,l^I\rangle=l^I|r^I,l^I\rangle\,,\tag{2.15}$$

with $l^I\in\Lambda$ and $r^I\in\Lambda^*$. Denoting the eigenvalues of p_{L}^I and p_{R}^I by w_{L}^I and w_{R}^I (related via eq. (2.14) to r^I and l^I) we get the mass formula

$$\tfrac{1}{4}\alpha' m_{\mathrm{R}}^2=\tfrac{1}{2}w_{\mathrm{R}}^2+\mathcal{N}-1\,,\qquad \tfrac{1}{4}\alpha' m_{\mathrm{L}}^2=\tfrac{1}{2}w_{\mathrm{L}}^2+\tilde{\mathcal{N}}-1\,.\tag{2.16}$$

An interesting feature of torus compactification in string theory is that compactification on very large tori is equivalent to compactification on very small ones. More precisely, if $B^{IJ}=0$, eq. (2.14) shows that the mass formula is not modified if one replaces the lattice Λ by $\alpha'\Lambda^*$ (the dual of Λ scaled by α'), except for an irrelevant sign change of w_L^I. Obviously, if Λ describes a torus of very large radius ($l^I \gg 1$), its dual describes a very small torus. (In the presence of a background B-field this "duality" is still present in the form of a sign change of w_L^I, but looks a bit more complicated in terms of the compactification lattice.) This large-scale/small-scale duality is not a special property of torus compactification, but merely a manifestation of the fact that length scales much smaller than the Planck length cannot be probed in string theory.

2.3. Self-dual Lorentzian lattices

To interpret the result it is convenient to express the left and right momenta in terms of the basis vectors e_a^I of the lattice (see appendix A):

$$l^I = e_a^I n_a = e_a^{*I} g_{ab} n_b \,, \qquad r^I = e_a^{*I} m_a \,, \qquad B^{IJ} = e_a^{*I} B_{ab} e_b^{*J} \,.$$

Then we get

$$\begin{aligned} w_R^I &= e_a^{I*} \frac{1}{\sqrt{2}} \left(\sqrt{\alpha'} m_a + \frac{1}{\sqrt{\alpha'}} (g_{ab} - B_{ab}) n_b \right), \\ w_L^I &= e_a^{I*} \frac{1}{\sqrt{2}} \left(\sqrt{\alpha'} m_a - \frac{1}{\sqrt{\alpha'}} (g_{ab} + B_{ab}) n_b \right). \end{aligned} \tag{2.17}$$

(Upper and lower indices have no special significance here.) Just as we did for B, we can also regard g as a constant background field. This becomes evident if one performs this change of basis directly in the action (2.1): the effect is to change the flat metric δ_{IJ} (in the compactified dimensions) to g_{IJ}.

It is now instructive to calculate the inner product of the $2N$-dimensional vectors $P = (w_L; w_R)$ and $P' = (w_L'; w_R')$ (the latter is expressed in terms of n' and m') with respect to the *Lorentzian* signature $[(-)^N; (+)^N]$. The result is

$$\boldsymbol{P} \cdot \boldsymbol{P}' = nm' + mn' \,.$$

From this expression we can learn many things. First of all, the lattice $\Gamma_{N;N}$ consisting of the vectors P defined above is an integral and even Lorentzian lattice. Secondly, the inner products do not depend on the background fields g and B. Therefore g and B must be parameters of a Lorentz transformation. Finally, using this background independence, we can make the choice $e_a^I = \delta^I{}_a$, $g_{IJ} = \delta_{IJ}$, $B_{IJ} = 0$. In this special case the lattice is $(\mathrm{D}_1 \times \mathrm{D}_1)^N$ (see appendix A) with conjugacy classes $(0,0) + (v,v) + (s,c) + (c,s)$ in each factor, which is manifestly Lorentzian self-dual. Because Lorentz transformations preserve Lorentzian self-duality, we conclude that $\Gamma_{N;N}$ is Lorentzian self-dual for any value of the background fields.

Thus compactification of the bosonic string on a torus $\mathbb{R}^N/\Lambda$, with an additional back-ground field B_{IJ}, can be described in terms of a new lattice $\Gamma_{N;N}$ which for *any* Λ and B is Lorentzian self-dual. It should be clear that the Lorentzian signature is a mathematical device for characterizing the lattice; it has nothing to do with the metric of space–time and does not lead to ghosts.

The number of parameters of g and B is $\frac{1}{2}N(N+1) + \frac{1}{2}N(N-1) = N^2$. One can verify that they are precisely the N^2 parameters of the coset $\mathrm{SO}(N,N)/(\mathrm{SO}(N) \times \mathrm{SO}(N))$, which is the set of Lorentz transformations of the lattice modulo the rotations of the left and right sector. The latter are a symmetry of the theory, because all masses depend on w_L and w_R only via their norms. The genuine Lorentz transformations mixing w_L and w_R on the other hand are not symmetries. Although the self-duality condition is invariant under such Lorentz transformations, the spectrum is not. So we have an N^2 parameter family of *different* string theories.

For special values of these parameters the compactification is called *rational*. Given the vectors $(w_L^I; w_R^I)$ of $\Gamma_{N;N}$, one can consider the set of all left-moving components w_L^I as vectors in $\mathbb{R}^N$. This set of vectors closes under addition, but does in general not form a lattice. The definition of a lattice in terms of basis vectors e_a^I implies in particular that there is a finite neighbourhood of each lattice point that does not contain any other lattice point. But if one takes for example the Lorentzian even self-dual lattice $\Gamma_{1;1}$ consisting of points $(\frac{1}{2}nR + m/R; \frac{1}{2}nR - m/R)$, one sees very easily that for arbitrary real R the left components can be chosen arbitrarily close to each other. In general, a torus compactification is called rational if that is not the case, i.e. if the left-moving (or, equivalently, the right-moving) components form separately a lattice, which we denote as Δ_L^* (Δ_R^*) (the purpose of the $*$ will be clear in a moment). The lattice $\Gamma_{1;1}$ in the example is rational if and only if R^2 is a rational number.

Equivalently, rational torus compactifications can be characterized by the existence of vectors of the form $(w_L^I; 0)$, spanning $\mathbb{R}^N$. The left-moving components of these vectors form a lattice that is the dual, Δ_L, of Δ_L^*, and analogously for the right-movers. The lattices Δ_L and Δ_R are obviously even lattices, and Δ_L^* and Δ_R^* can be decomposed in cosets with respect to them. The original even self-dual lattice $\Gamma_{N;N}$ can then be fully specified by Δ_L and Δ_R plus a finite list of pairs $(t_L; t_R)$, where t_L and t_R are coset representatives of Δ_L^*/Δ_L and Δ_R^*/Δ_R respectively. The simplest example of rational torus compactifications is to take for Δ_L and Δ_R the root lattice of a simply laced Lie algebra of rank N, but this is certainly not the only possibility.

2.4. *The spectrum*

For generic even self-dual lattices $\Gamma_{N;N}$ the spectrum of the compactified string is not very interesting. In the uncompactified case the lowest states are the tachyonic ground state $|0\rangle$ with $\frac{1}{4}\alpha' m^2 = -1$, and the set of massless light-cone states $\tilde{\alpha}^i_{-1}\alpha^j_{-1}|0\rangle$, which upon decomposition to irreducible representations of the Lorentz group yield the graviton, a rank-2 anti-symmetric tensor ($B_{\mu\nu}$) and a scalar, the dilaton. Here μ and ν are 26-dimensional space–time indices.

In the compactified case one obtains at least all these fields, except that the N components of a vector-index are to be regarded as internal. Thus one gets again a graviton, a $B_{\mu\nu}$ tensor (where μ and ν are $(26-N)$-dimensional space–time indices) and a dilaton, as well as tensors B_{IJ} and g_{IJ}, whose background values parametrize the torus compactifications discussed above. In addition we get the mixed states $\tilde{\alpha}^i_{-1}\alpha^J_{-1}|0\rangle$ and vice versa, which are vector bosons. When one examines their three-point couplings one learns that they interact like U(1) vector bosons, a conclusion which can be understood more easily in terms of dimensional reduction: these bosons are simply the components g_{iJ} and B_{iJ} of the 26-dimensional tensors, and correspond to the Kaluza–Klein modes one expects in torus compactification.

When one considers the mass formula (2.16) one notices that there is in principle another way of getting massless vector bosons, namely if there are vectors $(\boldsymbol{w}_L; \boldsymbol{w}_R)$ on the lattice such that $\boldsymbol{w}_L^2 = 2$, $\boldsymbol{w}_R^2 = 0$ or $\boldsymbol{w}_L^2 = 0$, $\boldsymbol{w}_R^2 = 2$. By analogy with Euclidean lattices we will call such vectors *roots* of the Lorentzian lattice. Only for very special choices of the set of background parameters g_{IJ} and B_{IJ} does one have such roots. Indeed, since the conditions for the existence of roots are not Lorentz-invariant, any small generic Lorentz transformation of the lattice will destroy them. For any simply laced Lie algebra $\mathscr{G}_N$ of rank N there is at least one lattice that contains its roots on the left as well as the right, namely the Englert–Neveu (EN) lattice (see appendix A). For $N \leq 5$ these are the only Lorentzian even self-dual lattices with roots that span the entire lattice; for $N = 6$ there is an additional $\mathrm{SU}(2)^6_{\mathrm{L}}\mathrm{SU}(2)^6_{\mathrm{R}}$ lattice [18], and for larger values of N one can have lattices with different root systems on the left and right (for example $(\mathrm{E}_8 \times \mathrm{E}_8)_{\mathrm{L}}(\mathrm{D}_{16})_{\mathrm{R}}$).

The extra vector bosons cannot be understood in terms of Kaluza–Klein compactification. Their origin is essentially "stringy", since they are related to strings winding around the compactification manifold. By examining their three-point interactions one learns that they gauge a group isomorphic to $\mathscr{G}_{\mathrm{L}} \times \mathscr{G}_{\mathrm{R}}$, if the left and right lattices contain the roots of $\mathscr{G}_{\mathrm{L}}$ and $\mathscr{G}_{\mathrm{R}}$ respectively. The $\mathrm{U}(1)^N$ vector bosons described above provide the Cartan sub-algebras of these left and right-moving gauge algebras. The mechanism by which these gauge-algebras are generated (usually known as the Frenkel–Kac construction) will be discussed in more detail in section (4.4).

An explicit representation for the background fields B_{ab} and g_{ab} appearing in eq. (2.17) for EN lattices has been given in ref. [19]. Let $\boldsymbol{\beta}_a$, $a = 1, \ldots, N$ represent the simple roots (of norm 2) of the Lie algebra, and let $C_{ab} = \boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_b$ be the Cartan matrix. Then set $\alpha' = \frac{1}{2}$, and choose $\boldsymbol{e}_a = \frac{1}{2}\boldsymbol{\beta}_a$, so that $g_{ab} = \frac{1}{4}C_{ab}$. Finally, choose $B_{ab} = g_{ab}$ for $a > b$ and $B_{ab} = -g_{ab}$ for $a < b$. It is then easy to see that w^I_{L} and w^I_{R} in eq. (2.17) are weights of $\mathscr{G}_N$ ($\frac{1}{2}\boldsymbol{e}^*_a$ is a set of basis vectors for the weight lattice), while $w^I_{\mathrm{L}} - w^I_{\mathrm{R}}$ is a root. Hence w^I_{L} and w^I_{R} are weights from the same $\mathscr{G}_N$ conjugacy class, which is the definition of an EN lattice. It is a simple exercise to identify the roots.

Notice that for all Lie algebras $\mathscr{G}_N$, a B_{ab} field is required to obtain a $\mathscr{G}_N$ gauge algebra, with the exception of SU(2), for which $N = 1$ so that B_{ab} does not exist.

2.5. Vertex operators

Up to now we have dealt with free string theories. To discuss interactions one needs the *vertex operators*, $V(\psi, \sigma_0, \sigma_1)$ which give the amplitude for the emission of a state ψ in the string spectrum from a point σ_1 on the string at time σ_0. The tree amplitude for a process with external lines corresponding to states $\psi_1, \ldots, \psi_N$ is obtained by calculating the σ_0-ordered product (here $\boldsymbol{\sigma} = (\sigma_0, \sigma_1)$)

$$\langle \psi_1 | V(\psi_2, \boldsymbol{\sigma}_2) \cdots V(\psi_{N-1}, \boldsymbol{\sigma}_{N-1}) | \psi_N \rangle ,$$

integrating over all σ_1 and all σ_0 (respecting the ordering $(\sigma_0)_i < (\sigma_0)_{i+1}$) and summing over all σ_0 orderings.

The simplest vertex operator in the uncompactified bosonic string is the tachyon emission vertex $\exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{X})$. The vertex operators for all other states are combinations of such exponentials with derivatives of $\boldsymbol{X}$.

If we consider the naive compactified version of the tachyon emission vertex, we would get $\exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{X}) \exp(\mathrm{i}\boldsymbol{r} \cdot \boldsymbol{X}_{\mathrm{int}})$, where $\boldsymbol{k}$ is the space–time momentum and $\boldsymbol{r}$ the momentum in the internal dimensions, defined in section (2.2); we have denoted the internal components of the string field as $\boldsymbol{X}_{\mathrm{int}}$ (because of the contribution of internal momentum $\boldsymbol{r}$ to the mass, this operator, of course, does in general not describe the emission of tachyons). This is clearly not an adequate operator for describing emission of states $|r^I, l^I\rangle$ with $l^I \neq 0$. The problem is that the operator algebra we considered so far does not contain an operator that creates states with non-vanishing winding number. This suggests that we should add an operator Q^I which is canonically conjugate to L^I, i.e. [20]

$$[Q^I, L^J] = \mathrm{i}\delta^{IJ} . \tag{2.18}$$

The action of this operator should not interfere with the position and canonical momentum operator of the string, i.e. we should require

$$[Q^I, q^J] = [Q^I, \pi^J] = 0 .$$

Only then does it make sense to define the states $|r^I, l^I\rangle$. Using the operator Q^I we can create all such states from the vacuum

$$|r^I, l^I\rangle = \mathrm{e}^{\mathrm{i}r\cdot q}\, \mathrm{e}^{\mathrm{i}l\cdot Q}|0\rangle . \tag{2.19}$$

These states manifestly satisfy eq. (2.15).

We would like to modify the tachyon emission vertex to allow for the possibility of emission of such states. One is more or less naturally led to the following definitions

$$X_{\mathrm{int}}(\sigma_0, \sigma_1) = \sqrt{\alpha'/2}(X_{\mathrm{L}}(\sigma_0 - \sigma_1) + X_{\mathrm{R}}(\sigma_0 + \sigma_1)) ,$$

$$X^I_{\mathrm{L}}(\sigma_0 - \sigma_1) = q^I_{\mathrm{L}} + 2p^I_{\mathrm{L}}(\sigma_0 - \sigma_1) + \mathrm{i} \sum_{n\neq 0} \frac{1}{n}\, \tilde{\alpha}^I_n\, \mathrm{e}^{-2\mathrm{i}n(\sigma_0 - \sigma_1)} ,$$

$$X^I_{\mathrm{R}}(\sigma_0 + \sigma_1) = q^I_{\mathrm{R}} + 2p^{\mathrm{I}}_{\mathrm{R}}(\sigma_0 + \sigma_1) + \mathrm{i} \sum_{n\neq 0} \frac{1}{n}\, \alpha^I_n\, \mathrm{e}^{-2\mathrm{i}n(\sigma_0 + \sigma_1)} ,$$

$$q^I_{\mathrm{R}} = \frac{1}{\sqrt{2\alpha'}}\,(\delta_{IJ} - B_{IJ})q^J + \sqrt{\frac{\alpha'}{2}}\, Q^I ,$$

$$q^I_{\mathrm{L}} = \frac{1}{\sqrt{2\alpha'}}\,(\delta_{IJ} + B_{IJ})q^J - \sqrt{\frac{\alpha'}{2}}\, Q^I .$$

Notice that because of eq. (2.18) the operators we have just defined satisfy the following commutators with the left and right momentum operators (2.14),

$$[q^I_{\mathrm{L}}, p^J_{\mathrm{L}}] = [q^I_{\mathrm{R}}, p^J_{\mathrm{R}}] = \mathrm{i}\delta^{IJ} ,$$

while all other commutators vanish. Thus the string fields X_{L} and X_{R} commute.

These definitions have no influence on the compactified tachyon emission vertex, since Q cancels in the sum of X_{L} and X_{R}. To accommodate the possibility of emission of winding states, a vertex operator for the compactified dimensions should contain a factor $\exp(\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{Q})$. A heuristic way to get such a vertex operator is to observe that in the case of uncompactified strings the tachyon emission vertex is nothing but the operator $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q})$ that creates momentum states of momentum $\boldsymbol{k}$ in first quantized point particle theory, with the position operator $\boldsymbol{q}$ replaced by the string field $\boldsymbol{X}$. We can try to mimic that procedure for compactified strings. We would like the zero-mode part of the vertex operator to be the operator in eq. (2.19). It is easy to show that

$$\mathrm{e}^{\mathrm{i}r\cdot q}\, \mathrm{e}^{\mathrm{i}l\cdot Q} = \mathrm{e}^{\mathrm{i}w_{\mathrm{L}}\cdot q_{\mathrm{L}}}\, \mathrm{e}^{\mathrm{i}w_{\mathrm{R}}\cdot q_{\mathrm{R}}} .$$

This suggests the following vertex operator

$$e^{\mathrm{i}w_{\mathrm{L}}\cdot X_{\mathrm{L}}}\, e^{\mathrm{i}w_{\mathrm{R}}\cdot X_{\mathrm{R}}}\,.$$

Notice that this operator is completely factorized in terms of left-movers and right-movers. This is not possible for uncompactified dimensions, since one does not have the operator Q. It can be shown that this inspired guess for the vertex operator for the emission of internal states satisfies the necessary conditions imposed on it by reparametrization invariance.

The left–right factorization implies that the propagators for the compactified and uncompactified string fields are somewhat different. It is convenient to go to Euclidean space by replacing σ_0 by $-\mathrm{i}\sigma_0$, and to define new, complex variables $z = e^{2(\sigma_0+\mathrm{i}\sigma_1)}$ (the resulting functions $X(z)$ are called *Fubini–Veneziano fields*). Between vacuum states satisfying $\alpha_n|0\rangle = \tilde{\alpha}_n|0\rangle = 0$, $n>0$ and $p^i|0\rangle = p^I|0\rangle = 0$ one gets for their propagators

$$\begin{aligned}&\langle X^I_{\mathrm{R}}(z)X^J_{\mathrm{R}}(w)\rangle = -\delta^{IJ}\log(z-w)\,, \qquad \langle X^I_{\mathrm{L}}(\bar{z})X^J_{\mathrm{L}}(\bar{w})\rangle = -\delta^{IJ}\log(\bar{z}-\bar{w})\,,\\ &\langle X^i(z,\bar{z})X^j(w,\bar{w})\rangle = -\alpha'\delta^{ij}\log(|z-w|)\,.\end{aligned} \tag{2.20}$$

Notice that with the standard choice $\alpha' = \frac{1}{2}$ the normalization of the uncompactified bosonic string propagator differs by a factor of four from the compactified one. For that reason it is more convenient to choose $\alpha' = 2$ so that the discussions of correlation functions of compactified and uncompactified bosons are more or less on the same footing. Therefore in the discussion of conformal field theory in sections 4 and 5 we will set $\alpha' = 2$, and we simply leave α' as a free parameter in the few cases where it appears in other sections. There still remains a difference between the compactified and uncompactified bosons, since it is not possible to split the uncompactified bosons $X^i(z,\bar{z})$ into left-moving and right-moving (anti-holomorphic and holomorphic) components, because of their zero-modes. Of course one never needs such a split, since one only encounters operators of the form $\exp(\mathrm{i}\boldsymbol{k}\cdot \boldsymbol{X}(z,\bar{z}))$ or (anti)-holomorphic derivatives such as $\partial_z X(z,\bar{z})$, whose correlation functions *can* be evaluated with eq. (2.20).

3. Partition functions

The foregoing discussion has not yet given any insight in the relevance of the self-dual lattice which emerges in the construction. The property of the lattice is crucial for the construction of fermionic strings that follows, and is the key to modular invariance for these theories. To understand this, we have to discuss in some detail the partition function of the compactified bosonic string.

3.1. Path integrals

A multi-loop closed bosonic string amplitude without external lines is given by an expression of the form [21]

$$A_\gamma = \int \frac{\mathscr{D}X\,\mathscr{D}g}{V_{\mathrm{Diff}}V_{\mathrm{C}}} \exp(-S_{\mathrm{E}}(g,X))\,, \tag{3.1}$$

where γ denotes the genus (i.e. the number of handles) of the surface, and g is the metric of the surface; $V_{\rm Diff}$ and $V_{\rm C}$ denote the volume of the set of local diffeomorphisms and of the conformal transformations respectively, and $S_{\rm E}$ is the bosonic string action, obtained by continuing (2.1) and (2.13) to Euclidean space:

$$S_{\rm E} = \frac{1}{4\pi\alpha'}\left[\int \partial_\alpha X_\mu \partial^\alpha X^\mu - \mathrm{i}\int \varepsilon^{\alpha\beta} B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right]. \tag{3.2}$$

What is meant by eq. (3.1) is that one should integrate over all values of X and the world sheet metric on the surface that are not equivalent by reparametrizations and conformal transformations. This is the analog of the process of gauge-fixing in gauge theories, and this problem can be tackled with very similar techniques. After gauge fixing one is left with an integral over those metrics that cannot be related by local coordinate transformations. The parameters that specify those metrics (i.e. that parametrize the inequivalent shapes of the surface) are called the moduli. A genus-γ surface has $3(\gamma-1)$ moduli for $\gamma \geq 2$, and one modulus for $\gamma = 1$.

For simplicity we consider first the one-loop case. A one-loop diagram is a torus, and it can be represented by points in the complex plane modulo a two-dimensional lattice. This corresponds to a choice of coordinates with a flat, constant metric on the torus. To regard the complex plane modulo a lattice simply means that points that differ by a lattice vector are identified with each other. A two-dimensional lattice has two basis vectors, chosen so that all other vectors are integral linear combinations of them. By rotations and scale transformations one can always align one of the basis vectors with the real axis, and scale it to have length 1. The second one can always be put into the positive upper half-plane, where it can be specified by a complex number τ, with $\mathrm{Im}(\tau) > 0$ (see fig. 1). We have now used all the transformations at our disposal to bring the metric in the simplest possible form. The remaining free parameter of the torus, τ, parametrizes metrics that cannot be mapped into each other by general coordinate transformations or scale transformations.

At one loop, the path integral for the uncompactified closed bosonic string takes the following form after gauge fixing (this expression is really only correct for $d = 26$, but we show the dependence on d for future purposes):

$$A_1 = \left(\frac{1}{2\pi\sqrt{\alpha'}}\right)^d \int \mathrm{d}^d X_0 \int \frac{\mathrm{d}\tau}{(\mathrm{Im}\,\tau)^2}\,(\mathrm{Im}\,\tau)^{(2-d)/2}$$
$$\times \int \mathscr{D}'X\,\mathscr{D}b\,\mathscr{D}c\,\exp[-S_{\rm E}(g(\tau), X) - S_{\rm ghost}]\,. \tag{3.3}$$

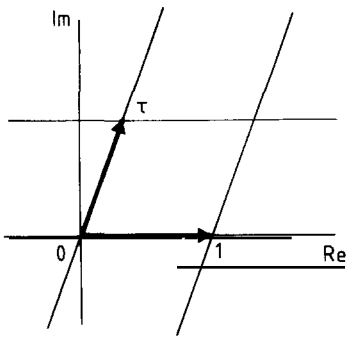

Fig. 1. Definition of τ.

Here we have separated the X-integral into an integral over the zero-modes, X_0, and an integral over the remaining modes, denoted $\mathscr{D}'X$. This separation requires some care, at least if one is interested in the overall normalization. One can determine the normalization by considering a harmonic oscillator in the zero-frequency limit. One finds that for each zero-mode one needs a normalization factor $1/\sqrt{2\pi\beta}$, where β is the Euclidean time interval. Furthermore one gets a factor $\sqrt{c}$ if the kinetic term in the action is of the form $\frac{1}{2}c\int(\partial X)^2$. Thus altogether each zero-mode contributes a factor

$$(2\pi\sqrt{\alpha' \operatorname{Im}\tau})^{-1}\, \mathrm{d}X_0 \,.$$

(There is an additional factor $(\operatorname{Im}\tau)^{-1}$ in eq. (3.3) as a remnant of the gauge fixing procedure.)

The X_0-integration over uncompactified space–time would of course diverge, rendering the complete expression infinite unless the integrand happens to vanish. This infinity is easy to understand: since the action vanishes for constant X, one of the infinitely many integrals in $\mathscr{D}X$ is simply an integral of a constant over all of space–time. This infinite factor is nothing to worry about, and is simply to be regarded as a momentum conservation delta function with argument zero, which appears because we are considering a diagram without external lines.

The remaining X-integral is a simple Gaussian, and the result may be written as

$$\int \mathscr{D}'X\, \mathrm{e}^{-S_\mathrm{E}} = \left(\frac{1}{\det'(-\Delta)}\right)^{d/2}, \tag{3.4}$$

where Δ is the Laplacian on the torus, $\Delta = g_{\alpha\beta}\,\partial_\alpha\,\partial_\beta$, and the prime on the determinant indicates that the zero-mode should be omitted. Finally there is a contribution from the integral over the reparametrization ghosts b and c, which at one loop cancels one of the factors $\det'(-\Delta)$ in eq. (3.4). For more details we refer to ref. [22].

3.2. *The one-loop partition function*

In general we define the partition function $\mathscr{F}(\tau)$ of a string theory to be the integrand of the τ-integral, with the zero-mode integral and $(\operatorname{Im}\tau)^{1-d/2}$ factored out. Here d is the dimension of *uncompactified* space–time. For the uncompactified bosonic string $\mathscr{F}(\tau)$ is thus given by eq. (3.4), with d replaced by $d-2$ due to the ghost contribution.

It is possible to evaluate the determinants directly by regularization of the infinite product of eigenvalues of Δ (see for example the appendix of ref. [22]), but at one loop there is a more intuitive way. Let us first recall a result for a one-dimensional bosonic system, specified by a coordinate $X(t)$ and an action $S(X(t))$. One can prove (if the action satisfies certain conditions) the following relation between the path integral for a Euclidean action S_E and the trace of the exponential of the Hamiltonian H of the system

$$\int_{\mathrm{PBC}} \mathscr{D}X\, \mathrm{e}^{-S_\mathrm{E}} = \operatorname{Tr} \mathrm{e}^{-\beta H} \,.$$

Here "PBC" means that $X(t)$ must satisfy periodic boundary conditions on the interval $[0, \beta]$. For a derivation of this relation see for example [23]. The result is easily generalized to X's having more components, or even an infinite number of components. The latter case is relevant for us, because the scalar field $X(\sigma_0, \sigma_1)$ appearing in string theory can be regarded as having an infinite number of components labeled by σ_1 (times a finite number of components because of the space–time index, which we omitted here). To use this relation, one should think of $\mathrm{Im}\,\tau$ as β. There is a small complication if $\mathrm{Re}\,\tau \neq 0$, because then the periodicity of X in $\mathrm{Im}\,\tau$ depends on $\mathrm{Re}\,\tau$. This can be taken into account by including a σ_1 shift operator in the trace, so that states at $\sigma_0 = 0$ are not identified with the same states at $\sigma_0 = \mathrm{Im}\,\tau$, but with states of a string which is rotated in the σ_1 direction by an amount $\mathrm{Re}\,\tau$. The generator of infinitesimal σ_1 translations is of course P, so that one gets

$$\mathcal{F}(\tau) = \int_{T(\tau)} \mathcal{D}'X\, \mathrm{e}^{-S_\mathrm{E}} = \mathrm{Tr}\, \mathrm{e}^{-\pi\, \mathrm{Im}\,\tau H'}\, \mathrm{e}^{\mathrm{i}\pi\, \mathrm{Re}\,\tau P} = \mathrm{Tr}\, \mathrm{e}^{2\mathrm{i}\pi\tau H_\mathrm{R}}\, \mathrm{e}^{-2\mathrm{i}\pi\bar{\tau} H_\mathrm{L}}\,. \tag{3.5}$$

The integral is over all two-dimensional fields X that are periodic on a torus T specified by a modular parameter τ. On the right-hand side we have used $\pi\,\mathrm{Im}\,\tau$ and $\pi\,\mathrm{Re}\,\tau$ for the Euclidean time and the σ_1-shift because the Hamiltonian and momentum operator are defined for a cylinder with periodicity π in σ_1. The usual convention for the torus is to take $0 \leq \sigma_1 < 1$, and the periodicity along the other cycle is usually defined with τ instead of $\pi\tau$. Because of scale-invariance of the action, the result does not change if we scale σ_1 and σ_0 simultaneously, so that eq. (3.5) gives the correct answer for the conventional definition of the torus.

This way of writing the partition function admits a simple interpretation. Expanding it in terms of $q = \mathrm{e}^{2\mathrm{i}\pi\tau}$ one gets a power series of the form $\mathcal{F}(\tau) = \Sigma\, d_{nm} q^n \bar{q}^m$. Obviously, d_{nm} is simply the number of states of left mass*) m and right mass n.

It is usually a simple combinatorial exercise to read off the partition function from the expressions for H_L and H_R. Every *term* in the Hamiltonian yields a *factor* in the partition function, so that one can derive the partition function term-by-term. Take for example a single bosonic oscillator α_{-n}, $n > 0$. It contributes to the right Hamiltonian via a term $\alpha_{-n}\alpha_n$, and when it acts it increases the mass of a state by n. Thus it contributes to the partition function a factor

$$(1 + q^n + q^{2n} + q^{3n} + \cdots) = 1/(1 - q^n)\,,$$

where on the right the term q^{kn} corresponds to the contribution of $(\alpha_{-n})^k$. The partition function of the bosonic string contains such a factor for every oscillator.

For the uncompactified string we leave out the zero-modes in H_L and H_R as discussed before. The remaining trace, i.e. the contribution from the bosonic oscillators to the partition function, is equal to

$$\mathcal{F}_\mathrm{B}(\tau) = \mathrm{Tr}\, q^{H_\mathrm{R}} \bar{q}^{H_\mathrm{L}} = |\eta(\tau)|^{-48}\,, \tag{3.6}$$

where η is the Dedekind η-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,.$$

Note that one can only express the result in terms of η-functions if $d = 26$ because of the zero-point subtraction -1 in H_L and H_R. This will prove to be crucial for modular invariance, which would be

*) By "mass" we mean here and in the following the value of the right-hand side of eq. (2.16).

spoiled by left-over factors of q (if one were to modify this zero-point subtraction, the graviton would become massive or tachyonic, and one would lose Lorentz invariance).

Since the oscillator contribution to the mass formula is the same for compactified and uncompactified coordinates, the partition function for the compactified string contains the same factor. The only difference is that the zero-modes of the compactified coordinates are now included in the determinant as well as the Hamiltonian. So first of all we have to multiply the uncompactified string result by $2\pi\sqrt{\alpha' \operatorname{Im} \tau}$ for every compactified coordinate to remove the zero-mode normalizations in eq. (3.3). Secondly, we have to take into account the contribution to the trace due to the extra momentum terms in the mass formula (2.16). Since this is simply an extra term in the mass formulas, it produces an extra factor in the partition function, which together with the $\sqrt{\operatorname{Im} \tau}$ factors is called the *correction factor*, because it gives the correction in the bosonic partition function for the fact that part of the zero-modes are compactified. From the mass formula (2.16) we get for the correction factor $C(\Gamma_{N;N})$ for a lattice $\Gamma_{N;N}$

$$C(\Gamma_{N;N}) = \mathscr{L}_{N;N}(\tau)(\operatorname{Im} \tau)^{N/2} , \tag{3.7}$$

where $\mathscr{L}_{N;N}$ is the *lattice partition function*:

$$\mathscr{L}_{N;N} = \sum_{(w_{\mathrm{L}};w_{\mathrm{R}})} \mathrm{e}^{\mathrm{i}\pi\tau w_{\mathrm{R}}^2}\, \mathrm{e}^{-\mathrm{i}\pi\bar{\tau} w_{\mathrm{L}}^2} . \tag{3.8}$$

Here the sum is over all the vectors on the lattice.

In the case of rational torus compactifications (defined in section (2.3)) the lattice sum can be rewritten as a *finite* sum of the form $\Sigma_i\, \chi_i^{\mathrm{L}}(\bar{\tau})\chi_i^{\mathrm{R}}(\tau)$, where χ_i^{L} is anti-holomorphic and χ_i^{R} holomorphic. These functions are the contribution to the partition function of one entire coset or (for Lie algebra lattices) conjugacy class.

3.3. One-loop modular invariance

Not all different values of τ correspond to different metrics. Although different values of τ cannot be related by *local* transformations, some can be related by *global* ones. For example, if one defines a torus by $\tau' = \tau + 1$ one makes the same identification in the complex plane, but the metric would look rather different, i.e. have a different but equivalent parametrization. On a real-life, donut-shaped torus this transformation corresponds to cutting the torus along a cycle which cannot be continuously deformed to a point, rotating one edge by 2π, and gluing the edges back together. A torus has two independent cycles of this type, and hence there is a second transformation of τ which leads to equivalent metrics. For this transformation one can take for example $\tau \to -1/\tau$. Together these transformations generate the group of *modular transformations*, which is the set of transformations of the form

$$\tau \to \frac{a\tau + b}{c\tau + d} , \qquad a, b, c, d \in \mathbb{Z} , \qquad ad - bc = 1 .$$

This group is isomorphic to $\mathrm{SL}_2(\mathbb{Z})/\mathbb{Z}_2$. In practice one needs usually only its generators, which are the two transformations we have already described,

$$\tau \to \tau + 1\,, \qquad \tau \to -\frac{1}{\tau}\,. \tag{3.9}$$

As explained above, the expression for a one-loop diagram is of the form

$$\int \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2}\,\mathscr{F}(\tau)\,,$$

where $\mathscr{F}$ is some function which depends on the theory. The issue of modular invariance centers on the region of the τ-integration. If one were to integrate over the entire positive half-plane, one would encounter the same metric an infinite number of times. One would like to restrict the integration to a *fundamental domain*, which is defined as a region in the upper half-plane which contains every metric just once (the canonical choice is $-\frac{1}{2} \leq \mathrm{Re}\,\tau < \frac{1}{2}$, $|\tau| \geq 1$, and is shown in fig. 2), but this can only be done in a meaningful way if $\mathscr{F}(\tau)$ does not depend on the region one chooses, since otherwise the result would be ambiguous. Thus $\mathscr{F}$ should be invariant under eq. (3.9). This is what is meant by modular invariance.

Checking invariance under $\tau \to \tau + 1$ is rather easy, because this is true if and only if the partition function contains only terms of the form $q^x \bar{q}^y$, with $|x - y| \in \mathbb{Z}$. This is the case for both compactified and uncompactified strings, and it is clearly essential that $\Gamma_{N;N}$ is even. The second modular transformation is a bit harder to check, and requires a trick, which goes by the name of "Poisson resummation formula". It is derived in appendix C, and applied there to proving modular invariance for Euclidean lattice partition functions.

The fact that we have here a Lorentzian lattice and a non-holomorphic partition function is no problem though. Consider the lattice contribution to the partition function, described by the function

$$\mathscr{L}_{N;N}(\tau) = \sum_{(w_\mathrm{L};w_\mathrm{R})\in\Gamma} \mathrm{e}^{\mathrm{i}\pi\tau w_\mathrm{R}^2}\,\mathrm{e}^{-\mathrm{i}\pi\bar{\tau} w_\mathrm{L}^2} \equiv \sum_{(w_\mathrm{L};w_\mathrm{R})\in\Gamma} f(w_\mathrm{L};w_\mathrm{R})\,. \tag{3.10}$$

It is not difficult to show that the Poisson formula gives in this case

$$\mathscr{L}_{N;N}(-1/\tau) = \sum_{(w_\mathrm{L};w_\mathrm{R})\in\Gamma} \mathrm{e}^{(-\mathrm{i}\pi/\tau) w_\mathrm{R}^2}\,\mathrm{e}^{(\mathrm{i}\pi/\bar{\tau}) w_\mathrm{L}^2} = \sum_{(v_\mathrm{L};v_\mathrm{R})\in\Gamma^*} f^*(v_\mathrm{L};v_\mathrm{R})\,,$$

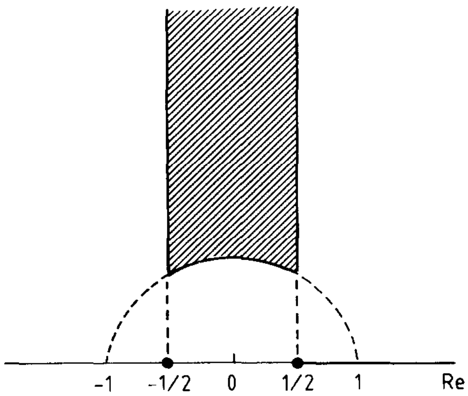

Fig. 2. Fundamental modular domain.

where

$$f^*(v_L; v_R) = \int_{\mathbb{R}^{N,N}} \mathrm{d}w_R\, \mathrm{d}w_L\, \mathrm{e}^{2\pi \mathrm{i} v_R \cdot w_R}\, \mathrm{e}^{-2\pi \mathrm{i} v_L \cdot w_L}\, \mathrm{e}^{(-\mathrm{i}\pi/\tau) w_R^2}\, \mathrm{e}^{(\mathrm{i}\pi/\bar\tau) w_L^2}$$
$$= \mathrm{e}^{\mathrm{i}\pi\tau v_R^2}\, \mathrm{e}^{-\mathrm{i}\pi\bar\tau v_L^2} |\tau|^N ,$$

by a simple Gaussian integration. Here we used unimodularity of the lattice, since otherwise there would be an additional volume factor. Using $\Gamma = \Gamma^*$ one derives immediately

$$\mathscr{L}_{N;N}(-1/\tau) = |\tau|^N \mathscr{L}_{N;N}(\tau) . \tag{3.11}$$

Combining the uncompactified string partition function with the correction factor, we find for the vacuum amplitude of the compactified bosonic string

$$A_1' = \int \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2} (\mathrm{Im}\,\tau)^{(2-d)/2} |\eta(\tau)|^{-48} \mathscr{L}_{N;N}(\tau) , \tag{3.12}$$

where the prime indicates the omission of the zero-mode integral. To check the modular invariance of the integrand one has to verify that the factors $|\tau|$ in eq. (3.11) (which are called *modular weight* factors) cancel. They do, because there are also contributions from the transformation of $\mathrm{Im}\,\tau$ and $\eta(\tau)$. The transformation of $\mathrm{Im}\,\tau$ is straightforward:

$$\mathrm{Im}(-1/\tau) = |\tau|^{-2}\, \mathrm{Im}\,\tau .$$

Furthermore one needs

$$\eta(-1/\tau) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau) .$$

Clearly one needs $d = 26 - N$ in eq. (3.12) to cancel the modular weight factors. This is essentially a consequence of conformal invariance, since the weight factors can be thought of as a scale transformation. The *phases* in the modular transformations cannot fail to cancel for the *uncompactified* bosonic string, since its partition function is "non-chiral", i.e. depends in the same way on left- and right-movers. Hence all phases cancel automatically between left and right. Thus, a priori, only the correction factor could have violated modular invariance. This remark is also valid for higher genus, and we will check explicitly that also in that case there is no problem if the correction factor comes from an even self-dual Lorentzian lattice.

For future purposes it is useful to extend the notion of lattice partition functions to *character-valued* partition functions. They are defined by modifying eq. (3.10) as follows

$$\mathscr{L}_{N;N}(x_L, x_R|\tau) = \sum_{(w_L;w_R)\in\Gamma} \exp(\mathrm{i}\pi\tau w_R^2 + 2\pi\mathrm{i} w_R \cdot x_R) \exp(-\mathrm{i}\pi\bar\tau w_L^2 - 2\pi\mathrm{i} w_L \cdot x_L) . \tag{3.13}$$

The modular transformation properties of this function can be derived in a straightforward way by using Poisson resummation exactly as above. One only gets a nice transformation if the arguments x_L and x_R

are rescaled. The result is

$$\mathscr{L}_{N;N}\left(\frac{x_{\mathrm{L}}}{c\bar{\tau}+d}, \frac{x_{\mathrm{R}}}{c\tau+d}\,\middle|\,\frac{a\tau+b}{c\tau+d}\right) = |c\tau+d|^{N} \exp\left\{\mathrm{i}\pi c\left(\frac{x_{\mathrm{R}}^{2}}{c\tau+d} - \frac{x_{\mathrm{L}}^{2}}{c\bar{\tau}+d}\right)\right\}\mathscr{L}_{N;N}(x_{\mathrm{L}}, x_{\mathrm{R}}|\tau)\,. \tag{3.14}$$

Here we have also displayed the dependence on the parameters a, b, c and d of the modular transformations, generated by repeated action of $\tau\to\tau+1$ and $\tau\to -1/\tau$.

3.4. *Soliton sums*

There is a second way to determine the one-loop partition function, and that is to integrate explicitly over the zero-modes in the compactified dimensions. This has the advantage that, unlike the Hamiltonian approach, it has a straightforward generalization to higher genus.

The zero-mode contribution to the path-integral in the compactified dimensions comes from "soliton" states (since we shall consider such states for *Euclidean* world sheets, "instanton" would perhaps be a better terminology.) These are states with non-zero winding number in either the σ_0 or the σ_1 direction on the torus, i.e.

$$\begin{aligned} &X^{I}(\sigma_0, \sigma_1+1) = X^{I}(\sigma_0,\sigma_1) + 2\pi k_a^{I}\,,\\ &X^{I}(\sigma_0+\operatorname{Im}\tau, \sigma_1+\operatorname{Re}\tau) = X^{I}(\sigma_0,\sigma_1) + 2\pi k_b^{I}\,, \end{aligned} \tag{3.15}$$

where k_b^I and k_a^I are vectors on the lattice Λ on which N dimensions are compactified (the corresponding coordinates are labeled by I). In terms of the parameters L^I and p^I in the string field (2.10) this implies

$$L^{I} = \pi k_a^{I}\,, \qquad \alpha' p^{I} = \frac{\pi}{\operatorname{Im}\tau}(k_b^{I} - k_a^{I}\operatorname{Re}\tau)\,.$$

The zero-mode contribution to the action (3.2) is

$$S(\boldsymbol{k}_b, \boldsymbol{k}_a) = \frac{\pi}{\alpha'\operatorname{Im}\tau}\,[\boldsymbol{k}_b^{2} + \boldsymbol{k}_a^{2}|\tau|^{2} - 2\boldsymbol{k}_b\cdot\boldsymbol{k}_a\operatorname{Re}\tau - 2\mathrm{i}k_b^{I}B_{IJ}k_a^{J}\operatorname{Im}\tau]\,.$$

Substituted into eq. (3.3) this gives for the contribution of the zero-modes in the compactified dimensions

$$\mathscr{L}_{N;N} = \left(\frac{1}{2\pi\sqrt{\alpha'\operatorname{Im}\tau}}\right)^{N}\int \mathrm{d}^{N}X_0 \sum_{k_b,k_a} \mathrm{e}^{-S(k_b,k_a)}\,. \tag{3.16}$$

In the compact dimensions, the X_0-integral is finite, and yields a factor $(2\pi)^N\times\operatorname{vol}(\Lambda)$, the volume of the compact space.

The sum is over all $k_b, k_a\in\Lambda$. Using Poisson resummation we can write it as follows

$$\sum_{k_b,k_a\in\Lambda} \mathrm{e}^{-S(k_b,k_a)} = \frac{1}{\operatorname{vol}(\Lambda)} \sum_{q\in\Lambda^*, k_a\in\Lambda} \mathscr{F}(q, k_a)\,,$$

where $\mathscr{F}$ is obtained by Fourier-transforming in k_b:

$$\mathscr{F}(q, k_a) = \int_{\mathbb{R}^N} \mathrm{d}k_b\, \mathrm{e}^{2\pi \mathrm{i} q\cdot k_b}\, \mathrm{e}^{-S(k_b, k_a)}$$
$$= (\alpha' \operatorname{Im}\tau)^{N/2} \exp\{[\pi/(\alpha' \operatorname{Im}\tau)][(k_a \operatorname{Re}\tau + \mathrm{i}Bk_a \operatorname{Im}\tau + \mathrm{i}\alpha' q \operatorname{Im}\tau)^2 - k_a^2|\tau|^2]\}\,.$$

Substituting this into eq. (3.16) one obtains the expression (3.10) for $\mathscr{L}_{N;N}$ with w_L and w_R given by (2.14), with π replaced by q and L by k_a.

3.5. Partition functions and modular invariance at higher genus

All the above can be generalized quite easily to genus-γ surfaces (see also refs. [24–27] for discussions of higher genus partition functions). Some of the necessary mathematics is reviewed in appendix C.

The generalizations of the soliton states on the torus are field configurations with periodicities $2\pi \boldsymbol{k}_b^i$ along the b_i-cycle and $2\pi \boldsymbol{k}_a^i$ along the a_i-cycle, with the cycles defined as in fig. 3. (Here we view $\boldsymbol{k}_b^i$ and $\boldsymbol{k}_a^i$ as vectors in the compactified dimensions.) Such a configuration can be written down explicitly. Write X^I in terms of complex coordinates $z = \sigma_0 + \mathrm{i}\sigma_1$, where σ_0 and σ_1 are the Euclidean world sheet coordinates. In these coordinates the action is (with $\alpha' = \frac{1}{2}$ from now on)

$$S_\mathrm{E} = \frac{2}{\pi} \int \mathrm{d}^2 z\, \partial_z X^\mu \partial_{\bar z} X^\nu (g_{\mu\nu} + B_{\mu\nu})\,. \tag{3.17}$$

As before we choose $g_{\mu\nu} = \eta_{\mu\nu}$, and we put the information about the angles and lengths of the basic vectors of the lattice in the boundary conditions of X^I rather than in the background metric. A soliton state defined by the sets of vectors $(\boldsymbol{k}_a^i, \boldsymbol{k}_b^i)$ is defined to transform as follows when z is moved along a homology cycle

$$\boldsymbol{X}(z + a_i, \boldsymbol{k}_b^i, \boldsymbol{k}_a^i) = \boldsymbol{X}(z, \boldsymbol{k}_b^i, \boldsymbol{k}_a^i) + 2\pi \boldsymbol{k}_a\,,$$
$$\boldsymbol{X}(z + b_i, \boldsymbol{k}_b^i, \boldsymbol{k}_a^i) = \boldsymbol{X}(z, \boldsymbol{k}_b^i, \boldsymbol{k}_a^i) + 2\pi \boldsymbol{k}_b\,.$$

It is easy to see that the following function has the required property

$$\boldsymbol{X}(z, \boldsymbol{k}_b^i, \boldsymbol{k}_a^i) = \mathrm{i}\pi(\boldsymbol{k}_a \bar\Omega - \boldsymbol{k}_b)(\operatorname{Im}\Omega)^{-1} \int_{z_0}^{z} \omega + \text{c.c.}\,,$$

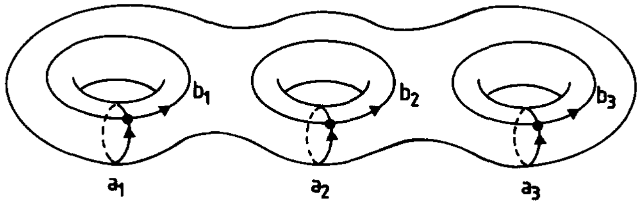

Fig. 3. Homology cycles for a genus-3 Riemann surface.

where z_0 is an arbitrary fixed reference point on the surface. Substituting this into the action we get

$$S_{\mathrm{E}}(k_b^i, k_a^i) = 2\pi(k_b - k_a\bar{\Omega}) \cdot (\mathbf{1} + \boldsymbol{B}) \cdot \frac{1}{\operatorname{Im}\Omega}(k_b - k_a\Omega)\,,$$

where the genus-labels are contracted in the obvious way. When restricted to genus 1 all these results reduce to those of the previous section.

To write the expression in a more convenient way one can perform a Poisson resummation on k_b, regarding it as a vector in the direct sum of γ copies of $\mathbb{R}^N$. For the higher loop generalization of the correction factor we get then

$$\begin{aligned} C_\gamma(\Gamma) &= \sum_{(w_{\mathrm{L}}^i; w_{\mathrm{R}}^i)\in\Gamma} \exp(\mathrm{i}\pi w_{\mathrm{R}}^i \Omega_{ij} \cdot w_{\mathrm{R}}^j - \mathrm{i}\pi w_{\mathrm{L}}^i \bar{\Omega}_{ij} \cdot w_{\mathrm{L}}^j)[\det(\operatorname{Im}\Omega)]^{N/2} \\ &\equiv \mathscr{L}^\gamma_{N;N}(\Gamma)[\det(\operatorname{Im}\Omega)]^{N/2}\,. \end{aligned} \tag{3.18}$$

As in the one-loop case, we have passed here from a lattice Λ of dimension N to a Lorentzian lattice $\Gamma_{N;N}$. The vectors on the latter lattice are related to those on Λ and to B^{IJ} as in eq. (2.14), and the sum is over γ copies of that lattice.

The complete partition function is obtained by multiplying the correction factor with the uncompactified bosonic string partition function. The latter is non-chiral on the world sheet, and hence insensitive to modular phases. To check multi-loop modular invariance we have to check therefore only the invariance of the correction factor.

This works exactly as in appendix C, apart from a generalization to Lorentzian partition functions as discussed in section (3.3). The phase factors in the modular transformations are trivial in this case because the lattice is even and integral. The weight factor $|\Omega_{11}|^N$ appearing in the transformation S_1 cancels against a similar factor from $[\det(\operatorname{Im}\Omega)]^{N/2}$, so that the genus-$\gamma$ correction factor is modular invariant. It is amusing to note that proving modular invariance with respect to the transformation S_1 involves modular transformations of lower genus character-valued partition functions. Thus by requiring multi-loop modular invariance for the higher genus vacuum amplitude, we could have determined the transformation property of a lower genus character-valued partition function. Although we have seen this here for the special case of lattices, it is an example of a far more general situation.

3.6. *Chiral bosons*

For heterotic strings one needs partition functions for different numbers of left- and right-moving bosons. Actions for such chiral bosons can be written down by starting with N left and right-moving bosons, and introducing an extra Lagrange multiplier term to set some of the right-moving degrees of freedom to zero [28], [1]. This extra term makes it rather difficult to calculate the multi-loop partition function directly using the path integral.

In practice we need only bosons on lattices $\Gamma_{p;q}$ which are Lorentzian self-dual and even. Such lattices exist only for $p - q = 0 \bmod 8$, and this fact allows us to write down the lattice partition function without using the path integral directly (the following is a variation on an argument by Ginsparg and Vafa [27]). If $p = q + 8k$, we simply consider first the lattice $\Gamma'_{p,p} = \Gamma_{p;q} \times (\mathrm{E}_8)^k$. This lattice has equal left and right dimensions, and its partition function can be derived by introducing background B_{IJ} fields. The contribution of the p left- and right-moving bosons to the partition function is

$$\mathscr{P}(\Gamma') = \frac{\mathscr{L}^{\gamma}_{p;p}(\Gamma')}{[\det'(-\Delta)]^{p/2}} .$$

We can get the partition function for the lattice $\Gamma_{p,q}$ out of this one if we can divide out the E_8 lattice contributions. This is possible, because the E_8 partition function can be written in terms of chiral fermion determinants, which can be expressed in terms of ϑ functions

$$\mathscr{P}(E_8) = (\mathscr{R}(\Omega))^{-4} \sum_{\alpha} \vartheta^8_{\alpha}(0|\Omega) . \tag{3.19}$$

Here $\mathscr{R}(\Omega)$ is a holomorphic square root of $\det'(-\Delta)$, $\mathscr{R}(\Omega)\mathscr{R}(\bar{\Omega}) = \det'(-\Delta)$. Using the definition of the ϑ-functions (see appendix C) one can write the sum in eq. (3.19) as a sum over vectors on the E_8 lattice:

$$\sum_{\alpha} \vartheta^8_{\alpha}(0|\Omega) = \sum_{p \in E_8} e^{i\pi p^i \Omega_{ij} \cdot p^j} .$$

This cancels the E_8 contribution to the lattice sum in $\mathscr{P}(\Gamma')$, so that we can conclude that the result is what one might have expected,

$$\mathscr{P}(\Gamma_{p;q}) = \mathscr{R}(\bar{\Omega})^{-p/2} \mathscr{R}(\Omega)^{-q/2} \sum_{(w^i_L; w^i_R) \in \Gamma_{p,q}} \exp(i\pi w^i_R \Omega_{ij} \cdot w^j_R - i\pi w^i_L \bar{\Omega}_{ij} \cdot w^j_L) .$$

At one loop, the exponential expression can also be derived more directly using background fields. In addition to g_{IJ} and B_{IJ} background fields (the indices refer to the p non-chiral, left–right paired bosons) one has to introduce fields A^a_I, where a labels the $p-q$ chiral bosons [17].

4. Bosonization and conformal field theory on lattices

In this section we recall elementary concepts of conformal field theory [29, 30]. More detailed reviews can, for example, be found in refs. [31–37]. We emphasize that conformal field theory is a priori a more general notion than string theory: strings moving in a specific d-dimensional space–time represent particular examples of conformal field theories. We will restrict the discussion mainly to that type of conformal field theories relevant for lattices.

4.1. *Operator product algebras*

The basic objects are *conformal* or *primary fields* $\Phi(z, \bar{z})$, which are defined to transform under conformal transformations $z \to f(z)$ (cf. section 1.2) as

$$\Phi(z, \bar{z}) = (\partial f/\partial z)^h (\partial \bar{f}/\partial \bar{z})^{\bar{h}} \Phi(f, \bar{f}) , \tag{4.1}$$

and hence $\Phi(z, \bar{z})(dz)^h(d\bar{z})^{\bar{h}}$ is conformally invariant. The complex variable parametrizes either the world sheet cylinder continued to Euclidean space (in which case we use instead the variable $w = 2\sigma_0 + 2i\sigma_1$) or the conformally equivalent punctured complex plane (with $z = e^w$). Fields on the cylinder and on the plane are related by

$$\Phi_{\text{plane}}(z, \bar{z}) = z^{-h}\bar{z}^{-\bar{h}}\Phi_{\text{cyl}}(w, \bar{w}) . \tag{4.2}$$

Any conformal field is thus characterized by a *conformal weight* $(h, \bar{h})$; the scaling weight is then given by $h + \bar{h}$ and the two-dimensional spin by $h - \bar{h}$ (in two dimensions, the concept of spin is quite delicate: there can exist objects with arbitrary fractional spin and statistics). All relevant (local) properties of a given conformal field are characterized by operator product expansions with other fields:

$$\Phi_i(z, \bar{z}) \cdot \Phi_j(w, \bar{w}) = \sum_k (z - w)^{-h_i - h_j + h_k}(\bar{z} - \bar{w})^{-\bar{h}_i - \bar{h}_j + \bar{h}_k}\alpha_{ijk}\Phi_k(w, \bar{w}) . \tag{4.3}$$

Expressions of this kind are valid if the arguments are *radially ordered* in the complex plane (i.e. $|z| > |w|$), which corresponds to time-ordering on the cylinder. The coefficients α_{ijk} define the *operator product algebra*, which is closely related to the fusion rules of the theory [29, 38]. Unless all powers in $(z - w)$ in all possible operator products are integral $(h_k - h_j - h_i - \bar{h}_k + \bar{h}_j + \bar{h}_i \in \mathbb{Z})$, the theory is ill-defined (*non-local*), that is, the correlation functions cannot be unambiguously integrated because of branch cuts. As we will see, requiring locality is a strong constraint.

If $\bar{h} = 0$ $(h = 0)$, a field is called *chiral* or *right-(left-)*moving; it depends then only (anti-)analytically on z. In some simple cases (such as the free boson theories discussed in this report), conformal fields can be written as a tensor product of an analytic and an anti-analytic part. Then the Hilbert space decomposes into a product of left- and right-moving Hilbert spaces. These correspond to *chiral* conformal field theories; for a given theory, the left- and right-moving conformal field theories need not be identical. If a chiral conformal field theory is local by itself $(h_k - h_j - h_i \in \mathbb{Z})$, it is called *meromorphic*, and if there exists only a finite number of operators with the same conformal weight, it is called *unitary* [4]. In this section, we will mainly discuss unitary chiral conformal field theories.

The conformal weight h of any primary field Φ_h is determined by the operator product with the stress–energy tensor $T(z)$:

$$T(z) \cdot \Phi_h(w) = h\,\frac{\Phi_h(w)}{(z - w)^2} + \frac{\partial_w \Phi_h(w)}{(z - w)} + \text{regular terms} . \tag{4.4}$$

Note that $T(z)$ is the generator of infinitesimal conformal transformations of the form (4.1): the action of the operator $\int (\mathrm{d}z/2\pi\mathrm{i})\varepsilon(z)T(z)$ on a conformal field produces the transformation $z \to z + \varepsilon(z)$. Conformal fields have the special property that they obey (4.4) without creating higher poles in $z - w$. For example, the fields appearing in the non-leading and regular terms in eq. (4.4) are *not* conformal fields. They are basically derivatives of the conformal fields Φ_k and called *descendants*. More generally, taking the operator product (4.3) of any two conformal fields creates a series of other operators, which may be partly conformal fields and partly descendants. As the descendants are trivially related to the conformal fields, the theory is completely characterized by all possible operator products between the primary fields. Therefore we will only consider primary fields in the following. The operator $T(z)$ itself has weight $h = 2$,

$$T(z) \cdot T(w) = \frac{c/2}{(z - w)^4} + 2\,\frac{T(w)}{(z - w)^2} + \frac{\partial_w T(w)}{(z - w)} + \text{regular terms} , \tag{4.5}$$

but for $c \neq 0$ it is not a conformal field due to the "anomalous" higher pole. The coefficient c is the conformal anomaly, discussed in section (1.2). If one considers the stress–energy tensor of a subset of fields in the two-dimensional theory one usually finds $c \neq 0$, but in the string theories we consider the sum of all contributions to c vanishes.

There is an intimate relationship between meromorphic chiral operator products and (anti-)commutation relations of the Fourier modes of Φ. On the plane, the modes are defined by*)

$$\Phi_h(z) = \sum_n z^{-n-h} \Phi_{h,n} , \qquad \Phi_{h,n} = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z\, z^{n+h-1} \Phi(z) . \tag{4.6}$$

Note that the factors z^h in these definitions cancel against the conformal factors when Φ is transformed back to the cylinder, so that one gets the canonical mode expansions on the cylinder. The information encoded in operator product expansions is equivalent to the collection of all (anti-)commutation relations among the modes Φ_n; operator products are just an economic way for writing these. The precise form of the (anti-) commutation relations depends on the precise form of the operator algebras. Upon exchanging $\Phi_i(z)$ with $\Phi_j(w)$ one easily finds whether a given pair of operators commutes or anti-commutes. Commutators and anti-commutators are naturally related to even and odd powers of $z - w$, as will become clear in the following examples.

Kac–Moody algebras

Consider commuting operators $\Phi(z) \equiv J^a(z)$ of conformal weight one in a unitary conformal field theory. As we want the right-hand side to be even under the simultaneous interchange $a \leftrightarrow b$ and $z \leftrightarrow w$, the most general operator product is

$$J^a(z) \cdot J^b(w) = k \frac{\delta^{ab}}{(z-w)^2} + \frac{1}{(z-w)} \mathrm{i} f^{ab}{}_c J^c(w) + \text{regular terms} , \tag{4.7}$$

where $f^{ab}{}_c$ is anti-symmetric in a and b. Actually, upon considering three- and four-point functions one finds that $f^{ab}{}_c$ must be structure constants of some Lie algebra [39]. Equation (4.7) has the special property that it closes on the same type of operators, thus yielding a finite operator product algebra for unitary conformal field theories. The commutation relations are obtained via the usual contour integration argument: $J^a_m J^b_n$ and $J^b_n J^a_m$ can be written as double integrals with the same integrand, but with different radial ordering of the integration contours. As the only poles are at z, $w = 0$ and $z = w$, the commutator is the same integrand integrated over the difference of the two contours, which by Cauchy's theorem is the same as

$$[J^a_m, J^b_n] = \oint_0 \frac{\mathrm{d}w}{2\pi \mathrm{i}} \oint_w \frac{\mathrm{d}z}{2\pi \mathrm{i}}\, w^n z^m J^a(z) J^b(w) . \tag{4.8}$$

Substituting (4.7) into (4.8) and performing the contour integrals, one obtains

$$[J^a_m, J^b_n] = \mathrm{i} f^{ab}{}_c J^c_{m+n} + k m \delta^{ab} \delta_{m,-n} , \tag{4.9}$$

*) The values of n in (4.6) must be such that the integral is well defined. For example if $h \in \mathbb{Z}$ and $\Phi(z)$ is single-valued around the origin, then n takes all integral values.

which is nothing but the *level-k Kac–Moody extension* of some Lie algebra with structure constants $f^{ab}{}_{c}$ (in the following, we will only encounter such algebras with $k = 1$).

Virasoro algebra

Similarly, defining

$$T(z) = \sum_n z^{-n-2} L_n , \tag{4.10}$$

leads to the following expression, equivalent to eq. (4.5)

$$[L_n, L_m] = (c/12)n(n^2 - 1)\delta_{n,-m} + (n - m)L_{n+m} . \tag{4.11}$$

Thus the conformal anomaly c of eq. (4.5) appears as a *central charge* term in the algebra. The zero-mode L_0 is (up to a constant) the Hamiltonian of the right-moving (analytic) sector of the theory (of course, the full theory contains in addition an isomorphic Virasoro algebra in the left-moving sector). For unitary conformal field theories, the spectrum of L_0 is bounded from below.

Super Virasoro algebra

As an example for an anti-commuting object, consider the superpartner $T_{\mathrm{F}}(z)$ (known as the *supercurrent*) of the stress–energy tensor, which plays an important role in *superconformal theories* (fermionic strings). It has weight $h = \frac{3}{2}$ (so that it is fermionic), and the corresponding equation (4.4) together with (4.5) and

$$T_{\mathrm{F}}(z) \cdot T_{\mathrm{F}}(w) = \frac{2c/3}{(z - w)^3} + \frac{2T(w)}{(z - w)} + \text{regular terms} \tag{4.12}$$

constitute the $N = 1$ superconformal algebra in operator product form. Equivalently, its mode components G_m satisfy

$$\{G_m, G_n\} = 2L_{n+m} + \tfrac{1}{3}c(m^2 - \tfrac{1}{4})\delta_{m,-n} . \tag{4.13}$$

If $m \in \mathbb{Z}$, one has the Ramond version, if $m \in \mathbb{Z} + \frac{1}{2}$, one has the Neveu–Schwarz version of the *super Virasoro algebra*. The zero-mode in the Ramond sector, $G_0 = (1/2\pi \mathrm{i}) \oint \mathrm{d}z\, z^{1/2} T_{\mathrm{F}}(z)$, is the two-dimensional supercharge, and obeys

$$G_0^2 = L_0 - \tfrac{1}{24}c . \tag{4.14}$$

In a unitary superconformal theory we have theory $G_0^2 \geq 0$, hence it follows from (4.14) that the Ramond ground state $|\alpha\rangle$ has weight $h = \frac{1}{24}c$:

$$L_0|\alpha\rangle = \tfrac{1}{24}c|\alpha\rangle . \tag{4.15}$$

4.2. Basic conformal fields and states

One of the most important fields is the derivative of a (chiral, free) boson, ∂X, which has $h = 1$. An explicit expression for such a free boson is the Fubini–Veneziano field described in section (2.5). The field X is itself not a conformal field, a fact that is related to its non-local operator product with itself:

$$X(z)\cdot X(w) = -\log(z-w)\,. \tag{4.16}$$

In contrast, ∂X is a local, conformal field:

$$\partial X(z)\cdot \partial X(w) = -\frac{1}{(z-w)^2} + \text{regular terms}\,. \tag{4.17}$$

For free bosons, the stress–energy tensor is

$$T(z) = -\tfrac{1}{2}{:}\partial X \partial X{:} = -\tfrac{1}{2}\lim_{w\to z}\left(\partial X(z)\cdot\partial X(w) + \frac{1}{(z-w)^2}\right), \tag{4.18}$$

where, on the right-hand side, we have explicitly subtracted the poles as required by normal ordering. Henceforth products of operators taken at the same point are always implicitly understood to be normal ordered, i.e., all singularities are subtracted. By inserting eq. (4.18) into (4.5) one learns that in this case $c=1$.

Another important type of primary field that can be built with a free boson is given by $\Phi_\lambda(z) = {:}\mathrm{e}^{\mathrm{i}\lambda X}(z){:}$. One finds $T(z)\cdot\Phi_\lambda(w) = \frac{1}{2}\lambda^2(z-w)^{-2}\Phi_\lambda(w)+\cdots$ and hence $h=\frac{1}{2}\lambda^2$. Furthermore, using Wick contractions and eq. (4.16), we have the fundamental formula

$$\begin{aligned}
&{:}\mathrm{e}^{\mathrm{i}\lambda\cdot X}{:}(z)\cdot{:}\mathrm{e}^{\mathrm{i}\lambda'\cdot X}{:}(w) = (z-w)^{\lambda\cdot\lambda'}{:}\mathrm{e}^{\mathrm{i}\lambda\cdot X(z)+\mathrm{i}\lambda'\cdot X(w)}{:}\\
&\quad = (z-w)^{\lambda\cdot\lambda'}{:}\mathrm{e}^{\mathrm{i}(\lambda+\lambda')\cdot X(w)}(1+\mathrm{i}(z-w)\lambda\cdot\partial X(w)+\cdots){:}\,.
\end{aligned} \tag{4.19}$$

Nothing in the foregoing changes if we replace λ and X by vectors $\boldsymbol{\lambda}$ and $\boldsymbol{X}$. For $2N$ bosons, $\mu=1,\ldots,2N$, $T(z) = -\frac{1}{2}\partial X^\mu\partial X_\mu$ and accordingly $c=2N$. Note that this operator product is local only if $\lambda\cdot\lambda'\in\mathbb{Z}$.

In constructions of conformal field theories, one often uses two-dimensional fermions. A particular example are the NSR fermions ψ^μ with space–time vector index. Although we concentrate on this case in the following, our discussion applies to fermions with "internal" indices as well. The fermions ψ^μ obey

$$\psi^\mu(z)\cdot\psi^\nu(w) = \frac{\delta^{\mu\nu}}{(z-w)} + J^{\mu\nu}(w) + \text{further regular terms}\,, \tag{4.20}$$

which, if we make a mode expansion*)

$$\psi^\mu(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} z^{-n-1/2}\psi^\mu_n\,, \tag{4.21}$$

is equivalent to $\{\psi^\mu_n, \psi^\nu_m\} = \delta^{\mu\nu}_{n+m}$. The free fermion stress energy tensor is $T_\psi = \frac{1}{2}(\partial\psi)\psi$, and from (4.20) we have

*) We take ψ here to be single valued on the complex plane. A more detailed discussion of fermion boundary conditions will be given in the next section.

$$T_\psi(z)\cdot\psi^\mu(w) = \frac{1}{2}\,\frac{\psi^\mu(w)}{(z-w)^2} + \frac{\partial\psi^\mu(w)}{(z-w)} + \text{regular terms}\,, \tag{4.22}$$

so that $h=\frac{1}{2}$. Furthermore, the operator product of T_ψ with itself shows that a collection of $2N$ fermions has $c=N$.

From these basic building blocks one can construct other conformal fields by considering operator products. For instance, the first regular term in the expansion (4.20) is given by

$$J^{\mu\nu}(z) = :\psi^\mu\psi^\nu:\,, \quad \mu,\nu = 1,\ldots,2N\,. \tag{4.23}$$

One has then the following operator products:

$$\begin{aligned} J^{\mu\nu}(z)\cdot J^{\rho\sigma}(w) &= \frac{1}{(z-w)^2}\{\delta^{\mu\rho}\delta^{\nu\sigma}-\delta^{\mu\sigma}\delta^{\nu\rho}\} - \frac{1}{(z-w)}\{\delta^{\mu\rho}J^{\nu\sigma}(w)-\delta^{\mu\sigma}J^{\nu\rho}(w) \\ &\quad -\delta^{\nu\rho}J^{\mu\sigma}(w)+\delta^{\nu\sigma}J^{\mu\rho}(w)\} + \text{regular terms}\,, \\ J^{\mu\nu}(z)\cdot\psi^\rho(w) &= -\frac{1}{(z-w)}\{\delta^{\mu\rho}\psi^\nu(w)-\delta^{\nu\rho}\psi^\mu(w)\} + \text{regular terms}\,. \end{aligned} \tag{4.24}$$

The first one is equivalent to a level $k=1$, $\hat{\mathrm{D}}_N \equiv \widehat{\mathrm{SO}(2N)}$ Kac–Moody algebra for the mode components $J_m^{\mu\nu} = (1/2\pi\mathrm{i})\oint \mathrm{d}z\, z^m J^{\mu\nu}(z)$. In particular, the zero-modes $J_0^{\mu\nu}$ are the usual D_N generators, under which ψ_m^μ transform as vectors. As $J^{\mu\nu}(z)$ is given above in terms of fermions, one calls this a *fermionic* construction of the Kac–Moody algebra. Later we will discuss a *bosonic* version.

One reason why *conformal* fields are special is that there is a one-to-one correspondence between conformal fields Φ_h of weight h and highest weight states*) $|h\rangle = \Phi_h(0)|0\rangle$ of the Virasoro algebra, defined by

$$L_0|h\rangle = h|h\rangle\,, \qquad L_n|h\rangle = 0 \quad (n>0)\,. \tag{4.25}$$

The eigenvalue h determines the mass of the state $|h\rangle$. Such highest weight states characterize representations of the Virasoro algebra. The other states in a given representation (the descendants, which fill out the *Verma module*) are obtained by acting with certain products of L_{-n} to $|h\rangle$; the corresponding operators (e.g., derivatives of Φ_h) are not conformal fields, as they create higher poles in eq. (4.4).

Physical states in a string theory are generated by certain combinations of conformal fields called *vertex operators*. They describe the emission of states from the world sheet, i.e., the in-coming or out-going asymptotic states. For closed strings, which are described by a combination of a right-moving (z-dependent) and a left-moving ($\bar{z}$-dependent) conformal field theory, we require vertex operators to have conformal weight one in the left- and the right-moving sector in order to produce conformally invariant correlation functions $\langle V(z_1)\cdots V(z_k)\rangle$. Consequently, they consist of a left- and a right-moving piece together with the exponentiated momentum,

$$V(\bar{z},z) = \Phi_h(z)\Phi'_{\bar{h}}(\bar{z})\,\mathrm{e}^{\mathrm{i}k\cdot X}(\bar{z},z)\,. \tag{4.26}$$

*) $\Phi_h(0)$ on the plane corresponds to an incoming state at $t=-\infty$ on the world sheet cylinder.

Only if the on-shell condition $\frac{1}{2}k^\mu k_\mu = 1 - h = 1 - \bar{h}$ is satisfied (plus, in general, additional on-shell polarization conditions to prevent higher poles in eq. (4.4)) is $V(z)$ a good vertex operator and $|\Phi, k\rangle = V(0)|0\rangle$ a highest weight state. The mass m of such a state is given by $\frac{1}{2}m^2 = h - 1$ (note that we are using $\alpha' = 2$ here). The fields Φ_h and $\Phi'_{\bar{h}}$ belong to independent left- and right-moving sectors, and are, a priori, not related to each other. The simplest bosonic string vertex operator, the tachyon emission vertex is given by eq. (4.26) with $\Phi(z) \equiv \Phi'(\bar{z}) \equiv 1$, ($k^\mu k_\mu = -m^2 = 2$). Some more examples can be found in section 4.5.

Consider now string theories involving not only some bosons, X^μ, but also a collection of world sheet fermions ψ^μ, where μ is a space–time index ($\mu = 1, \ldots, 2N$). In addition to states created by purely bosonic operators (i.e., by ∂X and exponentials of X), there also exist highest weight states of the form (see eq. (4.21))

$$(\psi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_k})(0)|0\rangle = \psi^{\mu_1}_{-1/2}\psi^{\mu_2}_{-1/2}\cdots\psi^{\mu_k}_{-1/2}|0\rangle\ , \tag{4.27}$$

corresponding to anti-symmetric tensors of rank k ($h = \frac{1}{2}k$). Taking $\Phi_h = \psi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_k}$ in (4.26) yields vertex operators describing the emission of anti-symmetric tensor states; in particular,

$$V = \xi^\mu\psi_\mu\ \mathrm{e}^{\mathrm{i}k\cdot X}\ , \tag{4.28}$$

(combined with a left-moving operator $\Phi'_{\bar{h}}$ of conformal weight one) is a vector boson emission vertex. One can also define the G-parity operator

$$G_{\mathrm{NS}} = (-1)^F\ , \tag{4.29}$$

(F is the two-dimensional fermion number) which is $+1$ (-1) on even (odd) anti-symmetric tensors (4.27). The operator G_{NS} commutes with $T(z)$.

The crucial difference between the bosonic and fermionic Hilbert spaces is that the latter splits into two parts, called the *Neveu–Schwarz* (NS) and *Ramond* (R) sectors. This is due to the fact that there are two *spin structures* on the cylinder: in the NS-sector the fermions ψ have anti-periodic, while in the R-sector they have periodic boundary conditions. By performing a conformal transformation $z = \mathrm{e}^w$ we see that (because of the factor $\mathrm{e}^{-w/2} = z^{-1/2}$ in eq. (4.2)) fermions in the NS-sector are periodic (single-valued), and those in the R-sector anti-periodic (double-valued) along cycles enclosing the origin of the complex plane. Thus in the NS-sector the mode numbers n in the expansion (4.21) of $\psi(z)$ are half-integral, and in the R-sector integral. In particular, the zero-modes in the R-sector obey

$$\{\psi^\mu_0, \psi^\nu_0\} = \delta^{\mu\nu}\ , \tag{4.30}$$

so they can be represented by space–time SO($2N$) γ-matrices. Accordingly the representation space on which ψ^μ act must be spinorial, which can be achieved by choosing the ground state $|\alpha\rangle$ to be a spinor (thus α is an SO($2N$) spinor index). Then a generic highest weight R-state (corresponding to (4.27)) looks like

$$|\mu_1\mu_2\cdots\mu_k\alpha\rangle = (\psi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_k})(0)|\alpha\rangle\ . \tag{4.31}$$

Clearly, while highest weight states in the NS-sector (4.27) correspond to space–time antisymmetric *tensors*, R-states (4.31) lead to space–time *spinors*. As in the NS-sector, one can introduce the G-parity operator

$$G_{\mathrm{R}} = \gamma_{2N+1} \otimes (-1)^F , \tag{4.32}$$

whose eigenvalues are ± 1, and which now takes into account the space–time helicity.

One now wishes to create states in the R-sector from states in the NS-sector via vertex operators. If we view $\psi(z)$ as an analytic function on the plane, we can describe the double-valuedness around the origin (in the R-sector) as due to a square-root branch cut emanating from there: it flips the boundary conditions between periodic and anti-periodic. This cut corresponds to an operator insertion at $z = 0$ (which describes an incoming state at $t = -\infty$ on the cylinder). This operator has to be a conformal field $S^\alpha(z)$ that takes the NS-vacuum to the Ramond highest weight state $|\alpha\rangle$ that satisfies (4.15)

$$|\alpha\rangle = S^\alpha(0)|0\rangle . \tag{4.33}$$

The operator S^α is called a *spin* field and it has a very complicated description in terms of the fundamental field ψ^μ. In the next section we will, however, give a simple description of S^α. The spin field is indispensable in the calculation of scattering amplitudes in fermionic string theories, since it appears in the vertex operator for the emission of space–time spinors.

The property that S^α produces a branch cut with ψ is imposed by requiring that the operator product expansion of $\psi(z)$ with $S^\alpha(w)$ contains half-integral powers in $(z - w)$:

$$\psi^\mu(z) \cdot S^\alpha(w) = \frac{1}{(z-w)^{1/2}} (\gamma^\mu)^\alpha_{\dot\beta} S^{\dot\beta}(w) + \text{regular terms} . \tag{4.34}$$

Thus, the very presence of this cut naively renders the theory *non-local* and as such ill-defined. We will discuss later how these difficulties are resolved.

The operator S^α is a special case of what one calls a *twist* operator: the R-sector is obtained from the NS-sector by a Z_2-twist. One can consider more general, e.g., Z_M-twists, that can also act on the bosons ∂X. One encounters generalized twist operators in orbifold constructions.

4.3. Basics of bosonization

Bosonization is a feature that is very special to two dimensions, and simply amounts to expressing fermionic field operators like ψ in terms of bosonic ones. The crucial point is that two conformal field theories are physically equivalent, i.e., indistinguishable, if all correlation functions are identical. Now, the singular parts of all correlation functions are characterized by all possible operator product expansions (4.3) between the various fields. Thus bosonizing a theory essentially amounts to expressing any given two-dimensional fermionic operator by a certain bosonic expression in such a way that it reproduces all local operator product expansions. In particular, the realization of certain symmetries (for example, those generated by $J^{\mu\nu}$ or T_{F}) depends only on operator products involving the symmetry currents and not on how these are represented.

To bosonize ψ^μ, $\mu = 1, \dots, 2N$, we first switch to a complex basis by defining N pairs

$$\begin{aligned} &(1/\sqrt{2})(\psi^1 + \mathrm{i}\psi^2) \rightarrow \psi^1 , \qquad (1/\sqrt{2})(\psi^1 - \mathrm{i}\psi^2) \rightarrow \psi^{-1} , \\ &(1/\sqrt{2})(\psi^3 + \mathrm{i}\psi^4) \rightarrow \psi^2 , \qquad (1/\sqrt{2})(\psi^3 - \mathrm{i}\psi^4) \rightarrow \psi^{-2} , \end{aligned} \tag{4.35}$$

and so on. Then eq. (4.20) takes the form (j, $k = 1, \ldots, N$):

$$\psi^k(z) \cdot \psi^{-j}(w) = \frac{\delta_{kj}}{(z-w)} + J^{k,-j}(w) + (z-w)\delta_{k,-j}P^k(w) + \text{terms containing descendant fields}\,. \tag{4.36}$$

Using the operator product formula (4.19) for exponentials, it is easy to see that we can express our $2N$ fermions by N free bosons H_k in a way such that (4.36) is reproduced:

$$\psi^k = :\mathrm{e}^{\mathrm{i}H_k} c_k:\,, \qquad \psi^{-k} = :\mathrm{e}^{-\mathrm{i}H_k} c_{-k}:\,. \tag{4.37}$$

(c_k are certain cocycle-phase generating operators that ensure that fermions with different indices really anti-commute; for explicit realizations of the operators c_k see for example, refs. [40, 36]. A very simple representation is obtained by taking $c_i = c_{-1} = \gamma_i$, $i = 1, \ldots, N$, where γ_i are the SO(N) γ-matrices. In the following, we often will not write such factors explicitly). Both conformal systems $(\{\psi^\mu\}, \{H_k\})$ have central charge $c = N$.

One can easily bosonize $J^{\mu\nu}$ by just reading it off from eq. (4.36). Switching to the Cartan–Weyl basis, we have for the off-diagonal ($k \neq j$) currents

$$J^{\pm k,\pm j}(z) \equiv :\psi^{\pm k}\psi^{\pm j}:(z) = :\mathrm{e}^{\pm\mathrm{i}H_k \pm \mathrm{i}H_j}:(z)\,\varepsilon(\pm k, \pm j)c_{\pm k \pm j}\,, \tag{4.38}$$

and for the diagonal currents, which span the Cartan sub-algebra:

$$\begin{aligned} J^{k,-k}(z) &= :\mathrm{e}^{\mathrm{i}H_k}\,\mathrm{e}^{-\mathrm{i}H_k}:(z) \\ &= \lim_{w \to z} (\mathrm{e}^{\mathrm{i}H_k}(z)\,\mathrm{e}^{-\mathrm{i}H_k}(w) - 1/(z-w)) = \mathrm{i}\partial H_k(z)\,. \end{aligned} \tag{4.39}$$

Note that in eq. (4.38) one generates a cocycle factor for the current from the cocycle factors of the individual fermions. In general one has $c_k c_l = \varepsilon(k, l)c_{k+l}$, where $\varepsilon(k, l) = \pm 1$ (the sign depends on the precise realization one chooses for the cocycles, but is not relevant for our present purpose). Furthermore we have used $c_k c_{-k} = 1$, an identity that the cocycles are required to satisfy. It is easy to check that these currents represent a *bosonic* realization of the level one $\hat{\mathrm{D}}_N$-Kac–Moody algebra (4.24). Note that the price to pay for bosonization is that the Kac–Moody currents are not all on the same footing: the off-diagonal currents are realized by solitons, i.e., circle-valued bosons, while the diagonal, Abelian currents take the canonical form. Equations (4.38) and (4.39) provide only a particular example of the more general *Frenkel–Kac* construction, which is described in more detail in the next section.

Similarly, the operator $P^k(z)$ appearing in eq. (4.36) is seen to have the form

$$P^k(z) = :\mathrm{e}^{2\mathrm{i}H_k}:(z)c_{2k}\,. \tag{4.40}$$

In the last section we have given further examples of operators in the fermionic sector. For instance, in bosonic language the operator corresponding to $\psi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_p}$ (4.27) is (a linear combination of operators) of the form

$$S^\alpha(z) = \mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}_s \cdot H}(z)\,, \qquad \boldsymbol{\lambda}_s = (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \ldots, \pm\tfrac{1}{2})\,. \tag{4.44}$$

where $\boldsymbol{\lambda}$ is an N-dimensional vector with p components equal to ± 1 and the remaining $N-p$ equal to zero. More generally, we can construct primary fields where $\boldsymbol{\lambda}$ has arbitrary integral components. The corresponding (highest weight) states are then best written as $|\boldsymbol{\lambda}\rangle = \Phi_\lambda(0)|0\rangle$. For instance, $|(\pm 2, 0, \ldots, 0)\rangle = P^{\pm 1}(0)|0\rangle$. On the other hand, in fermionic language, this state is $|(\pm 2, 0, \ldots, 0)\rangle = \psi^{\pm 1}_{-1/2}\psi^{\pm 1}_{-3/2}|0\rangle$.

The G-parity operator (4.29) takes the form

$$G|\boldsymbol{\lambda}\rangle = \exp\left[\mathrm{i}\pi\left(\sum_{k=1}^{n} \lambda_k\right)\right]|\boldsymbol{\lambda}\rangle\,, \tag{4.42}$$

whose eigenvalue is ± 1 depending on whether $\Sigma\,\lambda_i$ is even or odd. Operators describing (anti-) commuting objects are related to vectors $\boldsymbol{\lambda}$ of even (odd) length-squares (one can easily see this from eq. (4.19) upon exchanging z and w).

It is clear that the collection of all operators of the form $\mathrm{e}^{\mathrm{i}\lambda\cdot H}$ is related to a *lattice*. This follows simply from the fact that in any operator product (4.19) the vectors add to give a new one:

$$\Phi_\lambda(z)\cdot\Phi_{\lambda'}(w) = \Phi_{\lambda+\lambda'}(z-w)^{\lambda\cdot\lambda'}(1+\mathrm{O}(z-w))\,. \tag{4.43}$$

Consistency, that is, closure of the operator product algebra (4.3), then requires that the vertex operator $\Phi_{\lambda+\lambda'}$ exists and generates a state $|\boldsymbol{\lambda}+\boldsymbol{\lambda}'\rangle$ of the theory. The lattice is thus formed by the set of vectors $\boldsymbol{\lambda}$ obtained by all possible additions of some given basic vectors. Thus, for theories based on lattices, one can use the information carried by the lattice to characterize the operator product algebra*).

For the lattice generated by bosonizing the fermions, as described above, operator products preserve a natural Z_2-grading, that is, a split into vectors of even and odd length-squares, or equivalently, a split into eigenvalues ± 1 of G_{NS}. More specifically, the lattice characterized above is precisely the weight lattice of D_N generated by the conjugacy classes (0) and (v). In particular, the non-zero roots of D_N $(\pm e_i \pm e_j)$, which appear in the bosonized currents $J^{\pm i \pm j}(z)$ (4.38), as well as the vectors $\boldsymbol{\lambda} = (0, \ldots, \pm 2, \ldots, 0)$, associated with $P^{\pm k}(z)$, belong to the neutral class (0), while the fermions $\psi^{\pm i}$ (as well as all other odd anti-symmetric tensors) belong to (v).

In this framework it is now very easy to describe states in the R-sector. As discussed in the previous section, states in the R-sector are generated via spin fields $S^\alpha(z)$. They have the property that they create square-root branch cuts (4.34) with the NSR-fermions ψ^μ. In bosonic language, they have a particularly simple form: it follows from eq. (4.43) that spin fields have to be associated with lattice vectors $\boldsymbol{\lambda}_s$ with entries $\pm\frac{1}{2}$:

$$S^\alpha(z) = \mathrm{e}^{\mathrm{i}\lambda_s\cdot H}(z)\,, \qquad \boldsymbol{\lambda}_s = (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \ldots, \pm\tfrac{1}{2})\,. \tag{4.44}$$

In fact, there are two spin fields, S^α and $S^{\dot\alpha}$, with opposite D_N helicity belonging to (s) and (c) of D_N; they are distinguished by an even or odd number of $-$ signs in $\boldsymbol{\lambda}_s$, respectively. The number 2^{N-1} of possible sign choices for $\boldsymbol{\lambda}_s$ in each case is precisely the dimension of the two chiral spinor representations of D_N: hence α, $\dot\alpha = 1, \ldots, 2^{N-1}$.

*) In the operator product algebra there also appear fields involving derivatives ∂H, which are not represented on the lattice; however, these play only a trivial role in this context.

Using world sheet supersymmetry to relate the $2N$ fermions ψ^μ to $2N$ bosons X^μ we can check these results with the ground-state energy (4.15). The conformal weights of S^α, $S^{\dot\alpha}$ are $h_s = \frac{1}{2}\boldsymbol{\lambda}_s^2 = \frac{1}{8}N$. The total value of the central charge of the fermions and the bosons is $3N$, so that (4.15) gives us precisely the value h_s.

This bosonic realization of the spin fields means that, in lattice language, the R-sector is described by the spinor conjugacy classes (s) and (c) of D_N. To which particular spinor class a given state $|\boldsymbol{\lambda}\rangle$ belongs is determined by the eigenvalue of G_{R} (4.32) which is, in bosonic language, and up to an overall phase, identical to G_{NS} given in eq. (4.42). Every given conjugacy class can be represented by a particular vector belonging to that class; all other vectors in the same class are obtained by adding the root lattice, i.e., all vectors of the neutral class (0). Thus all states (up to derivatives) in the R-sector are of the form

$$:\mathrm{e}^{\mathrm{i}(\lambda+\lambda_s)\cdot H}:(0)|0\rangle\,, \quad \boldsymbol{\lambda}\in(0) \text{ of } \mathrm{D}_N\,, \tag{4.45}$$

and we can think of the ground states of the R-sector $|\alpha\rangle = S^\alpha(0)|0\rangle$ as being generated by shift vectors $\boldsymbol{\lambda}_s$ of minimal lengths in the spinor conjugacy classes of D_N.

4.4. Lattices and Kac–Moody algebras

The foregoing discussion dealt only with periodic and anti-periodic fermionic boundary conditions, i.e., with Z_2-twisted fermions. We have just seen that these can be described, in bosonic language, in terms of the weight lattice D_N of $\mathrm{SO}(2N)$. One can generalize these concepts to other lattices, for instance to the weight lattices A_N of $\mathrm{SU}(N+1)$, or to lattices that are products of weight lattices of several Lie algebras. (There are of course also lattices that are not related to Lie algebras at all.) Such lattices may be described in terms of free fermions with more complicated boundary conditions combined with generalized spin (or twist) fields. (Alternatively, they may be described by fermions with Thirring interactions. See, e.g., ref. [41] for a discussion of these matters.) These twist fields need not be either fermionic or bosonic. Like the spin fields, such operators are quite difficult to describe in terms of the fermionic variables ψ^μ. In bosonic language, however, all operators have the same (exponentiated) form, and ψ^μ are no more fundamental than S^α. Clearly, the bosonic lattice formulation provides a convenient and elegant description of such objects.

Of course, such theories based on lattices are not the most general conformal field theories, but certainly the simplest ones. More precisely, we consider here only conformal field theories involving *free*, *untwisted* bosons, such that the set of primary fields is given by the operators ∂H and $\mathrm{e}^{\mathrm{i}\lambda\cdot H}$. The descendant fields are simply all possible derivatives of these. Thus any theory of this type is completely characterized by specifying the underlying lattice; in the following, we will mainly consider lattices that are products of Lie algebra lattices, and which therefore have a *finite* number of conjugacy classes.

As mentioned in the last section, there is a close relation between the operator product algebra and vector addition. In general, the full operator product algebra contains infinitely many primary fields. For certain lattice theories, it is however possible to isolate closed, finite sub-algebras of the full operator product algebra that characterize the full theory. They involve operators with lowest conformal weights and thus are related to lattice vectors with smallest lengths. It is well-known (cf. appendix A) that a simply laced Lie algebra $\mathscr{G}$ is characterized by its vectors of norm two, that is, by its roots (we consider here only Euclidean lattices with $\boldsymbol{\lambda}^2 \geq 0$). These are associated with conformal operators of weight one

$$J^{\lambda}(z) = e^{i\lambda\cdot H} c_{\lambda}(z)\,, \qquad \boldsymbol{\lambda}^2 = 2\,, \tag{4.46}$$

where c_λ is a cocycle factor, to be discussed below. These currents characterize the conformal field theory in exactly the same way as the roots characterize the Lie algebra. Together with ∂H_i, $i = 1, \ldots, \operatorname{rank} \mathcal{G}$ they form a closed operator sub-algebra (4.7), which is equivalent to a Kac–Moody algebra $\hat{\mathcal{G}}$ (4.9). The identification (4.46) is the Halpern–Frenkel–Kac–Segal construction of level-one Kac–Moody algebras [42, 43], which works for all simply-laced Lie algebras. In the Cartan–Weyl basis, eq. (4.9) takes the form

$$[(\partial H^i)_m, (\partial H^j)_n] = m\delta^{ij}\delta_{m,-n}\,, \qquad [(\partial H^i)_m, J^{\lambda}_n] = \lambda^i J^{\lambda}_{m+n}\,,$$

$$[J^{\lambda}_m, J^{\lambda'}_n] = \begin{cases} 0\,, & \lambda\cdot\lambda' \geq \ (\boldsymbol{\lambda}+\boldsymbol{\lambda}' \text{ is not a root})\,, \\ \varepsilon(\lambda,\lambda') J^{\lambda+\lambda'}_{m+n}\,, & \lambda\cdot\lambda' = -1\ (\boldsymbol{\lambda}+\boldsymbol{\lambda}' \text{ is a root})\,, \\ \lambda\cdot(\partial H)_{m+n} + m\delta_{m,-n}\,, & \lambda\cdot\lambda' = -2\ (\boldsymbol{\lambda}+\boldsymbol{\lambda}' = 0)\,. \end{cases} \tag{4.47}$$

If one checks the contour integral manipulations analogous to eq. (4.8), one finds that the currents commute properly only if the cocycle factors are chosen to satisfy $c_\lambda c_{\lambda'} = (-1)^{\lambda\cdot\lambda'} c_{\lambda'} c_\lambda$. In order to get a closed algebra one needs furthermore $c_\lambda c_{\lambda'} = \varepsilon(\lambda, \lambda') c_{\lambda+\lambda'}$, where $\varepsilon(\lambda,\lambda')$ can be ± 1. Such cocycle factors can be constructed for all root lattices of simply laced algebras. If one constructs a bosonic representation for $\hat{\mathrm{D}}_N$ currents as in eq. (4.38), the cocycle factors of the currents are obtained directly from those of the free fermions.

Clearly, $(\partial H^i)_0$ represent the generators of the Cartan sub-algebra and J^{λ}_0 the off-diagonal generators of $\mathcal{G}$. Note that this algebra closes into a finite set of conformal operators because the right-hand side of (4.47) vanishes if $\boldsymbol{\lambda}+\boldsymbol{\lambda}'$ is not a root. In contrast, the full operator algebra of a string theory does not necessarily close into a finite set of conformal operators. There are also closed sub-algebras other than Kac–Moody algebras: one example is the world sheet superconformal algebra in fermionic strings.

By using the formula (4.19) for operator products of exponentials, many properties of such conformal field theories on Lie algebra lattices can be determined solely in terms of the addition and multiplication (i.e. inner product) rules of the lattice conjugacy classes. In other words, one can classify the primary fields in terms of Lie algebra conjugacy classes. This feature is quite useful; for instance, the addition rules of conjugacy classes represent superselection rules of the operator product algebra, and the multiplication rules determine the (anti-) commutation properties of the various fields. As will be discussed below, these addition and multiplication rules also determine the modular behavior of the corresponding character functions. For example, in the foregoing section the lattice in question was D_N with the possible conjugacy classes (0), (v), (s) and (c). The twisted boundary conditions for fermions in the R-sector reflect in the branch cut arising from $(z-w)^{\lambda\cdot\lambda'}$ in (4.19) by the multiplication rule $(v)\cdot(s) \in \mathbb{Z} + \frac{1}{2}$. This is nicely illustrated by eq. (4.34), which also reflects the fusion rule $(v) + (s) = (c)$. On the other hand, the operator product expansion (4.36) of two fermions $\psi^i \in (v)$, which contains J^{ij} and P^i in its regular terms, represents $(v) + (v) = (0)$. (Anti-)commutation properties are also easily determined from (4.19) upon exchanging z and w; checking the powers of $z - w$, one finds that while all operators in (0) commute and those in (v) anti-commute, states in the spinor conjugacy classes have (for generic N) no definite (anti-)commutation properties.

The foregoing does not imply that all lattice conjugacy classes actually do occur in a given theory. Rather, a given lattice theory is characterized by the "root" lattice plus a list of the specific conjugacy classes actually present. One condition is of course that the given set of conjugacy classes closes under addition, i.e., that the operator product algebra is consistent, and that it is also local. The latter means that the inner product of any two lattice vectors has to be integral. At quantum level, there are further constraints on the conjugacy classes due to modular invariance of the partition function.

Generically, the one-loop partition function takes the form*)

$$\mathcal{F}(\tau,\bar\tau) = \mathrm{Tr}\, q^{L_0 - c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} = \sum_{i,j} c_{ij}\bar\chi_i(\bar\tau)\chi_j(\tau)\,, \tag{4.48}$$

where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, and c_{ij} are some integral coefficients. The building block χ_i describes the contribution of the ith Verma module, i.e., the contribution of the ith primary field together with its infinitely many descendants. In other words, eq. (4.48) represents a decomposition of the Hilbert space in terms of irreducible representations of the left- and right-moving Virasoro algebras. In general, the foregoing sum extends over infinitely many such building blocks. A simplification occurs if the primary fields can be grouped into representations of certain algebras. This is precisely the case for theories based on Lie algebra lattices with a finite number of conjugacy classes. Then the partition function can be expressed as a finite sum**) over characters $\mathrm{Ch}_{[i]}$, where (i) denotes a conjugacy class and combines the contribution of the infinitely many Verma modules belonging to that class. For theories based on free bosons, $\mathrm{Ch}_{[i]}$ are the character functions of level one Kac–Moody algebras $\hat{\mathrm{G}}$:

$$\mathrm{Ch}^{\mathrm{G}}_{[i]}(\tau) = \mathrm{Tr}_{(i)}\, q^{L_0-c/24} = \frac{1}{\eta^d(\tau)} \sum_{\boldsymbol\lambda\in(i)} q^{\boldsymbol\lambda^2/2}\,. \tag{4.49}$$

The sum runs over all vectors of the d-dimensional weight lattice of $\hat{\mathrm{G}}$ that belong to (i).

Consider, for example, the weight lattices of D_m. As discussed above, they correspond in the fermionic language to theories with Z_2-valued (periodic and anti-periodic) boundary conditions. The $\hat{\mathrm{D}}_m$ characters can thus be represented entirely in terms of the well-known ϑ-functions (see appendix C.6; the first argument of the ϑ-function is omitted here, since in all formulas in this section it is zero):

$$\begin{aligned}
\mathrm{Ch}^{\mathrm{D}_m}_{[0]}(\tau) &= \frac{1}{2}\,\frac{1}{\eta^m(\tau)}\,\{\vartheta_3(\tau)^m + \vartheta_4(\tau)^m\}\,,\\
\mathrm{Ch}^{\mathrm{D}_m}_{[v]}(\tau) &= \frac{1}{2}\,\frac{1}{\eta^m(\tau)}\,\{\vartheta_3(\tau)^m - \vartheta_4(\tau)^m\}\\
\mathrm{Ch}^{\mathrm{D}_m}_{[s]}(\tau) = \mathrm{Ch}^{\mathrm{D}_m}_{[c]}(\tau) &= \frac{1}{2}\,\frac{1}{\eta^m(\tau)}\,\vartheta_2(\tau)^m\,.
\end{aligned} \tag{4.50}$$

In this framework, it is easy to describe the modular properties of the partition function in terms of its constituent characters. The behavior of Lie algebra characters under the modular transformation $\tau\to -1/\tau$ is quite simple [38] (cf. appendix C):

$$\mathrm{Ch}^{\mathrm{G}}_{[k]}(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{(l)} \mathrm{e}^{2\pi\mathrm{i}(k)\cdot(l)}\,\mathrm{Ch}^{\mathrm{G}}_{[l]}(\tau)\,, \tag{4.51}$$

*) The constants $c/24$ are the zero-point energy subtractions, analogous to the "-1" for the light-cone Hamiltonian of the bosonic string (which has $c=24$). If one would not subtract these constants, the partition function would not have good modular properties. For example for the bosonic string this constant ensures that the partition function can be expressed in terms of η-functions (see section (3.2)).

**) Such theories made out of a finite number of building blocks are called *rational* conformal field theories in the literature. The rational torus compactifications discussed in section (2.3) are examples of rational conformal field theories.

where the sum runs over all possible N (=the order of the center) conjugacy classes (defined by the dual of the root lattice), and the inner product represents their multiplication rules. Basically, eq. (4.51) is nothing but a Fourier transformation in the space of conjugacy classes.

It follows from eq. (4.51) that characters of theories with the same fusion rules behave in the same way; for instance, the characters of A_1 and E_7 (or A_2 and E_6 etc.) behave identically under $\tau \to -1/\tau$. (The "euclideanization" trick in appendix A.4 is based on this). For D_m ($m = 4k$, $k \in \mathbb{Z}$) we get

$$
\begin{aligned}
\mathrm{Ch}^{D_m}_{[0]}(-1/\tau) &= \tfrac{1}{2}\{\mathrm{Ch}^{D_m}_{[0]}(\tau) + \mathrm{Ch}^{D_m}_{[v]}(\tau) + \mathrm{Ch}^{D_m}_{[s]}(\tau) + \mathrm{Ch}^{D_m}_{[c]}(\tau)\} \\
\mathrm{Ch}^{D_m}_{[v]}(-1/\tau) &= \tfrac{1}{2}\{\mathrm{Ch}^{D_m}_{[0]}(\tau) + \mathrm{Ch}^{D_m}_{]v]}(\tau) - \mathrm{Ch}^{D_m}_{[s]}(\tau) - \mathrm{Ch}^{D_m}_{[c]}(\tau)\} \\
\mathrm{Ch}^{D_m}_{[s]}(-1/\tau) &= \tfrac{1}{2}\{\mathrm{Ch}^{D_m}_{[0]}(\tau) - \mathrm{Ch}^{D_m}_{[v]}(\tau) + \mathrm{Ch}^{D_m}_{[s]}(\tau) - \mathrm{Ch}^{D_m}_{[c]}(\tau)\}\,,
\end{aligned}
\tag{4.52}
$$

so that e.g. $\mathrm{Ch}^{D_m}_{[0]}(\tau) + \mathrm{Ch}^{D_m}_{[s]}(\tau)$ transforms into itself.

At this stage we can make contact between the lattice partition functions of section (3.3) (where we considered compactification of the bosonic string) and the discussion here. In a lattice-compactified theory, the partition function is given by a lattice sum. For Lie algebra lattices, this sum can also be expressed as a finite sum over the characters of the various conjugacy classes that are present on the lattice. Modular transformations mix these characters in the manner described above. The theory is invariant under $\tau \to -1/\tau$ precisely if the choice of conjugacy classes is such that the lattice is self-dual.

In table 1, we summarize the corresondence between lattices and conformal field theories (the table was inspired by ref. [4]).

4.5. *Conformal field theory and torus compactification*

To make all this more explicit, and to clarify the relation with the results of sections 2 and 3, consider, for example, a torus compactification of the 26-dimensional closed bosonic string on an 8-dimensional even Lorentzian self-dual lattice $(E_8)_L \times (E_8)_R$. This is the same as $(D_8)_L \times (D_8)_R$ with conjugacy classes $((0) + (s); (0) + (s))$. Accordingly, the contribution of the compactified bosons to the partition function (cf. eq. (3.12)) can be written in terms of D_8 characters:

Table 1
Translation table between lattice and conformal field theory language.

Lattice	Conformal field theory
dimension d	central charge c
lattice point $\boldsymbol{\lambda}$	primary field Φ_λ
length of $\boldsymbol{\lambda}$	conformal weight
vector addition	operator product
roots	Kac–Moody currents
conjugacy classes	Hilbert space sectors
even lattice points	commuting operators
odd lattice points	anti-commuting operators
integral	meromorphic
self-dual	invariant under $\tau \to -1/\tau$
even	invariant under $\tau \to \tau + 1$

$$\mathscr{F}(\tau, \bar\tau) = \frac{1}{\bar\eta^8(\bar\tau)\eta^8(\tau)} \sum_{(\boldsymbol\lambda_L;\boldsymbol\lambda_R)\in E_8\times E_8} \bar q^{\boldsymbol\lambda_L^2/2} q^{\boldsymbol\lambda_R^2/2} \equiv \mathrm{Ch}^{E_8}_{[0]}(\tau)\cdot \mathrm{Ch}^{E_8}_{[0]}(\bar\tau)\,,$$
$$\mathrm{Ch}^{E_8}_{[0]}(\tau) = \mathrm{Ch}^{D_8}_{[0]}(\tau) + \mathrm{Ch}^{D_8}_{[s]}(\tau) = \frac{1}{2}\,\frac{1}{\eta^8(\tau)}\,\{\vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau)\}\,. \tag{4.53}$$

It is easy to discuss the conformal field theory of this model in terms of operators like the Kac–Moody currents $J^{ab}(\bar z)$, $J^{a'b'}(z)$ of $(E_8)_L \times (E_8)_R$. The currents belonging to (0) of the subgroup $D_8 \sim SO(16)$ are precisely given by (4.38) and (4.39); the currents (4.39) belonging to the Cartan sub-algebra correspond to Kaluza–Klein states, the off-diagonal ones to winding states, i.e., solitons. The remaining currents belonging to (s), i.e., to E_8/D_8, are realized by spin fields (4.44) of D_8. This is possible since a spinor weight of D_8 has length-square two, so that it can represent a root vector.

The massless sector of the conformal field theory describing the compactified bosonic string theory above is characterized by the vertex operators

$$V^{ab}_\mu(\bar z, z) = J^{ab}(\bar z)\partial X_\mu(z)\,e^{ik\cdot X}\,, \quad \mu = 1,\ldots,18\,, \tag{4.54}$$

which describe gauge vector bosons belonging to the adjoint representation of the left-moving E_8, whereas

$$V^{a'b'}_\mu(\bar z, z) = J^{a'b'}(z)\bar\partial X_\mu(\bar z)\,e^{ik\cdot X} \tag{4.55}$$

describes the emission of a vector boson of the right-moving E_8. Furthermore,

$$V_{\mu\nu}(\bar z, z) = \bar\partial X_\nu(\bar z)\partial X_\mu(z)\,e^{ik\cdot X} \tag{4.56}$$

describes the emission of gravitons, anti-symmetric tensor fields and dilatons in eighteen dimensions, and

$$V^{abc'd'}(\bar z, z) = J^{ab}(\bar z)J^{c'd'}(z)\,e^{ik\cdot X} \tag{4.57}$$

represents 496×496 massless scalars. These vertex operators are conformal fields of weight one only if $k^\mu k_\mu = 0$, and if they are multiplied with polarization vectors satisfying certain transversality conditions. For example the polarization tensors ξ^μ for vector emission vertices must satisfy $k_\mu\xi^\mu = 0$.

According to the fermion–boson equivalence in two dimensions, in order to describe the same theory we could also start with a fermionic version of the foregoing objects. The D_8 currents are then represented by bilinears as in eq. (4.23), however, the currents represented by the spin fields have a very complicated, transcendental dependence on the fermions [44]. In this formulation, there is no trace of higher dimensions left.

We have seen that certain conformal theories can be described in terms of free compact bosons. It is sometimes possible to interpret these as torus-compactified higher dimensional coordinates. The example above represents a compactified theory in the literal sense, and this is true for the entire class of compactified bosonic strings discussed in section 2, provided that one includes a background $B_{\mu\nu}$-field. There exist however more general conformal field theories based on lattices, which do not admit any compactification interpretation. We will encounter many such theories when we discuss four-dimensional heterotic strings. One should therefore not confuse *compact* bosons with *compactified space–time coordinates* (which are a special case). From the two-dimensional point of view, the distinction between compactified and other internal degrees of freedom is irrelevant and misleading; the subclass of conformal field theories which have a compactified space–time interpretation is not essentially different from the generic case, for which such an interpretation is difficult or impossible. In the next section we will apply the ideas discussed here to bosonic lattice descriptions of heterotic strings, which in general do not admit such a compactification interpretation.

5. The NSR model, superstrings and heterotic strings

5.1. The covariant NSR model in bosonic lattice formulation

We will first introduce the covariant ten-dimensional NSR-model [45] (the *spinning string*) in bosonic language, which will allow a straightforward generalization to lower dimensions.

The covariant NSR-model can be described by a specific conformal field theory whose main variables are ten fermions ψ^μ and ten bosons X^μ, which play the role of space–time coordinates. There is a local $N=1$ world sheet supersymmetry relating X^μ to ψ^μ, generated by a supercurrent $T_F = \mathrm{i}\partial X^\mu \psi_\mu$, so that we have a superconformal field theory. The basic conformal operators are $\partial X^\mu(z)$ and $\psi^\mu(z)$, whose properties have been discussed already in the previous section. Since these fields depend *analytically* on z, we deal only with right-moving fields. What we are eventually after is however *closed*, and in particular, heterotic string theories. Thus we will have to pair the right-moving NSR-model with an a priori independent left-moving ($\bar{z}$-dependent) conformal field theory. This will be explained further below.

On a trivial background, and in superconformal gauge, the action takes the free form

$$S_{\mathrm{NSR}} \propto \int \mathrm{d}^2 z \, (\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \mathscr{L}_{\mathrm{ghost}}) \,. \tag{5.1}$$

We denote by c and b the (anti-commuting) ghost and anti-ghost fields for fixing conformal invariance, and by γ and β the (commuting) ghost and anti-ghost fields for fixing the fermionic part of superconformal invariance. Then the ghost Lagrangian is given by

$$\tfrac{1}{2} \mathscr{L}_{\mathrm{ghost}} = b \bar{\partial} c + \beta \bar{\partial} \gamma \,. \tag{5.2}$$

As indicated above, the part of the theory consisting of ∂X^μ and ψ^μ by itself suffers from severe deficiencies. For example, ψ^μ appears in the emission vertices for space–time vector bosons, but being a two-dimensional spinor, ψ^μ naturally anti-commutes and thus the vertex operator (4.28) apparently has the wrong space–time spin statistics. Another difficulty is that in order to describe the R-sector, we had to introduce the spin field S^α. This field has conformal weight $h = \frac{1}{8} N = \frac{5}{8}$ and not $h = 1$, which would be needed for a vertex operator describing massless space–time fermions. When inserted at some point on the complex plane, S^α creates a branch cut (4.34) in order to change the boundary conditions of ψ^μ. (In bosonic formulation, this is related to the rule $(v)\cdot(s) \in \mathbb{Z} + \frac{1}{2}$ for the conjugacy classes of D_5, which is the Lie algebra of the Wick-rotated Lorentz group in ten dimensions). But the very presence of this cut makes the theory non-local. Another problem we encounter is that S^α does not anti-commute with itself as it should as a space–time fermion. This fact is reflected by the rule $(s)\cdot(s)\in 2N+\frac{5}{4}$, $N\in\mathbb{Z}$ in D_5, which also reveals the presence of even worse cuts.

It is important to notice that many of these and other unpleasant features can be traced back to properties of the various conjugacy classes of the lattice D_5, if one uses the bosonic description of the ψ^μ. We will see that a resolution lies in the modification of this lattice.

The only new ingredient of the covariant NSR-model (5.1) compared to the conformal field theory of X^μ and ψ^μ as described in the last section is the ghost system. It is in fact a crucial ingredient, which can cure all above-mentioned diseases. The ghosts are conformal fields obeying $b(z)\cdot c(w)=c(w)\cdot b(z)=1/(z-w)$, $\beta(z)\cdot\gamma(w)=-\gamma(w)\cdot\beta(z)=-1/(z-w)$. Their conformal weights are

$$\begin{aligned} &c\colon\ h=-1\,, \qquad b\colon\ h=2\,,\\ &\gamma\colon\ h=-\tfrac{1}{2}\,, \qquad \beta\colon\ h=\tfrac{3}{2}\,. \end{aligned} \tag{5.3}$$

Their contribution to the full stress–energy tensor is $T_{\mathrm{gh}}=c\partial b+2\partial cb-\frac{1}{2}\gamma\partial\beta-\frac{3}{2}\partial\gamma\beta$ [30]. Substituting this into eq. (4.5), we learn that the contribution to the central charge of the b, c system is $c_{bc}=-26$, and similarly $c_{\beta\gamma}=11$; thus altogether we have $c_{\mathrm{ghost}}=-15$ and this cancels precisely the contribution of the ten X^μ plus ten ψ^μ. So the conformal anomaly disappears, which is crucial for the consistency of the theory. This does of course *not* imply that the string theory can only be consistent in ten dimensions. Rather, what counts here is only the *total* value of $c_{\mathrm{matter}}=15$, and not any particular decomposition in terms of contributions of X^μ and ψ^μ. Allowing for arbitrary internal degrees of freedom, which carry "missing" central charge, will allow us later to construct string theories in lower dimensions than ten that are not necessarily compactifications of ten-dimensional theories.

As we intend to discuss only the spectrum of the theory and not other aspects (like correlation functions), we will explain only aspects of the ghost system relevant here. The role of the b, c system is then quite simple: for our purposes, all that the b, c system does is to provide the correct vacuum energy. In section 5.2 we described how the states of the theory are created by acting with conformal fields on some vacuum. We did however not specify the vacuum itself. The correct vacuum (of the bosonic string) is $|0\rangle=c(0)|0\rangle_{\mathrm{SL}(2,\mathbb{C})}$, where $|0\rangle_{\mathrm{SL}(2,\mathbb{C})}$ is the true $\mathrm{SL}(2,\mathbb{C})$ invariant vacuum of the theory, which is annihilated by L_n, $n\geq -1$. (Another way of describing the same situation is to redefine the vertex operators of the theory by replacing V by cV; these objects then act on the $\mathrm{SL}(2,\mathbb{C})$ invariant vacuum state.) Since c has weight $h=-1$, the mass of $|0\rangle$ is given by $\frac{1}{2}m^2=-1$; it represents the tachyon state of the bosonic string. Note that in this formalism the action of the ghost field $c(0)$ produces the familiar zero-point mass shift -1 of the bosonic string. Massless states are built upon $|0\rangle$ using conformal fields of weight $h=1$. This gives us back the mass formula of the bosonic string (2.16) (where $\mathcal{N}$ counts the number of derivatives in a vertex operator).

In fermionic strings, we also have the commuting ghosts β, γ. Their role is much more involved. As we cannot discuss all issues in great detail, we will just give recipes for handling these ghosts (details can be found in ref. [30]). It is very convenient to bosonize these fields*)

$$\beta=\mathrm{e}^{-\mathrm{i}\phi}\,\partial\xi\,, \qquad \gamma=\mathrm{e}^{\mathrm{i}\phi}\,\eta\,. \tag{5.4}$$

*) One can also bosonize the b, c system. However their boundary conditions are always periodic, and as we will see later, the corresponding bosons would play no rôle in the lattice description.

Here, ϕ is a chiral boson with negative norm, $\phi(z)\cdot\phi(w)=\log(z-w)$, and ξ and η are auxiliary fermions, which will play no role in the following. They are needed in eq. (5.4) since one cannot bosonize the commuting fields β, γ solely in terms of an anti-commuting exponential of one boson.

One important feature of the ϕ-ghost is that the conformal field $e^{iq\phi}$ has a peculiar conformal weight:

$$T_{\beta\gamma}(z)\cdot e^{iq\phi}(w)=-\tfrac{1}{2}q(q+2)\,\frac{e^{iq\phi}(w)}{(z-w)^2}+\cdots\quad\rightarrow\quad h=-\tfrac{1}{2}q(q+2)\,. \tag{5.5}$$

The linear term $(-q)$ originates from an anomaly of the ghost number current $j=\beta\gamma$, and the negative sign is due to the negative metric of ϕ, which also implies

$$e^{iq\phi}(z)\cdot e^{ip\phi}(w)=(z-w)^{-qp}\,e^{i(q+p)\phi}(w)+\cdots\,. \tag{5.6}$$

Note that h-values in eq. (5.5) are not bounded from below, which shows that the β–γ system is not a unitary conformal field theory. Therefore we cannot rely on our experience with unitary theories to understand it. We do not know of a derivation of the properties of this system that we find entirely satisfactory. Most of the knowledge about the way this system works has been gained through trial and error, rather than by systematic derivation. Nevertheless, enough work has been done to check that what we do know is internally consistent.

Perhaps the most useful guiding principle is the old light-cone formalism of the NSR-model, which we should be able to reproduce. The natural starting point is to try to find the ground states of the Neveu–Schwarz and Ramond sector, which should have $\frac{1}{2}m^2=-\frac{1}{2}$ and $\frac{1}{2}m^2=0$ respectively. Since we just saw that the reparametrization ghosts contribute to the mass of the ground states, it is not unreasonable to expect the superghosts to contribute as well.

An important difference between the bc system and the $\beta\gamma$ system is that the former fields anti-commute, whereas the latter commute. It is easy to see that by acting with b and c oscillators one cannot lower the energy of the SL(2, $\mathbb{C}$) invariant vacuum beyond -1, since fermionic oscillators can act at most once. The bosonic oscillator modes of β and γ can however act an infinite number of times. Since there are modes that lower the energy (e.g. $\gamma_{1/2}$ in the NS-sector), it can be lowered to $-\infty$ by successive action of these oscillators, as one can also observe in eq. (5.5).

In view of this it is not too surprising that the correct ground state is not obtained by acting on the SL(2, $\mathbb{C}$) vacuum with, for example, $\gamma(0)$. Instead it turns out that we should consider (for reasons that will become a bit clearer in a moment) a set of states characterized by a certain *Bose ghost sea charge* q:

$$|q\rangle=e^{iq\phi}(0)|0\rangle\,. \tag{5.7}$$

To specify the possible values of q, note first that the $\beta\gamma$ ghosts are related to gauge degrees of the (non-dynamical) world sheet gravitino, hence should have the same periodicity properties. Since the gravitino couples to the supercurrent $T_F=i\psi\partial X$, it follows that because of world sheet supersymmetry the $\beta\gamma$ ghosts should have the same boundary conditions as the NSR-fermions ψ^μ (as ∂X is periodic). This implies that we need *different* vacua in the NS- and R-sectors of ψ^μ: in the NS-sector $q\in\mathbb{Z}$ while in the R-sector $q\in\mathbb{Z}+\frac{1}{2}$. States obtained from the $|q\rangle$ vacua by bosonized NSR operators $e^{i\lambda\cdot H}$, can be obtained from the SL(2, $\mathbb{C}$) invariant vacuum by acting with $V_q(\boldsymbol{\lambda})=e^{i\lambda\cdot H}\,e^{iq\phi}$. The fact that q is quantized differently for the NS- and R-sectors can be summarized elegantly using a trick, invented in ref. [6]: just consider the ghost charge q as part of an enlarged *covariant* lattice. That is, associate to any of the aforementioned (and all other possible) bosonized vertex operators (i.e., states) six-dimensional vectors $\boldsymbol{w} = (\boldsymbol{\lambda}|q)$. This lattice Γ_6 looks rather like an extension $D_5 \to D_6$, but we have to take into account the switched sign in the ghost metric. The natural inner product on this lattice is defined by the power of $z - w$ in operator product expansions, because that determines the various (anti-)commutation properties of the fields and measures locality. From (4.19) and (5.6) we have

$$V_\lambda\, e^{iq\phi}(z) \cdot V_{\lambda'}\, e^{iq'\phi}(w) = (z - w)^{\lambda\cdot\lambda' - qq'} V_{\lambda+\lambda'}\, e^{i(q+q')\phi}(w) + \cdots . \tag{5.8}$$

Therefore the natural inner product has *Lorentzian* metric $(+ + + + +|-)$:

$$\langle \boldsymbol{w}, \boldsymbol{w}' \rangle = \lambda \cdot \lambda' - qq' . \tag{5.9}$$

The same inner product appears also in the mass formula for arbitrary ghost charges:

$$\tfrac{1}{2}m^2 = \tfrac{1}{2}\langle \boldsymbol{w}, \boldsymbol{w} \rangle - q + \mathcal{N} - 1 . \tag{5.10}$$

Accordingly, we denote the Lorentzian lattice by $\Gamma_{5,1} = D_{5,1}$, with the comma indicating the sign flip in the metric.

The operator product (5.8) represents a *Lorentzian* algebra (also denoted by $D_{5,1}$). Because of the Lorentzian metric, the operator algebra does not close on a finite set of conformal operators. In particular, there exists an infinite number of operators with the same weight, which roughly correspond to the various ghost pictures explained below. This is in contrast to Euclidean lattices, and a reflection of the fact that the superconformal field theory of the ghost system is not unitary.

There exists now still an infinite ambiguity in what values of q we actually choose. Out of the fields X^μ, H_k (the bosons replacing ψ^μ) and the ghost field ϕ (not to mention the auxiliary fields ξ and η) one can construct a huge set of vertex operators, that vastly outnumbers the known set of light-cone states of the NSR-model. This set is reduced by superconformal invariance conditions (most conveniently expressed by requiring that the physical vertex operators commute with the BRST-operator), but it turns out that one still has an infinite number of vertex operators for any physical state, namely one for every value of q. That is, there is a copy of every vertex operator in every ghost sector; the different copies are also called *pictures* [30]. There exists an operation which maps a given vertex operator with ghost charge q into one with $q + 1$:

$$V_{q+1}(z) = {:}\mathcal{P}_{+1} V_q{:}(z) . \tag{5.11}$$

This *picture changing operator* [30] carries one unit of ghost charge, and is given by

$$\mathcal{P}_{+1}(z) = e^{i\phi} T_F(z) = i\, e^{i\phi}\, \psi \cdot \partial X(z) + \cdots . \tag{5.12}$$

Thus, apart from the ghost part, picture changing is more or less the same as performing a two-dimensional supersymmetry transformation*⁾.

Upon bosonization, we can represent $\mathcal{P}_{+1}$ by the vector $\boldsymbol{w}^{PC} = (0, \ldots, \pm 1, 0, \ldots |{+1}) \in D_{5,1}$.

*⁾ In fact, the ghost part is necessary in order to have a well-defined local action of T_F on states in the R-sector; remember the branch cut of ψ in T_F. Roughly speaking, one may regard $e^{i\phi}$ as a bosonized anti-commuting coordinate, playing the role of ε in $[\varepsilon Q^{SUSY}, \ldots]$.

Picture changing then acts on the lattice(-part of the theory) simply as shifting by w^{PC}. In order for this operation to be well-defined and local, w^{PC} has to have integral inner product with all other lattice vectors. Although trivially satisfied in the ten-dimensional NSR-model, this condition will prove quite non-trivial for the lower dimensional theories we shall consider later, as for these theories the supercurrent T_F (i.e. w^{PC}) can take a different form. This will be discussed in section 6.

Although one could in principle describe the properties of the NSR-model in any picture, there is one choice of the ghost charge q which is more convenient than all others. In this *canonical* ghost sector the description resembles the old light-cone NSR-formalism most closely. Consider first the NS ground state. A reasonable *ansatz* for its vertex operator is $e^{ik\cdot X}\, e^{iq\phi}$, which is supposed to create the NS-tachyon from the state $|0\rangle = c(0)|0\rangle_{SL(2,\mathbb{C})}$. If one requires the operator to have this form, there is just one possible value of q, namely $q=-1$. It is easy to check that for $q \geq 0$ the operator is not BRST invariant, whereas for $q \leq -2$ it vanishes when the picture changing operator acts on it, indicating that it contains no physical states. Thus the canonical NS-vacuum must be the state $|-1\rangle$. Having identified the NS-vacuum, we can easily construct all other states in this sector. They are simply created in the usual way, by primary fields constructed out of the bosons X^μ's and fermions ψ^μ on this vacuum, as in eq. (4.27).

It is customary and convenient to include the ϕ-ghost exponentials in the vertex operators, which act on the vacuum $|0\rangle$, as we did for the tachyon emission vertex above. The vector boson emission vertex is, for example

$$V^{\text{vector}}(z) = \xi_\mu \psi^\mu\, e^{-i\phi}\, e^{ik\cdot X}(z)\,. \tag{5.13}$$

In bosonic language the vertex operators in this picture are represented by $D_{5,1}$ lattice vectors of the form $w = (\boldsymbol{\lambda}|-1)$, where $\boldsymbol{\lambda}$ is a D_5 weight in the conjugacy classes (0) or (v). For example for the two lowest states we have discussed above one gets

$$w^{\text{vector}} = (0,\ldots,\pm1,\ldots,0,\ldots|-1)\,, \qquad w^{\text{tachyon}} = (0,0,0,0,0|-1)\,. \tag{5.14}$$

By eq. (5.10) the mass of such states is given by

$$\tfrac{1}{2}m^2 = h - 1\,, \tag{5.15}$$

$$h = \tfrac{1}{2}\boldsymbol{\lambda}^2 + \mathcal{N} + \tfrac{1}{2}\,. \tag{5.16}$$

This is the correct mass formula for the NS-states. Note for example that due to the ghost contribution the vector boson emission vertex (5.13) has conformal weight $h=1$ for $k^\mu k_\mu = 0$, rather than $h=\frac{1}{2}$. It is thus the emission vertex for a massless state. At the next level, one gets operators like $V^{\text{tensor}} = \xi_{\mu\nu}\psi^\mu\psi^\nu\, e^{-i\phi}\, e^{ik\cdot X}$, with mass $\frac{1}{2}m^2 = \frac{1}{2}$.

In the canonical picture, there is rather appealing similarity between modifications of the vacuum due to the bc ghosts and the $\beta\gamma$ ghosts. If we bosonize the bc ghost as $c = e^{-i\sigma}$, $b = e^{i\sigma}$, the vacuum state $|-1\rangle$ has the following relation with the SL(2, $\mathbb{C}$)-invariant vacuum

$$|-1\rangle = e^{-i\sigma}(0)\, e^{-i\phi}(0)|0\rangle_{SL(2,\mathbb{C})}\,. \tag{5.17}$$

Alternatively, one can try to re-express this vacuum state in terms of β and γ by "inverting" the bosonization (5.4). This is not at all straightforward, but in ref. [46] it was concluded that one should identify $e^{-i\phi}$ with $\delta(\gamma)$. Hence we have

$$|-1\rangle = \delta(\gamma(0))c(0)|0\rangle_{\mathrm{SL}(2,\mathbb{C})} \,. \tag{5.18}$$

Since a δ-function of a fermionic field, such as c, is the field itself, it is apparent that the differences between the two ghost systems can be attributed mainly to their different statistics. Finally we note that among the states $|q\rangle$ the canonical vacuum has the unique property of being annihilated by all positive modes of β and γ [34].

Knowing the vertex operators in the (-1)-picture, one can calculate the vertex operators in all other pictures with $q \geq 0$ by means of the picture changing operator. For instance, picture changing the tachyon and vector emission vertices in the (-1)-picture (5.13) leads to their copies in the (0)-picture:

$$\begin{aligned} &:\mathcal{P}_{+1} V^{\text{tachyon}}_{(-1)}:(z) \equiv V^{\text{tachyon}}_{(0)}(z) = \mathrm{i}k \cdot \psi \, e^{ik\cdot X}\,(z)\,, \\ &:\mathcal{P}_{+1} V^{\text{vector}}_{(-1)}:(z) \equiv V^{\text{vector}}_{(0)}(z) = \xi \cdot [\partial X + \mathrm{i}\psi(k\cdot\psi)]\, e^{ik\cdot X}(z)\,. \end{aligned} \tag{5.19}$$

In this picture the vector emission vertex is (apart from the ψ-dependent piece) the same as the vector emission vertex of the bosonic string (4.54). One easily checks that the mass formula (5.10) gives the same answer for both $V^{\text{vector}}_{(-1)}$ and $V^{\text{vector}}_{(0)}$. This already follows from the fact that the conformal weight h of $\mathcal{P}_{+1}(z)$ vanishes. In general, $\mathcal{P}_{+1}$ always acts in a way that compensates the effect of changing the q-charge. Thus the mass spectrum of the theory is bounded from below, a subtle feature which is not at all manifest in the mass formula (5.10).

The discussion of the R-sector is very similar. A reasonable *ansatz* for the fermion emission vertices for both chiralities is

$$V^{\text{spinor}(s)} = u_\alpha S^\alpha \, e^{iq\phi} \, e^{ik\cdot X}\,, \qquad V^{\text{spinor}(c)} = u_{\dot\alpha} S^{\dot\alpha} \, e^{iq\phi} \, e^{ik\cdot X}\,. \tag{5.20}$$

By arguments similar to the ones used for the NS-sector, one finds now that this is a good vertex operator for $q = -\frac{1}{2}$ (there is a second solution, $q = -\frac{3}{2}$, which is physically equivalent). We conclude that the ground state must have $q = -\frac{1}{2}$. In this picture all other Ramond sector vertex operators are of the form $V_{\mathrm{R}}\, e^{-i\phi/2}$, where V_{R} is a spin field in the D_5 sector. Note that there appears also a *ghost spin field* in this vertex operator. In bosonic language, the states created by these vertex operators have lattice momenta $(\boldsymbol{\lambda}|-\frac{1}{2})$, where $\boldsymbol{\lambda} \in (s)$ or (c) of D_5. For the ground states one finds

$$w^{\text{spinor}} = (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}|-\tfrac{1}{2})\,. \tag{5.21}$$

The mass formula has the same form as (5.15), but with

$$h = \tfrac{1}{2}\boldsymbol{\lambda}^2 + \mathcal{N} + \tfrac{3}{8}\,.$$

Therefore V^{spinor} is an operator of weight $h = 1$, and describes therefore the emission of massless particles in ten dimensions. Thus the ghost contribution cures one of the problems we mentioned earlier.

Again one can move to different pictures by applying the picture changing operator. Just as in the

NS-sector, one finds the structure of vertex operators in the $q=0$ picture is more complicated than that of eq. (5.13) (and even more complicated in higher pictures). In general, the canonical ghost sectors have the virtue of being simplest in the sense that the vertex operators are related to lattice vectors in the most simple way. In the other ghost sectors, there is no manifest relation between lattice vectors and physical vertex operators; they are in general given by more complicated linear combinations. For our purposes, however, these complications are irrelevant, as features like locality or (anti-)commutativity do not depend on them. That is, all relevant information about state classification is carried by the lattice vectors (oscillators also play a trivial role). We will therefore represent the spectrum of vertex operators in all ghost sectors by the lattice $D_{5,1}$. This certainly does *not* mean that the partition function of the physical states is given by the partition function of the lattice. The relation between lattice vectors and physical states is by no means one-to-one*). There exists however a method to count the physical degrees of freedom correctly: we split every lattice vector $w=(\boldsymbol{\lambda}|q)\in D_{5,1}$ into a four dimensional "light-cone" (describing the Wick-rotated transverse Lorentz group SO(8)) and a two-dimensional "longitudinal plus ghost" part, according to the decomposition

$$\begin{aligned} &D_{5,1}=D_4^{\text{light-cone}}\times D_{1,1}^{\text{ghost}}\,,\\ &\boldsymbol{w}=(\boldsymbol{u},\boldsymbol{x})\,,\qquad \boldsymbol{u}\in D_4^{\text{light-cone}}\,,\qquad \boldsymbol{x}\in D_{1,1}^{\text{ghost}}\,. \end{aligned} \tag{5.22}$$

Then the physical states are those associated with the following fixed entries in $D_{1,1}$:

$$\boldsymbol{x}=\boldsymbol{x}_0=\begin{cases}(0|-1)\,, & \text{NS-sector}\,,\\ (-\tfrac{1}{2}|-\tfrac{1}{2})\,, & \text{R-sector}\,.\end{cases} \tag{5.23}$$

This *physical state selection rule* [47, 36] is just a projection to light-cone states, in the canonical pictures. For the choice (5.23) of $\boldsymbol{x}_0$, the mass formula (5.10) turns precisely into the light-cone mass formula.

It is not difficult to see that states associated with $(\boldsymbol{u},\boldsymbol{x}_0)$ are precisely the light-cone states obtained by acting with transverse NS- and R-oscillators on the usual vacua. The light-cone components of the NSR-fermions form a unitary conformal field theory, and therefore we can appeal directly to the discussion of bosonization in section 4.3. In particular it follows that there is an exact correspondence between on the one hand the Fock space created by the modes of the fermions, and on the other hand states with momenta on the D_4 weight lattice, plus bosonic H_k oscillator excitations. To apply the result to the lattice $D_{5,1}$ we just have to remember that the ground states have a non-vanishing lattice momentum:

$$\boldsymbol{w}_0^{\text{NS}}=(0,0,0,0,0|-1)\,,\qquad \boldsymbol{w}_0^{\text{R}}=(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},-\tfrac{1}{2}|-\tfrac{1}{2})\,, \tag{5.24}$$

where for the R-sector we have chosen a fixed fifth entry in order to project on D_4 spinors. The bosonized light-cone components of the NSR-fermions act only on the first four components of the $D_{5,1}$ vectors, and shift the ground state momenta (5.24) by D_4 weights. Thus one gets precisely the states indicated in eqs. (5.22) and (5.23) (plus, of course, bosonic oscillator excitations).

*) For example, $\boldsymbol{w}^{\text{vector}}$ (5.14) associated with $V_{(-1)}^{\text{vector}}$ given in eq. (5.13) describes ten degrees of freedom (which is the correct count), while $\boldsymbol{w}^{\text{vector}}=(0,\ldots,0,\pm1,0,\ldots,0,\pm1,0,\ldots,0|0)$ associated with the picture changed, equivalent vertex operator $V_{(0)}^{\text{vector}}$ (5.19) has 40 components.

In the other ghost pictures one cannot read off the Lorentz representations of the states in such a straightforward way, although all states are present in all pictures. It is already clear for the tachyon vertex operator that the identification of states in the $q=0$ picture is less straightforward than for $q=-1$. This operator is basically the identity operator for $q=-1$, but contains a ψ^{μ} factor for $q=0$.

One can formulate the foregoing rules also in conjugacy class language: under the decomposition (5.22), the light-cone physical states are those associated with (v) or (s) of $\mathrm{D}_{1,1}^{\text{ghost}}$.

Having now everything reduced to the properties of the lattice $\mathrm{D}_{5,1}$, we split the set of vertex operators into cosets or conjugacy classes, much as we did for D_N in section (4.3). In particular, as is obvious from (5.14), V^{vector} belongs to (0) of $\mathrm{D}_{5,1}$, V^{tachyon} and V^{tensor} to (v) and V^{spinor} to (s) or (c). Any other vertex operator of the theory belongs to one of these four classes, and their lattice parts can be obtained by adding "root" vectors belonging to the neutral class (0) to the vectors (5.14); these are simply ground state (lowest mass) representatives of the conjugacy classes (note in particular that the action of the picture changing operator $\mathscr{P}_{+1}$ belonging to (0) does not change the class of a vertex operator).

Then, as with D_n, we can represent the (anti-)commutation relations, fusion rules and non-localities of the NSR-model in terms of addition and multiplication rules of the conjugacy classes of $\mathrm{D}_{5,1}$ (the dot denotes the Lorentzian inner product, and $i=v,\ s,\ c$):

$$\begin{aligned}
&(0)+(0)=(0)\,, && (0)\cdot(0)\in 2\mathbb{Z}\,,\\
&(0)+(i)=(i)\,, && (0)\cdot(i)\in \mathbb{Z}\,,\\
&(i)+(i)=(0)\,, && (i)\cdot(i)\in \mathbb{Z}\,,\\
&(i)+(j)=(k)\,, && (i)\cdot(j)\in \mathbb{Z}+\tfrac{1}{2}\quad (i\neq j\neq k)\,.
\end{aligned}\tag{5.25}$$

Note that all length(-squares) are integral, but not even; $\mathrm{D}_{5,1}$ is therefore an *odd* lattice. The list (5.25) tells us that operators in (0) (even lengths) commute, those in (s), (c) (odd lengths) anti-commute with each other. This is precisely what we like to have, since (0) contains odd rank, anti-symmetric tensors, while (s) and (c) contain spinors in space–time. Furthermore, $V^{\text{vector}}\in(0)$ and $V^{\text{spinor}}\in(s)$ are now local with respect to each other: the branch cut in $\psi(z)\cdot S(w)$ is cancelled by that of $\mathrm{e}^{-\mathrm{i}\phi}(z)\cdot\mathrm{e}^{-\mathrm{i}\phi/2}(w)$.

However, we are not yet completely finished: the even-rank anti-symmetric space–time tensors belonging to (v) anti-commute and thus violate spin-statistics, and they are also non-local with respect to the spinors (as can be inferred from the $\frac{1}{2}$'s in eq. (5.25)). Finally, the class (v) contains the tachyonic ground state of the NS-sector – an unpleasant feature. This is not surprising since what has been constructed so far is the spinning string, which is known to be inconsistent. One has to do a little more to really obtain the superstring.

5.2. Superstrings and heterotic strings

The remaining diseases of the spinning string are simply eliminated by restricting to a subset of the possible vertex operators, characterized by (0) and (s) (or (c)). One easily concludes from eq. (5.25) that all operator products are local and close within this subset. The spin-statistics relation is satisfied as (v) is now absent: tensors and spinors are related to even and odd vectors on $\mathrm{D}_{5,1}$. In particular, as massless states we keep $|V^{\text{vector}}\rangle$ and $|V^{\text{spinor}}\rangle$ which form a supersymmetry multiplet. Actually one can prove (we will do this explicitly in section 8) that the whole spectrum is supersymmetric. Accordingly, one calls this theory the superstring.

Keeping only states in (0) and (s) can also be expressed as projecting onto states $|w\rangle$ with "positive G-parity", $G=1$, where

$$G|w\rangle \equiv (-1)^{F+F^{\phi}}|w\rangle = \exp(2\pi i\langle w_{(s)}, w\rangle)|w\rangle\,, \tag{5.26}$$

and $w_{(s)}$ is in the (s) conjugacy class of $D_{5,1}$. This generalizes (4.42) by including the ϕ-ghost charge F^{ϕ}, and is called *GSO-projection* [48].

The GSO-projected lattice contains fewer conjugacy classes than the lattice we had before. We call it $E_{5,1}$, in analogy to E_8, which is given by $(0)+(s)$ of D_8. The lattice $E_{5,1}$ generates a hyperbolic superalgebra [49], which contains anti-commuting, spinorial generators. We will consider generalizations of $E_{5,1}$ in section 8.

The projected lattice $E_{5,1}$ is *self-dual* as one can easily check. This self-duality is actually necessary for the consistency of the superstring: it ensures modular invariance of the partition function, which contains a lattice factor

$$\mathcal{L}_{5,1}(\tau) = \sum_{w=(u,x)\in E_{5,1},\, x=x_0} e^{i\pi\tau u^2}\, e^{2\pi i q}\,. \tag{5.27}$$

The sum is only over the states satisfying the selection rule (5.23), and the phase counts space–time fermions and bosons with opposite signs. The physical state selection rule implies that instead summing over (0) and (s) of $E_{5,1}$, one rather sums $(v)-(s)$ of $D_4^{\text{light-cone}}$:

$$\begin{aligned}\frac{1}{\eta^4(\tau)}\,\mathcal{L}_{5,1} &= \mathrm{Ch}^{D_4}_{[v]}(\tau) - \mathrm{Ch}^{D_4}_{[s]}(\tau)\\ &= \frac{1}{2}\frac{1}{\eta^4(\tau)}\,\{\vartheta_3(\tau)^4 - \vartheta_4(\tau)^4 - \vartheta_2(\tau)^4\} = 0\,.\end{aligned} \tag{5.28}$$

The equation in the second line is the simplest example of a Riemann Identity and is a reflection of ten-dimensional space–time supersymmetry (we will encounter analogous identities for supersymmetry in lower dimensions in section 8).

Because of the physical state selection rules, $\mathcal{L}_{5,1}$ above is not a usual lattice partition function. It differs also in further ways from the lattice partition functions (3.8) we encountered earlier. In particular, the lattice is odd rather than even, and in writing the exponent above we used the peculiar mass formula (5.10). It was shown in ref. [47] that all these features conspire in precisely the right way to make the theory modular invariant. That the lattice $E_{5,1}$ is *odd* self-dual might seem surprising; usually (i.e., in the compactified bosonic string) modular invariance under $\tau\to\tau+1$ requires an *even* lattice. The difference lies in the mass formula (5.10). Invariance of $q^{L_0} = \exp[2\pi i\tau(\langle w, w\rangle - q + \mathcal{N} - 1)]$ under $\tau\to\tau+1$ requires

$$\langle w, w\rangle - q \in \mathbb{Z}\,. \tag{5.29}$$

This implies that vectors with even Lorentzian norm have integer ghost charge, while those with odd norm have half-integer ghost charge. In other words, states in the NS- (R)-sector should be (anti-)

commuting. This is precisely what the spin-statistics theorem demands. One can easily check that this condition is fulfilled for the GSO-projected $E_{5,1}$. All other theories we will consider below satisfy this condition by construction.

In the next section we will present an independent proof of modular invariance, which extends also to higher loops.

Knowing all this, we could have started originally with a self-dual lattice and would never have encountered the problems of the spinning string; locality is guaranteed by the fact that on a self-dual lattice all inner products are integral. Thus, in this bosonic formulation, we never need to consider GSO-projections in an explicit way. This will be our philosophy for constructing further string theories.

It is now elementary to apply the foregoing to heterotic strings. We simply have to combine the right-moving NSR-model with a left-moving bosonic string. As the conformal anomaly $c_{bc} = -26$ for the bosonic string, we need in addition to the ten space–time coordinates X^μ, 16 compact bosons $X^I(\bar z)$ $(I = 1, \ldots, 16)$

$$S_{\mathrm{het}} = -\frac{1}{2\pi}\int \mathrm{d}^2 z\,(\partial X^\mu \bar\partial X_\mu + \psi^\mu \bar\partial \psi_\mu + 2b\bar\partial c + 2\beta\bar\partial\gamma + \partial X^I \bar\partial X_I + 2\bar b \partial \bar c) \tag{5.30}$$

(one also has to add a Lagrange multiplier to ensure that X^I really are left-moving [28]). It is convenient to combine this 16-dimensional lattice Γ_{16} with that related to the right-moving NSR-model:

$$\Gamma_{16;5,1} \equiv (\Gamma_{16})_{\mathrm{L}} \times (\mathrm{D}_{5,1})_{\mathrm{R}}\,. \tag{5.31}$$

We emphasize that this does *not* necessarily represent a direct product decomposition: as (5.31) denotes only the root sublattices, there may still exist non-trivial correlations between the conjugacy classes of Γ_{16} and $\mathrm{D}_{5,1}$. In order to characterize $\Gamma_{16;5,1}$ completely, consider (in analogy to eq. (5.8)) the operator product of two *closed string* vertex operators (4.26):

$$\begin{aligned} &V_w(\bar z, z)\cdot V_{w'}(\bar y, y) = (\bar z - \bar y)^{w_{\mathrm{L}}\cdot w'_{\mathrm{L}}}(z-y)^{\lambda\cdot\lambda' - qq'} V_{w+w'}(\bar y, y) + \cdots\,,\\ &V_w(\bar z, z) = \mathrm{e}^{\mathrm{i}w^I_{\mathrm{L}}X^I}(\bar z)\,\mathrm{e}^{\mathrm{i}\lambda\cdot H}\,\mathrm{e}^{\mathrm{i}q\phi}(z)\,,\qquad \boldsymbol{w} = (\boldsymbol{w}_{\mathrm{L}}; \boldsymbol{w}_{\mathrm{R}})\,,\qquad \boldsymbol{w}_{\mathrm{R}} = (\boldsymbol{\lambda}|q)\,. \end{aligned} \tag{5.32}$$

Upon rewriting the z- and w-dependent part using $(\bar z - \bar w) = |z-w|^2/(z-w)$, the condition for locality, i.e., the absence of branch cuts becomes

$$\langle \boldsymbol{w}, \boldsymbol{w}'\rangle \in \mathbb{Z}\,, \tag{5.33}$$

where the inner product has metric $[(-)^{16}; (+)^5|(-)]$. Thus, $\Gamma_{16;5,1}$ has to be an integral, Lorentzian lattice with respect to this metric. Note however that the two sets of $(-)$-signs have completely different origins: the $(-)$-sign in the left-moving sector is the same as that arising in the lattice of a compactified closed bosonic string, as discussed in section 2.

The full partition function of such a theory has the form

$$\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\pi\tau H'_{\mathrm{R}}}\,\mathrm{e}^{-\mathrm{i}\pi\bar\tau H'_{\mathrm{L}}}\,\mathrm{e}^{4\pi\mathrm{i}F^\phi}] = \frac{1}{\eta^{12}\bar\eta^{24}} \sum_{\substack{\boldsymbol{w}=(\boldsymbol{w}_{\mathrm{L}};\boldsymbol{w}_{\mathrm{R}})\in\Gamma_{16;5,1}\\ \boldsymbol{w}_{\mathrm{R}}=(\boldsymbol{\lambda}|q)=(\boldsymbol{u},x_0)}} \mathrm{e}^{-\mathrm{i}\pi\bar\tau \boldsymbol{w}_{\mathrm{L}}^2}\,\mathrm{e}^{\mathrm{i}\pi\tau \boldsymbol{u}^2}\,\mathrm{e}^{2\pi\mathrm{i}q}\,. \tag{5.34}$$

Modular invariance, as we will see, forces again this lattice to be (odd) self-dual. This does certainly not imply that Γ_{16} and $D_{5,1}$ should separately be self-dual lattices. That would correspond only to the simplest cases: taking in the right-moving sector the odd self-dual lattice $E_{5,1}$ corresponding to the superstring, requires Γ_{16} to be even self-dual. As explained in appendix A, there exist only two such lattices, namely the root lattice of $E_8 \times E_8$ and the weight lattice of $\mathrm{Spin}(32)/Z_2$. These choices lead to the well-known supersymmetric string theories with gauge groups $E_8 \times E_8$ and SO(32) (the vertex operators for the emission of the corresponding gauge bosons are accordingly given by $V^{ab}(\bar{z}, z) = J^{ab}(\bar{z})V^{\text{vector}}_{(-1)}$).

Allowing for a non-trivial correlation between conjugacy classes, the two factors in eq. (5.31) need not be separately self-dual; only the combined lattice needs to be. This leads to further possibilities of constructing theories like the non-supersymmetric $SO(16) \times SO(16)$-string [50, 51]; it is given by a lattice $\Gamma_{16;5,1} = (D_8 D_8)_L (D_{5,1})_R$ equipped with a certain list of correlated conjugacy classes (see section (7.2), table 5). In this list, *all* conjugacy classes of $D_{5,1}$ of the spinning string do occur; however, the problems of the spinning string do not show up due to the particular, non-trivial interrelation with the left-moving sector. In particular, all branch cuts are cancelled by virtue of eq. (5.33).

The construction described above generalizes straightforwardly to the torus-compactified heterotic strings as introduced by ref. [9]. In reducing the number of dimensions, the lattice gets larger as we include also the $2N$ torus-compactified left- and right-moving space–time coordinates. More specifically, in $d = 2N = 10 - 2n$ dimensions we have an odd self-dual Lorentzian lattice of the direct product form

$$\Gamma_{16+2n;5+2n,1} = \Gamma_{16+2n;2n} \times E_{5,1} , \tag{5.35}$$

where $\Gamma_{16+2n;2n}$ is an even, Lorentzian self-dual lattice as described by Narain [9]. There exists a continuous variety of such lattices: for example, four-dimensional theories are characterized by 132 parameters. These are related to background values of the various fields, as for the torus-compactified bosonic string described in section 2. As $E_{5,1}$ in eq. (5.35) is the odd self-dual lattice corresponding to the superstring, it represents the torus-compactified fermionic degrees of the superstring. Therefore this kind of theory is non-chiral.

It is now clear how to construct more general, possibly chiral theories: we simply have to look for odd self-dual lattices (5.35), where the conjugacy classes of the left and right factors are correlated in a non-trivial way. These theories then cannot, in general, be described as compactifications in the literal sense. Of course we have to make sure that we correctly assign even and odd vectors to space–time tensors and spinors; furthermore in lower dimensional theories, there are additional constraints related to world sheet supersymmetry in the internal sector (this will be explained in sections 6 and 7). The structure of the covariant lattice is thus more general than that of eq. (5.35):

$$\Gamma_{16+2n;5+2n,1} = (\Gamma_{16+2n})_L \times (\Gamma^{\text{int}}_{3n} D^{\text{space-time}}_{5-n,1})_R \tag{5.36}$$

Having found this simple structure we have to look for, we can forget now all subtleties related to the superconformal ghost system.

What remains to be proved is the connection between (multi-loop) modular invariance of the partition function and self-duality of the lattice (5.36). Before attacking this, we will switch to an alternative, more convenient lattice formulation.

5.3. The bosonic string map

Although the covariant lattices introduced in the previous section specify the fermionic string theory completely, they are not very pleasant to deal with because of their rather cumbersome metric, and because some of their vectors have odd norm.

Here we will introduce a map of any such lattice to an *even* self-dual lattice. More generally, what we are going to do is to associate with any heterotic string theory (or type-II theory) a bosonic string theory, by mapping the right- (or left-) moving superconformal system to a bosonic one. This map will have the useful property that it preserves modular invariance of the partition function at all loop orders. The reason for its universal validity is that this map affects only the $D_{5-n,1}^{\text{space-time}}$ factor in eq. (5.36). Since this factor is present in *any* fermionic string theory in $10-2n$ dimensions, this *bosonic string map* is valid for any fermionic string theory, not just for lattice theories (there is a straightforward generalization to odd dimensions). However the *application* to be discussed in this section, namely the proof of modular invariance, works only for lattice theories. The map works in general only in one direction: although any heterotic or type-II theory can be mapped to a bosonic string, the opposite is not true.

Basically, the idea is very simple. Consider first, for simplicity, the ten-dimensional strings, described by lattices of the form (5.31). If we decompose one of these lattices with respect to the $D_{5,1}$ conjugacy classes, we see that it consists of four sets of vectors

$$(\Delta_0, 0) + (\Delta_v, v) + (\Delta_s, s) + (\Delta_c, c)\,, \tag{5.37}$$

where the second entry denotes the $D_{5,1}$ conjugacy classes, and the Δ's are simply the 16-dimensional vectors associated with them on the left lattice. Now define a new lattice $\Gamma_{16;8}$ by writing down a decomposition identical to eq. (5.37), but with the second entry interpreted as a conjugacy class of D_8 instead of $D_{5,1}$. That is, we map

$$D_{5,1} = D_4^{\text{light-cone}} D_{1,1}^{\text{ghost}} \leftrightarrow D_4^{\text{light-cone}} D_4^{\text{ghost}} = D_8\,. \tag{5.38}$$

Note that this is a map between conjugacy classes, and not between individual lattice vectors. The old and the new lattice have the same number of conjugacy classes, so that if the old lattice was unimodular, so is the new one. Now compute the list of norms and inner products of the conjugacy classes of D_8. In comparison with eq. (5.25) one finds that the only change is in the norm of the spinors, which change from odd into even length. But the spinors of $D_{5,1}$ were precisely the odd vectors on $\Gamma_{16;5,1}$. Hence the map preserves unimodularity and all inner products between distinct classes, but it maps all the odd vectors into even ones. The result is a lattice $\Gamma_{16;8}$ which is *even* self-dual with respect to the metric $[(-)^{16}; (+)^8]$.

Note that for the supersymmetric ten-dimensional strings the effect is that $E_{5,1}$ is replaced by E_8. This appearance of exceptional Lie algebras in supersymmetric string theories is a first manifestation of a more general phenomenon, which we will discuss in more detail in section 8.

If we extend the lattice $\Gamma_{16;8}$ with an E_8 factor on the right, we get an even self-dual $\Gamma_{16;16}$ lattice on which the bosonic string can be compactified (this can also be achieved by using D_{16} in the foregoing construction, but this is less useful for discussing the spectrum). The bosonic string compactified on the corresponding torus is multi-loop modular invariant. As we will see, this is not a coincidence. A similar map exists for *any* heterotic (or type-II) string theory in *any* dimension. We will now give the general prescription, and show that it always takes a modular invariant heterotic string theory to a modular invariant bosonic string theory.

Consider thus a generic heterotic string theory in $d = 10 - 2n$ dimensions (the generalization of the following arguments to type II is trivial). As discussed in section 1.3, such a theory consists of a space–time sector and some internal conformal field theory $\mathscr{C}_{16+2n;3n}$. The discussion of higher genus partition functions of fermionic strings is complicated by the appearance of supermoduli contributions. The supermoduli are anti-commuting parameters related to the world sheet gravitino, just as the ordinary moduli are related to the world sheet metric (in some of the literature, the word "supermoduli" is used for the complete set of commuting plus anti-commuting moduli). One approach to handling the supermoduli is to integrate them out, in which case one is left with correlation functions of the world sheet supercurrent. The resulting partition function has the following generic form

$$\mathscr{P}_H = \mathscr{X}(\Omega, \bar{\Omega}) \sum_{\alpha} P_{\alpha}^{16+2n;3n}(\Omega, \bar{\Omega})(\det D_{1/2})_{\alpha}^{5-n}(\det D_{3/2})_{\alpha}^{-1}\langle\cdots\rangle_{\alpha} . \tag{5.39}$$

Here $\mathscr{X}$ contains all spin-structure independent measure factors, the contributions of the bosons X^{μ} with space–time indices and the contribution of the reparametrization ghosts. The summation is over all $2^{2\gamma}$ spin-structures at genus γ, and Ω denotes the period matrix (see appendix C for a discussion of some of these quantities). Each factor $(\det D_{1/2})_{\alpha}$ represents the contribution of two Majorana–Weyl fermions. It is simply the determinant of the Dirac operator on a Riemann surface with spin structure α. At one loop, one has

$$(\det D_{1/2})_{\alpha} = \vartheta_{\alpha}(0|\tau)/\eta(\tau) ,$$

where α labels the four possible boundary conditions of the fermions along the two non-contractible cycles on the torus. Thus in eq. (5.39) the $\det D_{1/2}$ factors represent the contribution of the $10-2n$ NSR-fermions. The other determinant, $(\det D_{3/2})^{-1}$, is a more complicated object which represents the β–γ ghost contribution. Since all these determinants are due to right-moving fields, they depend only on Ω, and not on $\bar{\Omega}$, just as the correlators of picture changing operators originating from the supermoduli integration, denoted as $\langle\cdots\rangle$. Finally, the factor P_{α} denotes the contribution of all other world sheet degrees of freedom, i.e. it is a spin-structure dependent partition function of the $\mathscr{C}_{16+2n;3n}$ system. Unlike the other factors in the expression, this function depends on the specific string construction one chooses, and contains all the complicated construction-dependent structure of the theory.

The precise form of the fermionic string partition function depends on various gauge choices, and is furthermore ambiguous at higher genus. All these technical problems are universal, and affect all fermionic strings in essentially the same way. Our interest is not really in dealing with this problem, but in checking the modular properties of the internal sector of the theory. To simplify the problem we will therefore assume that in the discussion of modular invariance the supercurrent correlation functions can be ignored. This can be justified by observing, as was done in ref. [26], that with this approximation the ten-dimensional heterotic string partition functions are modular invariant. Since the supercurrent correlators have a universal modular behavior, their presence affects the modular properties of any heterotic string theory in the same way, i.e. not at all (assuming of course that the complete ten-dimensional heterotic string partition functions are modular invariant).

In other words, we will assume for simplicity that it is sufficient to check the modular transformation properties of the determinants of the NSR-fermions and the superghost system in (5.39). The modular properties of this determinant combination can be studied by means of the following trick. Define

$$Y_\alpha = \frac{(\det D_{1/2})_\alpha}{(\det D_{3/2})_\alpha} \tag{5.40}$$

(this notation was introduced in ref. [52]), and write the determinant combination as

$$\mathcal{P}_H(Y_\alpha) \equiv \mathcal{X}(\Omega, \bar{\Omega}) \sum_\alpha P_\alpha^{16+2n;3n}(\Omega, \bar{\Omega})(\det D_{1/2})_\alpha^{4-n} Y_\alpha \,. \tag{5.41}$$

Now define

$$X_\alpha = (\det D_{1/2})_\alpha^4 \sum_\beta (\det D_{1/2})_\beta^8 \,.$$

It is elementary to check, using the explicit transformation rules given in ref. [26], that under all higher genus modular transformations the functions Y_α transform in exactly the same way as the functions X_α. Therefore if we define

$$\mathcal{P}_B = \mathcal{P}_H(X_\alpha)$$

then $\mathcal{P}_H(Y_\alpha)$ is modular invariant if and only if $\mathcal{P}_B$ is modular invariant.

Now $\mathcal{P}_B$ has an interesting interpretation. It can be written as

$$\mathcal{P}_B = \mathcal{X} P^{16+2n;16+2n} \,,$$

where

$$P^{16+2n;16+2n} \equiv \sum_\alpha P_\alpha^{16+2n;3n} (\det D_{1/2})_\alpha^{8-n} \sum_\beta (\det D_{1/2})_\beta^8 \,, \tag{5.42}$$

is the multi-loop partition function of a some conformal field theory with central charges $c_\mathrm{L} = c_\mathrm{R} = 16 + 2n$, which we will denote as $\mathcal{C}_{16+2n;16+2n}$. We can describe it rather precisely. In addition to the partition function $P_\alpha^{16+2n;3n}$ of the internal sector of the heterotic string, one finds first of all a factor $(\det D_{1/2})_\alpha^{8-n}$. This factor may be attributed to $16-2n$ world sheet Majorana–Weyl fermions, all having the same periodic or anti-periodic boundary conditions. These fermions generate states in $\mathrm{SO}(16-2n)$ representations. By the arguments of section 4.3, this system has an equivalent description in terms of bosons with momenta on the D_{8-n} weight lattice. At one loop, the four possible spin-structures correspond to the four D_{8-n} conjugacy classes via a basis transformation (cf. eq. (4.50)). The last factor in eq. (5.42) represents by the same arguments a set of bosons on the D_8 lattice. The spin structure labels of this factor are summed separately, and according to eq. (4.53) this means that we have a D_8-root lattice with an additional spinor root, enlarging the Lie algebra to E_8.

Thus we have replaced the partition function of the non-unitary β–γ ghost system by the partition function of a unitary conformal field theory, which can be described by free bosons or fermions. Since the internal sector of the heterotic string is unitary by assumption, it follows that $\mathcal{C}_{16+2n;16+2n}$ is described by a unitary conformal field theory. Thus we can interpret (5.42) as the internal part of a multi-loop partition function of a compactified bosonic string.

Of course we are not saying here that the heterotic string *is* a bosonic string. The two theories are different and have different scattering amplitudes. To calculate fermionic string scattering amplitudes, one needs the full machinery of superconformal field theory, for which we do not know a formulation in terms of the bosonic string. A more extreme point of view, namely that all heterotic and type-II strings can be regarded in some sense as "vacua" of the bosonic string, was proposed in ref. [53]. Historically the bosonic string map originates from this paper, although it appeared there in a somewhat different form. The form of the map that we use here appeared first in ref. [20], and in ref. [35] it was shown that it preserves one-loop modular invariance. This was extended to higher loops in ref. [54].

Although the idea that heterotic strings are just vacua of the bosonic string remains attractive, a dynamical basis for it is lacking, and it has nothing to do with our goals. For us the map will merely be a useful tool for proving certain properties of fermionic strings. It avoids having to deal with the intricacies of the β–γ ghost system in situations where those intricacies are irrelevant. Instead of studying the fermionic string directly, we study instead its faithful bosonic string image. Of course everything one learns from studying this image can also be derived, with a little more effort, directly for the fermionic string. However, our intuition for unitary conformal field theories is usually further developed than that for non-unitary systems, which often allows us to derive the answer much more easily.

There are many applications of this map. It simplifies the discussion of fermionic string spectra, and we will make use of that later. More specifically, it leads to a remarkably simple characterization of space–time supersymmetry in string theory, a subject that we will discuss in section 8. If one applies the bosonic string map to both sectors of a type-II string, one obtains of course a bosonic string. However, one may also use it in just one sector, in which case the type-II string is mapped to a heterotic string. This application was exploited in ref. [16].

Another application is the proof of multi-loop modular invariance of the covariant lattice construction. Since the bosonic string map preserves modular invariance, proving multi-loop modular invariance of the original heterotic string is equivalent to proving modular invariance of the resulting bosonic string. This is true in general, but it is only useful if the compactified bosonic string is sufficiently manageable. The simplest examples of compactified bosonic strings are generalized torus compactifications on even self-dual Lorentzian lattices $\Gamma^{16+2n;16+2n}$, and we have explicitly checked the modular invariance of any such theory in section 3.5. Not any such bosonic lattice can be mapped back to a heterotic string. A necessary and sufficient*) condition for this is that the right lattice contains a factor $D_{8-n} \times E_8$, which can be identified with the Dirac determinants in eq. (5.42).

This suggests that the replacement of Y_α by X_α corresponds precisely to a map from the odd self-dual covariant lattice introduced in the previous section to an even self-dual lattice. To see this more explicitly it is better to go from spin-structure basis to D_n conjugacy class basis, which is a simple orthogonal transformation (up to normalization) explained in appendix C. At genus 1 we define, in analogy to eq. (C.10)

$$\begin{aligned} X_{[0]} &= \frac{1}{2}\left(X\begin{bmatrix}0\\0\end{bmatrix} + X\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}\right), \qquad X_{[v]} = \frac{1}{2}\left(X\begin{bmatrix}0\\0\end{bmatrix} - X\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}\right), \\ X_{[s]} &= \frac{1}{2}\left(X\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} + X\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right), \qquad X_{[c]} = \frac{1}{2}\left(X\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} - X\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right). \end{aligned} \tag{5.43}$$

*) Apart from world sheet supersymmetry, to be discussed in the next section.

Here we use the same phases as for D_4, since the soliton part of X_α corresponds to a $D_4(\times E_8)$ partition function*). The straightforward multi-loop generalization of the basis transformation (5.43) is discussed in appendix C.

Just as X_α corresponds to $D_4 \times E_8$, Y_α is associated with, although in a far more complicated way, the $D_{1,1}^{\text{ghost}}$-lattice defined in eq. (5.22). For Y_α we can define conjugacy class partition functions in exactly the same way as for X_α. If we now apply the same basis transformation to the functions $\mathcal{P}_\alpha$ multiplying Y_α in the partition function, we get

$$\sum_\alpha \mathcal{P}_\alpha Y_\alpha = 2^{2\gamma} \sum_{[x]} \mathcal{P}_{[x]} Y_{[x]} , \tag{5.44}$$

where $[x]$ denotes the conjugacy class labels. Of course the same identity holds with Y replaced by X, so that in conjugacy class basis the bosonic string map is simply the replacement of a $D_{1,1}^{\text{ghost}}$ conjugacy class of the covariant lattice by the identical D_4 conjugacy class (times an E_8 factor), just as in eq. (5.38). Because it originates from $D_{1,1}^{\text{ghost}}$, we will denote the D_4 factor as D_4^{ghost}.

In the complete covariant lattice of a $10-2n$ dimensional string theory the effect is to replace the $D_{5-n,1}^{\text{space-time}}$ conjugacy classes by the same $D_{8-n}^{\text{space-time}}$ conjugacy classes, that is

$$\begin{aligned} \Gamma_{16+2n;5+2n,1} &= (\Gamma_{16+2n})_{\mathrm{L}} (\Gamma_{3n}^{\text{int}} D_{5-n,1}^{\text{space-time}})_{\mathrm{R}} \\ &\leftrightarrow (\Gamma_{16+2n})_{\mathrm{L}} (\Gamma_{3n}^{\text{int}} D_{8-n}^{\text{space-time}})_{\mathrm{R}} = \Gamma_{16+2n;8+2n} . \end{aligned} \tag{5.45}$$

Using the foregoing we conclude that the following statements are equivalent:
– The covariant lattice $\Gamma_{16+2n;5+2n,1}$ is odd self-dual (with odd norms for the spinors, as discussed above).
– The lattice $\Gamma_{16+2n;8+2n}(\times E_8)$ obtained by means of the lattice map is even self-dual.
– The bosonic string, compactified on the latter lattice is multi-loop modular invariant.
– The fermionic string defined by the original odd self-dual covariant lattice is multi-loop modular invariant.

While the foregoing proof of multi-loop modular invariance applies only to covariant lattice theories, the bosonic string map itself is completely general. Therefore to describe the spectrum in bosonic string language we might as well return to the general case. The physical light-cone states of a general heterotic string are obtained by applying a set of truncation rules to the bosonic string spectrum. These rules are designed to mimic exactly the physical state selection rule (5.23). This selection rule tells us to keep only states with special components in the $D_{1,1}^{\text{ghost}}$ sublattice, namely those belonging to conjugacy classes (v) and (s) of that lattice. Since $D_{1,1}^{\text{ghost}}$ is mapped to D_4^{ghost} by the bosonic string map, we take the identical conjugacy classes of D_4^{ghost} to identify the physical states. Thus one physical state is represented by an entire conjugacy class of D_4^{ghost}. One can choose a representative of the D_4^{ghost} conjugacy class that makes all this even simpler. Namely, if one chooses some weight vector of 8_s and 8_v to represent the spinor and vector conjugacy class, then the mass of the light-cone states is given by

$$\begin{aligned} \tfrac{1}{4}\alpha' m_{\mathrm{R}}^2 &= \tfrac{1}{2}\boldsymbol{v}^2 + \mathcal{N}_X + \mathcal{N}_\psi + h_{3n} - 1 \\ \tfrac{1}{4}\alpha' m_{\mathrm{L}}^2 &= \mathcal{N}_X + h_{16+2n} - 1 , \end{aligned} \tag{5.46}$$

*) Strictly speaking $X[{}^{1/2}_{1/2}] = 0$ because the odd spin structure ϑ-function vanishes. These terms are shown here only for the purpose of the multi-loop generalization of this basis transformation. At one loop (5.43) is equivalent to (4.50) for D_4 (the E_8-factor plays a trivial role).

where $\mathcal{N}_X$ is the contribution of the $8-2n$ oscillators in X^i and $\mathcal{N}_\psi$ represents $4-n$ oscillators appearing in the bosonization of ψ^i; h_{16+2n}, h_{3n} are the contributions of excitations of the internal conformal system $\mathscr{C}_{16+2n;3n}$. Note that excitations of oscillators in the ghost sector do not appear; the bosonic string states they generate are not physical states of the superstring.

In eq. (5.46) $\boldsymbol{v}$ is a $\mathrm{D}_{8-n}^{\text{space-time}}$ weight, which must be chosen in such a way that the selection rule is satisfied. More concretely, this implies the following. Decompose $\mathrm{D}_{8-n}^{\text{space-time}} \subset \mathrm{D}_{4-n}^{\text{light-cone}} \times \mathrm{D}_4^{\text{ghost}}$, and write $\boldsymbol{v} = (\boldsymbol{y}, \boldsymbol{x})$ according to this composition. Now make an (arbitrary) choice of 8_v and 8_s weights. Then the physical state selection rule (5.23) for covariant lattices is replaced by

$$\boldsymbol{x} = \boldsymbol{x}_0 = \begin{cases} (0,0,0,1) & \text{NS-sector}\,, \\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) & \text{R-sector}\,. \end{cases} \tag{5.47}$$

It is somewhat mysterious that the norm of the fundamental spinor and vector of $\mathrm{D}_4^{\text{ghost}}$ precisely reproduces the correct zero-point for the fermionic string mass formula. To appreciate this, observe that if we would have mapped $\mathrm{D}_{5-n}^{\text{space-time}}$ to D_{16-n} instead of $\mathrm{D}_{8-n} \times \mathrm{E}_8$, modular invariance would also be preserved, but the fundamental D_{16-n} spinors have the wrong norm to give the correct mass formula.

The decomposition of $\mathrm{D}_{8-n}^{\text{space-time}} \to \mathrm{D}_{4-n}^{\text{light-cone}} \times \mathrm{D}_4^{\text{ghost}}$ is a standard one. The branching rules are as follows:

$$(0) \to (0,0) + (v,v)\,, \qquad (v) \to (v,0) + (0,v)\,,$$

$$(s) \to (c,c) + (s,s)\,, \qquad (c) \to (s,c) + (c,s)\,.$$

We have written them in such a way that the second term is the one that survives the physical state projection (5.47). From the first entry, representing $\mathrm{D}_{4-n}^{\text{light-cone}}$, we can read off the spin of the particles in each conjugacy class of $\mathrm{D}_{8-n}^{\text{space-time}}$. Notice that these rules imply a somewhat counter-intuitive interpretation of $\mathrm{D}_{8-n}^{\text{space-time}}$ conjugacy classes: the adjoint representation contains space–time vectors, whereas the vector representation contains scalars. Of course the Lorentz representation of a state can still be modified by oscillator excitations.

If the internal system is itself a lattice theory, h_{3n} and h_{16+2n} have the form

$$h_m = \tfrac{1}{2}\boldsymbol{u}_m^2 + \mathcal{N}_m\,,$$

where $\boldsymbol{u}_m$ are the internal components of the lattice vectors and $\mathcal{N}_m$ (or $\tilde{\mathcal{N}}_m$ for left-movers) includes the contribution of all the internal oscillators. For such a lattice theory we can define

$$\boldsymbol{w}_{\mathrm{R}} = (\boldsymbol{u}_{3n}, \boldsymbol{y}, \boldsymbol{x})\,, \qquad \boldsymbol{w}_{\mathrm{L}} = \boldsymbol{u}_{16+2n}\,,$$

and eq. (5.46) reduces to the mass formula for a compactified bosonic string (2.16), except that some states of the bosonic string (the ones not satisfying (5.47), all $\mathrm{D}_4^{\text{ghost}}$ oscillator excitations and all excitons in the extra right-moving E_8-factor) do not correspond to physical states of the superstring.

The physical state selection rule, derived here from conformal field theory, can also be obtained by studying the one-loop partition function. At one loop Y_α is a constant, and modular transformations can be used to evaluate it for different choices of α. The result is

$$Y\begin{bmatrix}0\\0\end{bmatrix} = -Y\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix} = -Y\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} = \pm Y\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = 1\,, \tag{5.48}$$

which looks more instructive when written in terms of conjugacy classes:

$$Y_{[0]} = Y_{[c]} = 0\,, \qquad Y_{[v]} = -Y_{[s]} = 1\,. \tag{5.49}$$

This obviously projects the one-loop partition function on the states described above, and furthermore provides a necessary −-sign for the fermion contribution. Note that the functions X_α do *not* satisfy these identities. Although they *transform* in the same way as Y_α, they are totally different objects.

6. World sheet supersymmetry

6.1. Conditions for world sheet supersymmetry

In the original formulation of the NSR-model in terms of bosons X^μ and fermions ψ^μ there is a manifest (local) $N=1$ symmetry on the world sheet, which transforms the fermions and bosons into each other. The transformation is

$$\delta_\varepsilon \psi^\mu = \mathrm{i}\varepsilon\partial X^\mu\,, \qquad \delta_\varepsilon X^\mu = -\mathrm{i}\varepsilon\psi^\mu\,, \tag{6.1}$$

where ε is an infinitesimal Grassmanian parameter. This transformation is generated by the operator

$$\mathcal{T}_\mathrm{F} = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, \varepsilon T_\mathrm{F}\,, \tag{6.2}$$

where

$$T_\mathrm{F} = \mathrm{i}\psi^\mu \partial X_\mu \tag{6.3}$$

is the world sheet supercurrent. The supersymmetry transformation of any conformal field $\Psi(z)$ is obtained by acting with $\mathcal{T}_\mathrm{F}$

$$\delta_\varepsilon \Psi = \mathcal{T}_\mathrm{F}\Psi\,.$$

One may check that on ψ^μ and X^μ, $\mathcal{T}_\mathrm{F}$ does indeed produce eq. (6.1).

In the bosonic formulation of the ten-dimensional NSR-model the 10 fermions ψ^μ are replaced by additional bosons H_i. This does not in itself destroy world sheet supersymmetry. In two dimensions one can realize supersymmetry entirely in terms of bosons. Indeed, by simply replacing ψ^μ in eq. (6.2) by its bosonized form (4.37) one obtains a generator of world sheet supersymmetry which is expressed entirely in terms of 10 bosons X^μ and 5 bosons H_i. In constructing four-dimensional strings one does however something more drastic than just bosonization: the internal bosons are allowed to mix freely with each other. In fact in lattice constructions one never really bosonizes anything, but one starts instead with as many bosons as are required by cancellation of the conformal anomaly. Clearly it is not automatically guaranteed that the result respects world sheet supersymmetry.

World sheet supersymmetry is an essential part of superconformal field theory, which is the foundation of fermionic string theory. To make this more concrete, one can identify several potential problems that are cured by having world sheet supersymmetry.

One of those problems is Lorentz-invariance in light-cone gauge. World sheet supersymmetry is used to remove unphysical degrees of freedom from the covariant action of fermionic strings, so as to write them entirely in terms of the physical, light-cone degrees of freedom. Although one can always write down such a light-cone action, it is important that it came from a covariant action. This makes it possible to write down a Lorentz-algebra which does not just generate the transverse rotations of the light-cone states, but extends to the full Lorentz algebra. This extension involves the familiar generators L^{i-}, whose commutators have to be checked to see if the algebra closes. In supersymmetric strings one of the terms appearing in the expression for L^{i-} is $\oint \psi^i T_{\mathrm{F}}$.

One of the things that can go wrong with Lorentz invariance is that $[L^{i-}, L^{j-}]$ simply does not yield the answer required by the algebra of the Lorentz group. This situation occurs in theories with a conformal anomaly, and can be avoided by choosing the field content so that the conformal anomaly cancels. This is one of the oldest ways of obtaining the critical dimension.

It might appear that nothing prevents us from simply using the "compactified" ten-dimensional Lorentz generators also in four dimensions, regarding six of the eight transverse Lorentz indices as internal. This algebra closes, and it is indeed a valid choice in torus compactifications. However, closure of the algebra is not enough. In addition, one has to require that L^{i-} takes physical states into physical states. Otherwise one faces a problem which is even worse than the conformal anomaly, namely the states of the theory will not even fit into representations of the Lorentz group. (In the presence of a conformal anomaly, the Lorentz algebra does not close, but for the bosonic string and the NSR-formulation of the superstring it is still possible to assemble the massive states into representations of $\mathrm{SO}(d-1)$, the little group of massive particles). This is indeed precisely what happens, and one sees this most easily in the chiral sector of chiral string theories. At first sight, chirality and string theory are almost contradictory, because one would think that chiral states can always be excited into massive states. Massive chiral states would however violate Lorentz invariance, so a really clever mechanism is needed to ensure that all massive states combine into non-chiral pairs, while the massless ones do not. This is achieved by world sheet supersymmetry. Indeed, every realization of world sheet supersymmetry implies a certain θ-function identity, which ensures that the one-loop chiral partition function*) does not depend on τ (the modular parameter for the right-moving, fermionic sector). Consequently the only chiral states satisfying the left–right mass condition of closed strings are precisely the massless ones. For example, the manifest X^μ–ψ^μ realization (6.3) yields the identity

$$\partial_\nu \theta_1(\nu|\tau)|_{\nu=0} = 2\eta(\tau)^3 ,$$

whereas one of the simplest supercurrents for four-dimensional chiral strings (namely the one defined in eq. (6.14) below) corresponds to the identity

$$\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau) = 2\eta(\tau)^3 .$$

For more details we refer to ref. [12], and for some other examples to ref. [18].

*) This is the partition function of the chiral states, which will be defined more precisely in section 9.2.

A second way in which world sheet supersymmetry enters is through the picture changing operator. This operator has the form $\mathcal{P}_{+1} = \mathrm{e}^{\mathrm{i}\phi}\, T_{\mathrm{F}}$, where ϕ is the bosonized ghost of section 5.1. It is essential for the calculation of scattering amplitudes, and it too should have a well-defined action, and map physical states in one ghost-charge sector into physical states in another ghost charge sector.

The problem of finding a bosonic realization of world sheet supersymmetry consists thus of two parts:
(1) Find a purely bosonic expression for T_{F} that satisfies the supersymmetry algebra.
(2) Make sure that it acts properly on all states.

The algebra that the supercurrent T_{F} must satisfy can be encoded in operator products, and reads (with finite terms omitted)

$$\begin{aligned} T(z)T_{\mathrm{F}}(w) &= \frac{3/2}{(z-w)^2}\, T_{\mathrm{F}}(w) + \frac{1}{(z-w)}\, \partial_w T_{\mathrm{F}}(w) \\ T_{\mathrm{F}}(z)T_{\mathrm{F}}(w) &= \frac{10}{(z-w)^3} + \frac{2}{(z-w)}\, T(w)\,. \end{aligned} \tag{6.4}$$

The first of these equations states that it must have conformal weight $\frac{3}{2}$. The second one is just eq. (4.12), with $c = 15$, the conformal anomaly of 10 bosons and 10 Majorana–Weyl fermions. The form of the supercurrent is further restricted because we want it to be Lorentz invariant in d dimensions. This means that it must have the form

$$T_{\mathrm{F}} = \mathrm{i}\psi^{\mu}\partial X_{\mu} + T_{\mathrm{F}}^{\mathrm{int}}\,,$$

where the first term is just eq. (6.3), restricted to d dimensions, and the second is some expression of conformal weight $\frac{3}{2}$, constructed out of the $3n$ internal bosons, needed to cancel the conformal anomaly (here we define n by $d = 10 - 2n$). These internal bosons will be denoted as X^{α}, $\alpha = 1, \ldots, 3n$. Subtracting the space–time part of the supercurrent in eq. (6.4), we see that $T_{\mathrm{F}}^{\mathrm{int}}$ has to satisfy it with "10" replaced by $10 - d$, and $T(w)$ by $T(w)^{\mathrm{int}}$, the energy momentum tensor of $3n$ bosons, which has the canonical form (4.18).

The first equation is quite easy to satisfy. The general solution has the form [55, 56]

$$T_{\mathrm{F}}^{\mathrm{int}} = \sum_{l} \mathrm{i}\boldsymbol{B}(\boldsymbol{l}) \cdot \partial \boldsymbol{X}\, \mathrm{e}^{\mathrm{i}\boldsymbol{l}\cdot \boldsymbol{X}} + \sum_{t} A(\boldsymbol{t})\, \mathrm{e}^{\mathrm{i}\boldsymbol{t}\cdot \boldsymbol{X}}\,, \tag{6.5}$$

where $\boldsymbol{l}^2 = 1$, $\boldsymbol{t}^2 = 3$, and $\boldsymbol{l} \cdot \boldsymbol{B}(\boldsymbol{l}) = 0$.

The second equation in (6.4) imposes more complicated conditions on A and $\boldsymbol{B}$, and we do not know its general solution. However, as we will show in a moment, we do know many special solutions. First however we want to address the second point mentioned above, the action of T_{F} on the states in the theory.

If an operator of the form $\exp(\mathrm{i}\boldsymbol{w}' \cdot \boldsymbol{X})$ acts on a state $|\boldsymbol{w}\rangle$ with lattice momentum $\boldsymbol{w}$, then it produces a state with momentum $\boldsymbol{w} + \boldsymbol{w}'$. Consider now the two operators L^{i-} and $\mathcal{P}_{+1}$ discussed above. Write the lattice $\Gamma_{16+2n;5+2n,1}$ as a direct sum of several components, as follows

$$\begin{aligned} \Gamma_{16+2n;5+2n,1} &= (\Gamma_{16+2n})_{\mathrm{L}}(\Gamma_{5+2n,1})_{\mathrm{R}} \\ &= (\Gamma_{16+2n})_{\mathrm{L}}(\Gamma_{3n}^{\mathrm{int}} \mathrm{D}_{5-n,1}^{\text{space–time}})_{\mathrm{R}} \\ &= (\Gamma_{16+2n})_{\mathrm{L}}(\Gamma_{3n}^{\mathrm{int}} \mathrm{D}_{5-n}^{\mathrm{Lorentz}} \Gamma_{1}^{\mathrm{ghost}})_{\mathrm{R}}\,. \end{aligned} \tag{6.6}$$

According to the four component lattices in the last line of this decomposition, we write every vector w as $(w_L; w_R)$ where w_R has the form

$$w_R = (\text{internal components, weight of } D_{5-n}^{\text{Lorentz}} | \text{ghost charge}) .$$

In this notation, the bosonic form of the operators L^{i-} and $\mathscr{P}_{+1}$ for a certain supercurrent T_F consists of vertex operators with lattice momenta $\boldsymbol{w}'$ equal to

$$\begin{aligned} w^{L^{i-}} &= (0; \boldsymbol{l}, \boldsymbol{v}|0) \text{ or } (0; \boldsymbol{t}, \boldsymbol{v}|0) , \qquad \text{for } L^{i-} \\ w^{PC} &= (0; \boldsymbol{l}, 0|-1) \text{ or } (0; \boldsymbol{t}, 0|-1) , \qquad \text{for } \mathscr{P}_{+1} . \end{aligned} \tag{6.7}$$

($\boldsymbol{v}$ denotes a vector weight of D_{5-n}^{Lorentz}). If adding a vector $\boldsymbol{w}'$ to a lattice vector should give another lattice vector, then $\boldsymbol{w}'$ should lie on the lattice itself. For the picture changing operator one should furthermore require that its action on all states is *local*, i.e. does not introduce any branch cuts. In other words, if $e^{i\boldsymbol{w}\cdot \boldsymbol{X}}$ is the vertex operator for some physical state with lattice momentum $\boldsymbol{w}$, then we should require that the operator product

$$e^{i\boldsymbol{w}'\cdot \boldsymbol{X}}\, e^{i\boldsymbol{w}\cdot \boldsymbol{X}} = (z-w)^{\boldsymbol{w}'\cdot \boldsymbol{w}}\, e^{i(\boldsymbol{w}'+\boldsymbol{w})\cdot \boldsymbol{X}} + \cdots$$

does not contain any fractional powers of $z - w$. This implies that $\boldsymbol{w}'$ must have integral dot product with all lattice vectors, so that it must lie on the *dual* of the lattice. Fortunately we are always dealing with self-dual lattices so that these requirements on $\boldsymbol{w}'$ are compatible.

The requirements one gets from L^{i-} and $\mathscr{P}_{+1}$ are also compatible with each other, because in covariant lattices the D_{5-n}^{Lorentz} components and the ghost charge belong to the same conjugacy class of $D_{5-n,1}^{\text{space-time}}$. What we conclude from this is that for every vector $\boldsymbol{l}$ and $\boldsymbol{t}$ appearing in T_F^{int} the lattice must contain vectors of the form

$$\boldsymbol{w}_R = (\boldsymbol{l}, (v)) \quad \text{and} \quad \boldsymbol{w}_R = (\boldsymbol{t}, (v)) ,$$

where (v) denotes generically the vector conjugacy class of $D_{5-n,1}^{\text{space-time}}$. This vector conjugacy class contains of course the vectors $(\boldsymbol{v}|0)$ and $(0|-1)$ appearing as the $D_{5-n,1}^{\text{space-time}}$ entries in eq. (6.7). We will call these vectors the *constraint vectors* of the supercurrent. The left component w_L of these vectors must vanish. The form of these vectors in the even lattice formulation is of course identical, except that (v) is now a $D_{8-n}^{\text{space-time}}$ conjugacy class instead of a $D_{5-n,1}^{\text{space-time}}$ conjugacy class.

It should be emphasized that the supercurrent itself is not the vertex operator of any physical state. Indeed, since it has conformal weight $\frac{3}{2}$, it cannot be physical (in the NSR-model it is in the sector that is removed by the GSO-projection). In other words, the action of the world sheet supersymmetry operator may change the boundary conditions of the field on which it acts. For example, in eq. (6.1) the fields X and ψ need not have the same boundary conditions. Any difference in boundary conditions can always be compensated by the boundary conditions of the fermionic parameter ε. If a supercurrent consists of more than one term, such as eq. (6.5), there are non-trivial consistency conditions, since there is just one choice one can make for the boundary conditions of ε for a given world sheet topology. There is no conflict if all terms in the supercurrent change all boundary conditions in precisely the same way. This implies that the difference of any two vectors $\boldsymbol{t}$ or $\boldsymbol{l}$ appearing in T_F should lie on the lattice.

Since the space–time part of T_F has vectors $\boldsymbol{v}$ in the vector conjugacy class of D_{5-n}^{Lorentz} this leads again to the same conditions we found above. This argument has the advantage of starting directly from the requirement of world sheet supersymmetry. It applies also to other, ungauged, world sheet supersymmetries that one might wish to consider.

6.2. *Lattice realizations: generalities*

Now we return to the problem of finding solutions to eq. (6.4). The kind of solution one needs depends on the kind of string theory one wants to construct. Two cases are of interest:

– Chiral heterotic strings. In this case one only needs a right-moving supercurrent. It will become clear when we discuss the spectrum that a necessary (though not sufficient) condition for chirality is that the lattice does not contain any vectors of the form $\boldsymbol{w}_L = 0$, $\boldsymbol{w}_R = (\boldsymbol{l}, (v))$, $\boldsymbol{l}^2 = 1$. In particular that means that all coefficients $\boldsymbol{B}$ in eq. (6.5) must vanish.

– Chiral type-II strings. In type-II strings one must get chirality from one of the two sectors (the right-moving one, say) and gauge-symmetries from the other, since these two properties are mutually exclusive within the same sector. For the right-moving sector one needs therefore the same kind of supercurrent as for chiral heterotic strings, whereas in the left-moving sector other possibilities may be considered.

Since the prospects for chiral type-II strings do not seem to be very bright [57], we will concentrate on the first case, and assume henceforth that $\boldsymbol{B}$ must vanish.

A general strategy for finding the complete set of solutions is the following. First one can try to construct all possible configurations of norm-3 vectors $\boldsymbol{t}$ with integral inner products. Thus one has to classify all N-dimensional odd integral lattices Ξ_N with norm-3 vectors spanning the entire lattice, and without norm-1 vectors. This is similar in principle to the classification of all configurations of norm-1 or norm-2 vectors. In the latter two cases the problem has been solved; for norm-1 the only solutions are Z_N lattices, and for norm-2 one gets the root lattices of the Lie algebras of types A, D or E. The corresponding problem for norm-3 appears to be considerably more complicated, and we have solved it completely only for 3 or fewer dimensions.

To actually determine the coefficients $A(\boldsymbol{t})$ in eq. (6.5), given such a lattice Ξ_N, can still involve a fair amount of work. From the operator algebra one gets a set of bilinear equations for these coefficients, one of which is

$$\sum_l |A(\boldsymbol{t}_l)|^2 t_l^i t_l^j = 2\delta^{ij} , \tag{6.8}$$

where the sum is over all norm-3 vectors in Ξ_N. This is an over-constrained set of linear equations for the squares of the coefficients. It has solutions (with positive values of $|A|^2$) only for rather special lattices Ξ_N. One should furthermore keep in mind that the coefficients are operators, and that one has to prove that a proper set of cocycles exists. Fortunately it is usually not necessary to solve these equations explicitly. For most purposes it is sufficient to know that solutions exist for some set of constraint vectors. This is sufficient to construct at least the lattices that admit a particular realization of world sheet supersymmetry (the explicit form of the supercurrent is only needed if one wants to calculate scattering amplitudes for which picture changing is required). We will discuss below two methods that provide us with sets of constraint vectors for which solutions to (6.8) (and the other equations) exist.

The coefficients $A(\boldsymbol{t})$ are not completely unique. One has at least the following ambiguity. The lattice

Ξ_N has a Frenkel–Kac group K associated with it, which contains at least $U(1)^N$. We call this the *invariance group* of the supercurrent. One can transform all states by elements k in that group. With respect to the new basis, the same theory is described by a supercurrent

$$T'_F = kT_F k^{-1} .$$

The $U(1)^N$ elements just change the coefficients by phases, but if Ξ_N contains norm-2 vectors, K is larger and one can obtain non-trivial transformations of the coefficients. Within a given string theory these transformations are irrelevant, since they simply amount to writing the theory in terms of a different basis of conformal fields: one has replaced some or all operators ∂X by linear combinations of the Frenkel–Kac generators. We will use such a "rebosonization" in the next subsection.

It may happen that for a special choice of coefficients $A(\boldsymbol{t})$ the supercurrent lives only on a sublattice Ξ'_N of Ξ_N. Clearly the set of constraint vectors of Ξ'_N is smaller than that of Ξ_N, so that the latter will allow fewer string theories than the former. By restricting to Ξ' one generically reduces the group K to a subgroup, thus reducing the degree of ambiguity. There is clearly no advantage to using the Ξ-supercurrent instead of the Ξ'-supercurrent in this case.

There is a logical possibility that a lattice Ξ_N admits two or more supercurrents that are physically *inequivalent*, i.e. that cannot be related by a change of conformal field basis. In that case one would be able to build two string theories described by the same covariant lattice, differing only in their realization of world sheet supersymmetry. This difference would not affect the spectrum, the gauge couplings or even the Yukawa couplings, but it would affect any interaction requiring picture changing, such as four-scalar couplings.

In section 6.5 we return to the general problem, and discuss the structure of lattices with a certain supersymmetry realization. To make all this more concrete, we will first discuss two classes of special solutions.

6.3. Orbifold supercurrents

The first class we consider is "orbifold-inspired" and yields by construction only supercurrents and corresponding lattice theories that can also be described as (asymmetric) orbifolds. A more detailed account of this class of supercurrents may be found in ref. [58].

Observe first of all that the canonical choice, $T_F^{\text{int}} = i\psi^a \partial X_a$, will always satisfy the operator product algebra (here a is an internal space index, $a = 1, \ldots, 2n$). However, upon bosonization of ψ^a we get precisely a supercurrent of the undesirable form, i.e. with $\boldsymbol{B} \neq 0$. This problem can be solved by "rebosonizing" ∂X. Let us first write the supercurrent in a complex basis ($i = 1, \ldots, n$):

$$T_F^{\text{int}} = \psi^{i-} \partial X_{i+} + \text{c.c.}$$

Now replace ∂X_{i+} by

$$\partial Y_{i+} = \sum_k c_k^i \, e^{i\boldsymbol{\alpha}_k^i \cdot \boldsymbol{X}} , \tag{6.9}$$

where the sum is over some set of norm-2 vectors $\boldsymbol{\alpha}_k^i$. The new supercurrent will have the same operator products (with itself and the energy–momentum tensor) as the old supercurrent if and only if $\partial Y_{i\pm}$ is a set of U(1) currents, with the same operator products as $\partial X_{i\pm}$.

A necessary condition is that the norm-2 vectors $\boldsymbol{\alpha}_k^i$ have integer inner products. Therefore they are part of the root system of some Lie algebra $\mathscr{G}$. It is also clear that this root-system must span the entire $2n$ dimensional internal space, since otherwise the operator product of $T_{\mathrm{F}}^{\mathrm{int}}$ with itself can never reproduce the full energy–momentum tensor in the internal dimensions.

There are a few more reasonable conditions that one can impose on the supercurrent. First of all, there should exist a lattice that contains the constraint vectors implied by it. To derive the constraint vectors we bosonize $\psi^{i\pm}$ in terms of n new bosons denoted $\boldsymbol{H}_i$. Then $\psi^{i+} = \exp(\mathrm{i}\boldsymbol{e}^i \cdot \boldsymbol{H})$, and in $(\boldsymbol{H}, \boldsymbol{X})$-space the norm-3 vectors $\boldsymbol{t}$ of eq. (6.5) are $(\boldsymbol{e}^i, \boldsymbol{\alpha}_k^i)$. Thus we need a lattice which contains all vectors of the form $(\boldsymbol{e}^i, \boldsymbol{\alpha}_k^i, (v))$ on the right-moving lattice $(\Gamma_{5+2n,1})_{\mathrm{R}}$.

This condition can always be satisfied by torus compactifications on the $\mathscr{G} \times \mathscr{G}$ torus, but such theories are not chiral. A necessary condition for chiral theories is that no vectors of the form $(\boldsymbol{e}^i, 0, (v))$ be present. Since self-dual lattices have the property that any vector which is not explicitly forbidden is allowed, it follows that the lattice must contain a vector of the form $\boldsymbol{s} = (\boldsymbol{\theta}, \boldsymbol{\sigma}, (0))$, such that

$$\theta_i \equiv \boldsymbol{e}^i \cdot \boldsymbol{\theta} \neq 0 \mod 1 .$$

This vector may appear in combination with anything on the left lattice*). Since $\boldsymbol{s}$ is a lattice vector, it must have integer inner product with the constraint vectors, which implies

$$\boldsymbol{\sigma} \cdot \boldsymbol{\alpha}_k^i = -\theta_i , \quad \text{for all } k .$$

Consider now the operator

$$R(\boldsymbol{\sigma}) = \exp\left(2\pi\mathrm{i}\boldsymbol{\sigma} \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}z}\, \partial \boldsymbol{X}(z)\right) . \tag{6.10}$$

It follows that

$$R(\boldsymbol{\sigma})\partial Y^{i+} R(\boldsymbol{\sigma})^{-1} = \mathrm{e}^{-2\pi\mathrm{i}\theta_i} Y^{i+} . \tag{6.11}$$

We observe now that $R(\boldsymbol{\sigma})$ is an element of the group G of which $\mathscr{G}$ is the Lie algebra, and furthermore that it rotates the Cartan sub-algebra formed by the $\partial Y^{i\pm}$ into itself. This means that $R(\boldsymbol{\sigma})$ is an element of the Weyl group of $\mathscr{G}$, lifted to G.

The canonical form of such lifted Weyl group elements is a product of basic reflections r_α in the plane orthogonal to some root $\boldsymbol{\alpha}$. An arbitrary Weyl group element can be written as a product of reflections r_α, and its canonical lift to the group G is the product of the corresponding lifted reflections. Any element of G, and in particular a lifted Weyl group element, can be conjugated into the maximal torus. This is the form in which we have found the group element $R(\boldsymbol{\sigma})$ defined above.

The classification of all supercurrents of this type amounts thus to a classification of all conjugacy classes of Weyl group elements of all simply-laced rank $2n$ algebras. For chiral theories we are only interested in those elements that act non-trivially on the entire Cartan sub-algebra ($\theta_i \neq 0 \bmod 1$), and we call such elements *non-degenerate*. Their classification is a mathematical problem that has been solved completely. The results, as well as several other properties of Weyl group conjugacy classes are summarized in appendix B.

*) One can assume without loss of generality that there exists a single vector $\boldsymbol{s}$ which achieves this, rather than several. One may also assume that it does not belong to the spinor conjugacy class of $\mathrm{D}_{8-n}^{\text{space-time}}$. For the rest of the argument, (0) and (v) are equivalent.

The most important piece of information that one needs to extract are the shift-vectors $\boldsymbol{\sigma}$ that define the maximal torus element $R(\boldsymbol{\sigma})$ representing the Weyl twist. For non-degenerate elements $\boldsymbol{\sigma}$ can be shown to be unique up to addition of roots and Weyl rotations. It is convenient to choose $\boldsymbol{\sigma}$ to have minimal length, and to lie in the positive Weyl chamber. Then every conjugacy class is represented by a unique "minimal" shift vector. For all non-degenerate Weyl group elements of simply laced algebras that are not Weyl group elements of sub-algebras, those minimal shift vectors are known explicitly, and are tabulated in appendix B.

The recipe for finding the Ξ-lattices is thus as follows. First choose a rank $2n$ Lie algebra $\mathscr{G}$ and a non-degenerate conjugacy class of its Weyl group. Then determine the inner product of its shift vector with all the roots. The roots $\boldsymbol{\alpha}^i_k$ that can appear in $\partial Y^{i\pm}$ are precisely those that have inner product $-\theta_i$ with $\boldsymbol{\sigma}$. Thus Ξ_{3n} is generated by

$$t^i_k = (\boldsymbol{e}^i, \boldsymbol{\alpha}^{i_k}) , \qquad \boldsymbol{\alpha}_{k^i} \cdot \boldsymbol{\sigma} = -\theta_i . \tag{6.12}$$

The invariance group K of the supercurrent is easy to determine in this case. It consists of two factors, K_ψ and K_X, the first of which is associated with the fermions ψ^a and the second with the bosons X^a. The group K_ψ is simply the subgroup of SO($2n$) that is left unbroken by the torus eigenvalues θ_i. If all θ_i are different and not equal to $\frac{1}{2}$, $\mathrm{K}_\psi = \mathrm{U}(1)^n$. The second factor, K_X, can be read off from the (extended) Dynkin diagram representation of the shift-vector $\boldsymbol{\sigma}$ [59, 60].

If the Weyl conjugacy class is not proper in $\mathscr{G}$ (i.e. if it lives in a sub-algebra $\mathscr{G}'$ of $\mathscr{G}$), there exists a solution that uses only roots of $\mathscr{G}'$. Thus the Ξ-lattice derived from $\mathscr{G}'$ is a sublattice of that of $\mathscr{G}$ for the same Weyl group conjugacy class. Then, as explained in the previous section, there is no advantage to using the bigger Ξ-lattice instead of its sublattice. Thus to classify the supercurrents of orbifold type it is sufficient to consider proper Weyl group conjugacy classes.

All supercurrents we have constructed in this section can be immediately recognized by the structure of their system of norm-3 vectors. They satisfy the condition that there exists a set of orthonormal vectors $\boldsymbol{e}_i$, $i = 1, n$, so that (1) for each vector $\boldsymbol{t}$ there is precisely one $\boldsymbol{e}_i$ for which $\boldsymbol{t} \cdot \boldsymbol{e}_i = \pm 1$, while the remaining $\boldsymbol{e}_j$ are orthogonal to $\boldsymbol{t}$; (2) for each $\boldsymbol{e}_i$ there must be some $\boldsymbol{t}$ for which $\boldsymbol{e}_j \cdot \boldsymbol{t} \neq 0$.

Conversely, any set of norm-3 vectors satisfying this condition can be shown to correspond to a supercurrent of orbifold type.

6.4. Free fermion supercurrents

A second method for obtaining bosonic supercurrents is to bosonize a non-trivial fermionic realization of world sheet supersymmetry. Such a free fermion supercurrent has the form

$$T_{\mathrm{F}}(z) = \tfrac{1}{6} f_{abc} \psi^a \psi^b \psi^c , \tag{6.13}$$

where the ψ's are $6n$ fermions related to the $3n$ internal bosons by bosonization. If f_{abc} are the structure constants of a semi-simple Lie algebra, $\mathscr{G}_{\mathrm{SC}}$, then one can check that eq. (6.4) is satisfied [61]. The generic choice for this Lie algebra for $6n$ fermions*) is $\mathrm{SU}(2)^{2n}$. In dimensions below five there are additional solutions (SU(4) for $d = 5$; SU(4) × SU(2) and SO(5) × SU(3) for $d = 4$, etc.) which are of

*) The solutions for other numbers of fermions may be of more than academic interest, since it might be possible to construct "hybrid" models, where only part of the supercurrent has a simple fermionic realization.

some interest for type-II strings, but not for chiral heterotic strings. To obtain a supercurrent of the form (6.5) one simply bosonizes the $6n$ fermions into $3n$ bosons using eq. (4.35).

There is a lot of freedom in performing this bosonization. First of all the $6n$ fermions can be rotated into each other by $O(6n)$ rotations. Generic $O(6n)$ rotations change the structure constants f_{abc}. Only $\mathrm{Ad}(\mathcal{G}_{SC})$ rotations and permutations of identical factors in $\mathcal{G}_{SC}$ leave them invariant. Given a choice of structure constants, one has to decide on a pairing of the $6n$ fermions, where each pair will be represented by one boson. This pairing ambiguity is not really anything new though: since all possible pairings are related to each other by permutations of the $6n$ fermions, and since those permutations are elements of $SO(6n)$, it is already included in the $SO(6n)$ ambiguity.

The following example illustrates the problem. Consider the $SU(2)^6$ supercurrent in its simplest form

$$T_F = N \sum_{i=1}^{6} [\psi_i^1 \psi_i^2 \psi_i^3]\,.$$

The simplest pairing is (ψ_i^m, ψ_{i+3}^m). The vectors t one gets are

$$\begin{gathered}(\pm1, \pm1, \pm1, 0, 0, 0, 0, 0, 0)\\(0, 0, 0, \pm1, \pm1, \pm1, 0, 0, 0)\\(0, 0, 0, 0, 0, 0, \pm1, \pm1, \pm1)\,,\end{gathered} \tag{6.14}$$

with all possible choices of plus and minus signs. This supercurrent was first used in ref. [10]. The resulting constraints involve triplets of fermions, and were therefore called "triplet constraints". It satisfies the conditions for orbifold supercurrents, listed above and it is obtained for $\mathcal{G} = (D_2)^3$, by using the Coxeter element of that algebra. The fact that this algebra is isomorphic to $\mathcal{G}_{SC}$ is more or less coincidental, as the following example demonstrates.

Suppose we rotate the fermions as follows

$$\psi_i^l = \sum_j S_{ij} \tilde{\psi}_j^l\,, \quad l = 1, 2, 3$$

where S_{ij} is the matrix

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{3} \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2}\sqrt{3} & -\frac{1}{2}\sqrt{3} \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -\frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{3} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2}\sqrt{3} & -\frac{1}{2}\sqrt{3} & 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

so that the $O(18)$ matrix is built out of three such blocks (this example appears in ref. [11] in a slightly different form). If we now pair $\tilde{\psi}$ exactly as we paired ψ before, then we end up with a very different set of constraint vectors. Again they satisfy conditions for orbifold supercurrents and in this case the Weyl group conjugacy class to be used turns out to be $D_6(a_2)$ (see appendix B.3 for a description of Weyl group conjugacy classes).

Although one might think that all supercurrents that satisfy eq. (6.4) can be written as a trilinear expression of free fermions, this is not the case. A counterexample, found in refs. [56] and [55] has vectors $\boldsymbol{t}$ of the form (for $n = 1$)

$$t_1 = (\sqrt{3}, 0, 0)\,, \qquad t_2 = (0, \sqrt{3}, 0)\,, \qquad t_3 = (0, 0, \sqrt{3})\,. \tag{6.15}$$

This pattern can be repeated n times if $n > 1$. By bosonizing the free fermionic supercurrent, one can only get vectors of the form $(\pm 1, \pm 1, \pm 1, 0, \ldots, 0)$, i.e., there must exist a basis in which three entries are ± 1, and the others are zero. Such a basis does not exist for eq. (6.15). This example does however have an orbifold interpretation. The vectors (6.15) can be rotated to $(1, \alpha)$, $(1, \beta)$, $(1, -\alpha, -\beta)$, where α and β are the simple roots of SU(3) [55]. Thus one obtains the SU(3) Coxeter supercurrent for any triplet of such building blocks.

At first sight it may appear paradoxical that not any bosonic supercurrent can be written in terms of free fermion trilinears, since in ref. [41] it was shown that the Fock space of any torus-compactified theory can always be written in terms of a fermionic representation. This may involve either fermions with Thirring interactions (to which the above does not apply, since one would not have a *free* fermion supercurrent), or free fermions with complicated boundary conditions. A bosonic theory described in this fermionic way will still have world sheet supersymmetry, since it is equivalent. But one cannot conclude from this argument that there must always exist a representation of the supercurrent of the trilinear form (6.13).

An illustrative example is a string theory on an E_8 lattice. The Fock space can be described either by means of fermionic oscillators with certain GSO-projections, or with bosonic oscillators and lattice momenta. The Frenkel–Kac generators of the D_8 sub-algebra of E_8 can also be described both bosonically and fermionically, the latter being fermion bilinears. But whereas the bosonic description extends trivially to the remaining spinor roots of $E_8/SO(16)$, the fermionic one does not. To describe these generators one would need a *transcendental* expression in terms of the fermions rather than a bilinear one [44]. Indeed, all orbifold supercurrents for D_n conjugacy classes can manifestly be written in terms of free fermion trilinears, since all generators can be written as bilinears. But for A- and E-type supercurrents this is clearly not the case.

This problem can be evaded by fermionizing the orbifold supercurrent $\psi^a \partial X_a$ *before* the rebosonization (6.9) of ∂X. Of course one can always replace ∂X_a by a fermion bilinear, but then the entire non-trivial structure of the theory is due to the boundary conditions, just as in orbifold constructions. These boundary conditions are in general non-commuting ones, which are not yet well-understood, but in this way it may be possible to obtain a fermionic description of lattice theories with A- and E-type supercurrents.

However, in this basis there is now a different set of operators that does not have a simple fermionic realization, namely the $3n$ internal U(1)-currents that are the essence of a lattice theory. In the Lie algebra basis before rebosonization, these U(1) currents are linear combinations of root-generators (see (6.9)), which in general cannot be written in terms of free fermion bilinears. In other words, the question is not whether one can find a simple fermionic expression for the supercurrent. This is not a meaningful question unless one species what else one would like to express simultaneously in a simple way in terms of fermions. Thus for A- and E-type supercurrents one has the option of fermionizing either the supercurrent or the U(1)'s, but not both. Of course it is precisely the simultaneous bosonization of both that makes the lattice description so simple.

6.5. The structure of lattices for chiral theories

If the supercurrent is made entirely with norm-3 vectors $\boldsymbol{t}$, then the structure of the right lattice is severely constrained. Denote, as before, the $3n$-dimensional lattice generated by the vectors $\boldsymbol{t}$ as Ξ. A covariant lattice must contain all constraint vectors of the form $(\boldsymbol{t}, (v))$, and since it must close under addition of these vectors, it follows that the right lattice must contain a sublattice Δ, defined as

$$\Delta = (\Xi_{\mathrm{odd}}, (v)) + (\Xi_{\mathrm{even}}, (0)) ,$$

where Ξ_{odd} and Ξ_{even} contain the odd and even vectors of Ξ respectively. Any other lattice vector must have integer inner product with all constraint vectors, so their right-moving components must lie on the dual Δ^* of Δ. It has the following decomposition with respect to $\Gamma_{3n}^{\mathrm{int}} \mathrm{D}_{8-n}^{\text{space-time}}$

$$\Delta^* = (\Xi^*_{\mathrm{NS}}, (0) + (v)) + (\Xi^*_{\mathrm{R}}, (s) + (c)) . \tag{6.16}$$

We distinguish two kinds of vectors on this lattice. The first, the Ramond sector Ξ^*_{R} consists of the vectors with half-integer inner product with Ξ_{odd} while the second (the Neveu–Schwarz sector Ξ^*_{NS}) contains the vectors with integer inner product with Ξ_{odd} (and hence is the dual of Ξ itself). The two sets of vectors together form the dual of Ξ_{even}, i.e.

$$\Xi^* = \Xi^*_{\mathrm{NS}} , \qquad \Xi^*_{\mathrm{even}} = \Xi^*_{\mathrm{NS}} + \Xi^*_{\mathrm{R}} .$$

Obviously, if $\boldsymbol{\xi}_{\mathrm{R}}$ is a vector in Ξ^*_{R}, we can write $\Xi^*_{\mathrm{R}} = \boldsymbol{\xi}_{\mathrm{R}} + \Xi^*_{\mathrm{NS}}$. Furthermore Ξ^*_{NS} can be decomposed into cosets (cf. appendix A.2) with respect to Ξ

$$\Xi^*_{\mathrm{NS}} = \biguplus_q (\boldsymbol{\xi}_q + \Xi) ,$$

where $\boldsymbol{\xi}_q$ is a set of coset representatives, chosen so that $\boldsymbol{\xi}_0 = 0$.

Similarly, the lattice Δ^* has a coset decomposition with respect to Δ:

$$\Delta^* = \biguplus_p (\boldsymbol{\delta}_p + \Delta) ,$$

where $\boldsymbol{\delta}_p$ is a coset representative and $\delta_0 = 0$. The coset representatives $\boldsymbol{\delta}_p$ are related to those of Ξ defined above. Since Δ has twice the volume of Ξ, Δ^* has four times as many conjugacy classes relative to Δ, and their representatives $\boldsymbol{\delta}_p$ can be chosen as follows

$$(\boldsymbol{\xi}_q, 0, \ldots, 0, 0) , \qquad (\boldsymbol{\xi}_q, 0, \ldots, 0, 1) ,$$

$$(\boldsymbol{\xi}_q + \boldsymbol{\xi}_{\mathrm{R}}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}) , \quad (\boldsymbol{\xi}_q + \boldsymbol{\xi}_{\mathrm{R}}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}) ,$$

where the last entries are $\mathrm{D}_{8-n}^{\text{space-time}}$ conjugacy class representatives. A general (even) lattice admitting the chosen representation of world sheet supersymmetry has then the form

$$\Gamma_{16+2n;8+2n} = \biguplus_p (\Delta_p; \boldsymbol{\delta}_p + \Delta) , \tag{6.17}$$

i.e. one associates with every coset on the right lattice a set of vectors Δ_p on the left lattice (Δ_p may be an empty set). The choice of Δ_p is of course restricted by self-duality. In particular we see that lattices describing chiral theories can always be written as a finite sum of building blocks, each of which is a direct product of a set of vectors on the left and the right lattice. Thus the partition function is a finite sum of holomorphically factorized terms (this is in general not true for non-chiral lattice theories, and also not for some chiral theories that cannot be described by lattices), i.e. a rational conformal field theory.

Each conjugacy class generator $\boldsymbol{\delta}_p$ belongs to either the NS or the R-sector. When expressed in terms of modes, every term in the supercurrent is half-integer moded on NS-states, and integer moded on R-states. In particular we see from eq. (6.16) that the moding of the space–time and internal part of the supercurrent is the same.

The existence of a zero-mode operator G_0 in the Ramond sector implies a lower bound of the energy in that sector, as was discussed in section 4.2. For the internal degrees of freedom, which have $c = 3n$, we find, using eq. (4.15)

$$L_0^{\text{int}} \geq n/8 \,. \tag{6.18}$$

This will be shown in section 7.2 to imply the absence of fermionic tachyons.

7. Heterotic and type-II spectra

7.1. Summary of lattice consistency conditions

To discuss the spectrum of covariant lattice theories we will use the even lattice formulation. The consistency conditions that such theories have to satisfy have been derived in the previous sections, and can be summarized as follows. To specify a covariant lattice theory in $d = 10 - 2n$ dimensions one must provide

– a lattice $\Gamma_{16+2n;8+2n}$ with left and right dimensions as indicated,

– a supercurrent that satisfies the $N = 1$ supersymmetry algebra, and is consistent with the lattice. The supercurrent consists of a space–time part and (for $d < 10$) an internal part. The latter has the form (6.5).

The lattice must satisfy the following requirements:

(1) It must be even and self-dual with respect to a Lorentzian metric of the form $\mathrm{diag}((-)^{16+2n}; (+)^{8+2n})$.

(2) The last $8 - n$ components of any vector must belong to one of the four conjugacy classes (0), (v), (s) or (c) of the Lie algebra $\mathrm{D}_{8-n}^{\text{space-time}}$.

(3) It must contain all vectors of the form $\boldsymbol{w}_{\mathrm{L}} = 0$, $\boldsymbol{w}_{\mathrm{R}} = (\boldsymbol{l}, (v))$ and $(\boldsymbol{t}, (v))$ for all $\boldsymbol{l}$ and $\boldsymbol{t}$ appearing in the supercurrent. (This condition applies obviously only to dimensions less than 10.)

The rules for obtaining the physical states from such a lattice have been formulated in section 5.3.

7.2. Lowest heterotic string states

Now we have all the ingredients necessary to determine the possible lowest mass states in the spectrum. Most features of the spectrum are not special to lattices, and we will therefore adopt the notation of eq. (5.46), denoting the number operator for the left-moving light-cone oscillators as $\tilde{\mathcal{N}}_X$, and for the contribution to the mass from the internal sector as $\mathcal{N}_{16+2n}$. Table 2 summarizes the possible massless states and tachyons. Note that right-moving oscillators cannot contribute to massless states. Masses are quoted here in units of $\frac{1}{4}\alpha' m^2$, so that the bosonic string tachyon has "mass" -1. To get the answer for the special case of lattice theories, one uses $h_{3n} = \frac{1}{2}\boldsymbol{u}_{3n}^2$ and $h_{16+2n} = \frac{1}{2}\boldsymbol{w}_{\mathrm{L}}^2 + \tilde{\mathcal{N}}_{16+2n}$. Here we denote vectors on the lattice as $\boldsymbol{w} = (\boldsymbol{w}_{\mathrm{L}}; \boldsymbol{u}_{3n}, \boldsymbol{v})$, where v is a $\mathrm{D}_{8-n}^{\text{space-time}}$ weight, and $\boldsymbol{u}_{3n}$ is a $3n$-vector in internal space.

Of course not all particles in this table appear necessarily in all string theories; only the graviton, $B_{\mu\nu}$ and the dilaton are always present. We will now discuss some aspects of the spectrum in more detail, concentrating on lattice theories, but indicating the simple generalizations.

CPT

Every lattice has at least one automorphism, namely the one that changes the sign of all vectors. It is easy to see that in any dimension this has precisely the effect of a *CPT* transformation. In four and eight dimensions it flips the chirality of a spinor, while in six and ten dimensions it preserves chirality. On the left lattice it maps weights of a representation (r) to weights of the complex conjugate representation $(\bar{r})$. In four dimensions every fermion $(r)_{\mathrm{L}}$ is thus automatically accompanied by its *CPT* conjugate $(\bar{r})_{\mathrm{R}}$. Together they count as one fermion, since in four dimensions one does not usually count *CPT* conjugates separately ("three generations" means three generations plus three *CPT* conjugates). The gravitini in four dimensions should thus be counted by just taking into account the ones from (s) of D_5. In six and ten dimensions however (s) and (c) of D_6 or D_8 correspond to different particles with opposite chirality. They should both be counted. This will be important in the discussion of supersymmetry in section 8. (Note that on the covariant lattice this automorphism maps states of canonical ghost charge (i.e. -1 and $-\frac{1}{2}$) to states with non-canonical ghost charge. Thus a discussion of *CPT* for the ghost states is not entirely straightforward, but the arguments given above do not depend on that; they are valid for the light-cone states obtained from the covariant lattice.)

In general conformal theories the matter system must be unitary, which implies in particular that any operator has a Hermitean conjugate. This generalizes the lattice automorphism and its role as a *CPT* transformation.

Table 2
Possible massless particles and tachyons in heterotic string spectra. In the first column one finds the $\mathrm{D}_{8-n}^{\text{space-time}}$ conjugacy classes

Class	h_{3n}	h_{16+2n}	$\tilde{\mathcal{N}}_X$	mass	particle type
(0)	0	1	0	0	$\mathscr{G}_{\mathrm{L}}$ gauge boson
	0	0	1	0	graviton, $B_{\mu\nu}$, dilaton
(v)	$0 \le h_{3n} < \frac{1}{2}$	$h_{3n} + \frac{1}{2}$	0	$h_{3n} - \frac{1}{2}$	scalar tachyon
	$\frac{1}{2}$	1	0	0	massless scalar
	$\frac{1}{2}$	0	1	0	$\mathscr{G}_{\mathrm{R}}$ gauge boson
(s)	$\frac{1}{8}n$	1	0	0	massless fermion (chirality +)
	$\frac{1}{8}n$	0	1	0	gravitino
(c)	$\frac{1}{8}n$	1	0	0	massless fermion (chirality −)
	$\frac{1}{8}n$	0	1	0	gravitino

Gravity

The appearance of gauge symmetries in the even lattice formulation is always related to roots of the lattice, i.e. to vectors with $w_L^2 = 2$, $w_R^2 = 0$ or with $w_L^2 = 0$, $w_R^2 = 2$. The graviton originates, together with the $B_{\mu\nu}$ field and the dilaton from the root lattice of $D_{8-n}^{\text{space-time}}$ (combined with a left-moving oscillator excitation). This $D_{8-n}^{\text{space-time}}$ root lattice is always present: Lorentz invariance dictates that the last, space–time components of every lattice vector belong to $D_{8-n}^{\text{space-time}}$ conjugacy classes, but then self-duality guarantees the presence of the roots, which have integer inner product with all conjugacy classes. This argument is valid for any string theory, because of universality of the space–time sector.

In lattice theories, the left-moving oscillators can have a vector index, but also an "internal" index. In the latter case one gets $16 + 2n$ commuting vector bosons, so that the gauge group has rank $16 + 2n$, the maximal value allowed for any string theory in $10 - 2n$ dimensions. This is the most important way in which lattice theories are special; in general heterotic string theories the rank of the gauge group can in principle have any value between 0 and $16 + 2n$.

Left gauge symmetries

If there are any roots on the left lattice, the rank $16 + 2n$ gauge algebra extends to the algebra $\mathcal{G}_L$ generated by the Frenkel–Kac mechanism (see section 4.4) for these roots. This works exactly as for the bosonic string, as discussed in section 2.4. The Kac–Moody algebras one gets in lattice theories are always of level 1, and its Sugawara energy–momentum tensor always has a central charge $16 + 2n$ so that it spans the entire left-moving internal sector of the theory. Neither of these last two properties holds in general.

Supergravity

As table 2 indicates, there may also be roots on the right lattice which are not orthogonal to the $D_{8-n}^{\text{space-time}}$ root system. Their $D_{8-n}^{\text{space-time}}$ components can thus be vectors or spinors of $D_{8-n}^{\text{space-time}}$. In the latter case these spinor roots, when exponentiated, give rise to vertex operators for gravitini. The possibilities for having such roots are severely limited by Cartan's classification of root systems. Indeed, for $n \leq 3$ (i.e. $d \geq 4$) the only possibilities are embeddings of $D_{8-n}^{\text{space-time}}$ in exceptional algebras. We will discuss this in more detail in section 8.

Right gauge symmetries

There is also a possibility of having extra roots that are vectors of $D_{8-n}^{\text{space-time}}$. In that case a new set of vector bosons appears, gauging an algebra whose structure is determined by the right lattice. The gauge bosons get their vector index from oscillator excitations on the left, and the algebra $\mathcal{G}_R$ they generate is determined by the vectors u_R, which have length 1. It is in fact not a Frenkel–Kac algebra, but a higher level sub-algebra of the Frenkel–Kac algebra that one may associate formally with Γ_{3n}^{int} [56]. The algebra $\mathcal{G}_R$ can be determined by computing the three gauge boson couplings. In the right sector this computation requires picture changing, and consequently $\mathcal{G}_R$ depends on the choice of the world sheet supercurrent. In heterotic strings such gauge bosons imply the absence of chiral fermions: the extra roots imply that $D_{8-n}^{\text{space-time}}$ is a regular sub-algebra of a larger Lie algebra, which in this case can only be D_m, $m > 8 - n$. But then the $D_{8-n}^{\text{space-time}}$ spinors come from the decomposition of D_m spinors, and hence appear automatically in non-chiral pairs. This is precisely the reason why torus-compactification destroys chirality, and indeed many theories with non-vanishing $\mathcal{G}_R$ can be regarded as torus compactifications of higher-dimensional ones.

Both the appearance of supergravity and of $\mathcal{G}_R$ gauge bosons generalizes in a simple way to any conformal field theory. In section 8 we will give a simple argument showing that such gauge symmetries associated with the right-moving sector correspond precisely to the embeddings of D_{8-n} in some larger Lie algebra, such that the Dynkin index of the embedding is 1. The only possibilities, other than the supersymmetric ones tabulated in section 8, are $SO(16-2n) \subset SO(M)$, $M \geq 16-2n$. (Here n may also have half-integer values, to allow odd space–time dimensions). One gets $M-16+2n$ $\mathcal{G}_R$ gauge bosons, which gauge a sub-algebra of a Kac–Moody algebra associated with the right-moving sector.

Massless fermions and scalars

The appearance of massless fermions and scalars is in general not governed by simple criteria, and to find them one has to search through the conjugacy classes of the lattice. Their $\mathcal{G}_L$ representation is determined by writing $\boldsymbol{w}_L$ in terms of $\mathcal{G}_L$ weights (or U(1) charges). Their $\mathcal{G}_R$ representation (if any) is determined in a more cumbersome way from $\boldsymbol{u}_R$, by using the embedding of $\mathcal{G}_R$ in the Frenkel–Kac group of Γ_{3n}^{int}.

Fermions or scalars in the *adjoint* of $\mathcal{G}_L$ always have a special significance. One can replace the Frenkel–Kac generator in their vertex operator by $\bar{\partial}X^\mu$ to get another vertex operator in the theory. Thus fermions in the adjoint representation imply the existence of gravitini and hence supergravity, while scalars in the adjoint representation imply the existence of $\mathcal{G}_R$ gauge bosons (and hence no chirality). For general heterotic strings the argument is a bit more subtle, and holds only if the Kac–Moody generators themselves are the only vertex operators of conformal weight 1 that belong to the adjoint representation. In that case one can always replace these currents by $\bar{\partial}X^\mu$ to change an adjoint fermion or scalar into a gravitino or gauge boson. This condition holds for all level-1 Kac–Moody algebras, but higher level ones may have additional operators that can be used as vertex operators for adjoint fermions or scalars.

Tachyons

As is clear from table 2 one can in general have tachyons that are Lorentz scalars. Whether they appear or not depends on the lattices one considers. In non-supersymmetric theories there is no reason why they should not be present, and typically they are, although there are many exceptions. In supersymmetric theories one never has tachyons. This is due to two facts, namely the fact that, as one might expect, all fermions and bosons fit in supermultiplets (see section 8), and the fact that there cannot be fermionic tachyons.

Space–time fermions are due to lattice vectors in the Ramond sector of $D_{8-n}^{\text{space-time}}$, which are then automatically in the Ramond sector of Γ_{3n}^{int} because of world sheet supersymmetry (see section 6.5). The internal space contribution to their mass is bounded from below by $\frac{1}{8}n$. This bound applies to the Ramond sector of *any* superconformal system, not just lattice theories. There is no such bound for the ghost system, which is not unitary and whose energy is not bounded from below if one chooses arbitrarily large ghost charges. But the origin of that problem is understood, and the solution is the physical state selection rule, assigning definite canonical ghost charges to bosons and fermions. In the even lattice formulation the minimal fermion mass is obtained for the fundamental $D_{8-n}^{\text{space-time}}$ spinor, and is according to (5.46) equal to $(8-n)/8-1=-n/8$. Together with the internal contribution we get then always a non-negative mass. Hence there are no fermionic tachyons.

7.3. Spectra of type-II theories

The spectrum of a type-II theory can be obtained by simply applying all the foregoing results for the

Table 3
Particles that can appear in the spectrum of Type-II theories for all possible combinations of left and right conjugacy classes

Right \ Left		(0)	(v)	(v)	(s) or (c)
	h_{3n}	0	$\frac{1}{2}$	$<\frac{1}{2}$	$\frac{1}{8}n$
(0)	0	graviton $B_{\mu\nu}$, ϕ	G_L-gauge boson	–	gravitino
(v)	$\frac{1}{2}$	G_R-gauge boson	massless scalar	–	massless fermion
(v)	$<\frac{1}{2}$	–	–	tachyon	–
(s) or (c)	$\frac{1}{8}n$	gravitino	massless fermion	–	anti-symmetric tensor

fermionic sector of the heterotic string, to both sectors of the type-II string. Thus we consider lattices of the form $\Gamma_{8+2n;8+2n}$ which have $D_{8-n}^{\text{space-time}}$ lattices on both sides. The possible massless states and tachyons depend on the 16 combinations of the four $D_{8-n}^{\text{space-time}}$ conjugacy classes that can occur, and on the contribution h_{3n} from the "internal" part of the lattice. The possibilities are listed in table 3.

Most of the properties of the states in the spectrum follow directly from the discussion for heterotic strings in the previous subsection. In particular it follows that gauge symmetries can only be of a type similar to $\mathcal{G}_R$ for heterotic strings. Hence to get chiral fermions one should have a non-trivial gauge group in only one sector. The other sector should then provide their spinor-index (in fact it is rather difficult to construct chiral four-dimensional type-II strings, and impossible to obtain the standard model [57]). Several proposals for lower-dimensional type-II theories can be found in refs. [62, 57, 56].

The most interesting new feature is the appearance of anti-symmetric tensors in the product of left and right spinors. The ranks of the tensors obtained in this way depends on the dimension and the chirality of the spinors, and is shown in table 4. Notice in particular that in four dimensions one can get additional massless vector bosons in this way. However, by computing the operator products of the corresponding vertex operators, one finds that they always gauge a product of U(1)'s.

7.4. Lattices for ten-dimensional strings

To obtain ten-dimensional strings we are looking for lattices $\Gamma_{16;8}$ where the right-moving components are D_8 weights. To find them, we can make use of a trick explained in appendix A. Given a Lorentzian even self-dual lattice of the required form, we can make an even self-dual *Euclidean* lattice out of it by simply changing the sign of the last eight components of the metric. This seemingly absurd operation works only because the right lattice factor is known to be D_8; certainly changing the sign of the metric on an arbitrary Lorentzian even self-dual does not preserve the self-duality. The result of this

Table 4
States in the tensor-product of left- and right-moving spinors. Anti-symmetric tensors of rank k are denoted as $[k]$. In 6 and 10 dimensions, the subscripts + and − distinguish (anti-) self-dual tensors of opposite chirality. In 4 and 8 dimensions $(s)\times(s)$ and $(c)\times(c)$ are *CPT* conjugates, and are both needed the produce the result indicated in this table

d	$(s)\times(s)$	$(c)\times(c)$	$(s)\times(c)$
4	[1]		[0]
6	$[0]+[2]_+$	$[0]+[2]_-$	[1]
8	$2\times[1]+[3]$		$[0]+[2]$
10	$[0]+[2]+[4]_+$	$[0]+[2]+[4]_-$	$[1]+[3]$

map is an even self-dual Euclidean lattice of dimension 24. Such lattices have been classified by Niemeier, and can be found in table 10 in appendix A.

Not every Niemeier lattice yields a good string theory. To go back to a Lorentzian lattice one has to be able to find a D_8 sub-algebra. Obviously this must be a regular sub-algebra, because we want to have a D_8 *sublattice*, not just a *subalgebra*. The solutions, which were obtained in ref. [63], are listed in table 5.

Notice that there is one Niemeier lattice, $D_{16} \times E_8$, that produces two different string theories, because there are two ways of embedding D_8. The first two theories in table 5 are supersymmetric, because $D_8 \subset E_8$. The next five are tachyonic, because D_8 is embedded in a D_n factor with $n > 8$. When the Niemeier lattice is mapped to $\Gamma_{16;8}$, this embedding leaves a trace in the form of lattice vectors $((v), (v))$, belonging to a $(D_{n-8})_L \times (D_8)_R$ sublattice originating from the D_n factor. Such vectors lead to tachyons. Finally, the last theory is the rather well-known $SO(16) \times SO(16)$ string, which does not have supersymmetry, and also no tachyons. When it was found it was the first of its kind, but in four dimensions such models are very common. There is just one more string theory known in ten dimensions, a tachyonic one with gauge group E_8. Since it has rank 8, it can obviously not be described by a lattice.

It is extremely simple to read off the spectra of these theories from table 5. For example, consider the $SO(16) \times SO(16)$ string, the last entry in the table. In combination with the conjugacy class (0) of $(D_8)_R$ one finds the $(D_8 \times D_8)_L$ conjugacy classes $(0, 0)$ and (c, c). The latter contains only massive states (the norm of the smallest vector in (c, c) is four) and from $(0, 0)$ one gets the $SO(16) \times SO(16)$ gauge bosons, since (0) of $(D_8)_R$ contains space–time vectors. The $(D_8)_R$ conjugacy class (v) contains space–time scalars, but since it is paired on the left with (s, v) and (s, v), which have minimal norm three, there are no massless scalars in the theory. Of the four $D_8 \times D_8$ conjugacy classes that appear in combination with spinors of $(D_8)_R$, three (namely (v, v), $(c, 0)$ and $(0, c)$) contain norm-2 vectors that

Table 5
Ten-dimensional rank-16 string theories from Niemeier lattices. The value of k is 0 or 1

		Conjugacy classes associated with $(D_8)_R$			
Niemeier lattice	$\mathcal{G}_L$	(0)	(v)	(s)	(c)
$(E_8)^3$	$(E_8)^2$	$(0,0)$	–	$(0,0)$	–
E_8D_{16}	D_{16}	$(0+s)$	–	$(0+s)$	–
E_8D_{16}	E_8D_8	$(0,0)$	$(0,v)$	$(0,s)$	$(0,c)$
D_{24}	D_{16}	(0)	(v)	(s)	(c)
$D_{12}D_{12}$	D_4D_{12}	$(0,0)$ (v,s)	$(v,0)$ $(0,s)$	(s,v) (c,c)	(s,c) (c,v)
$D_{10}(E_7)^2$	$D_2(E_7)^2$	$(0,0,0)$ $(v,1,1)$	$(v,0,0)$ $(0,1,1)$	$(s,1,0)$ $(c,0,1)$	$(c,1,0)$ $(s,0,1)$
D_9A_{15}	D_1A_{15}	$(0,8k)$ $(v,8k+4)$	$(v,8k)$ $(0,8k+4)$	$(s,8k+2)$ $(c,8k+6)$	$(c,8k+2)$ $(s,8k+6)$
$(D_8)^3$	$(D_8)^2$	$(0,0)$ (c,c)	(s,v) (v,s)	(v,v) (s,s)	$(c,0)$ $(0,c)$

can yield massless particles. The massless fermions are thus in the $SO(16)\times SO(16)$ representation

$$(16,16)_+ + (256,1)_- + (1,256)_- ,$$

where the subscripts denote the chirality.

It is also very simple to write down the partition functions for these theories. Taking as an example again the $SO(16)\times SO(16)$ theory, one gets at one loop for its partition function $P(\tau)$

$$P(\tau)=\frac{1}{\eta^{12}\bar{\eta}^{24}}\{\vartheta^4_{[v]}(\bar{\vartheta}^8_{[0]}\bar{\vartheta}^8_{[0]}+\bar{\vartheta}^8_{[c]}\bar{\vartheta}^8_{[c]})+\vartheta^4_{[0]}(\bar{\vartheta}^8_{[s]}\bar{\vartheta}^8_{[v]}+\bar{\vartheta}^8_{[v]}\bar{\vartheta}^8_{[s]})$$
$$-\vartheta^4_{[s]}(\bar{\vartheta}^8_{[v]}\bar{\vartheta}^8_{[v]}+\bar{\vartheta}^8_{[s]}\bar{\vartheta}^8_{[s]})-\vartheta^4_{[c]}(\bar{\vartheta}^8_{[c]}\bar{\vartheta}^8_{[0]}+\bar{\vartheta}^8_{[0]}\bar{\vartheta}^8_{[c]})\}\,,$$

where, for example, $\bar{\vartheta}_{[v]}$ denotes $\vartheta_{[v]}(0|\bar{\tau})$. It is a simple exercise to convert this expression to spin structure basis, so that one has it in terms of Jacobi ϑ-functions. Note the role of the physical state selection rule in the right-moving (τ-dependent) part of this partition function.

Type-II strings in ten dimensions can be classified in a very similar way. They are described by covariant lattices $\Gamma_{5,1;5,1}$ or equivalently by even lattices $\Gamma_{8,8}$. The latter can all be obtained from regular embeddings of $D_8\times D_8$ in Euclidean even self-dual lattices of dimension 16. There are just two such lattices, but because we can choose opposite or identical chiralities for the left- and right-moving spinors there are four string theories. From the $E_8\times E_8$ lattice one gets the type-IIA and type-IIB strings, and from D_{16} one gets two tachyonic theories first constructed in ref. [64].

7.5. Lattices for lower-dimensional strings

Below ten dimensions a complete classification of all lattice theories appears to be nearly impossible, since there are simply too many of them. By imposing additional phenomenological constraints (absence of tachyons, chirality, a reasonable gauge group and fermion spectrum with not too many generations, and perhaps space–time supersymmetry) one can cut the number of surviving lattice theories down significantly, but it appears very difficult to enumerate them without first obtaining the complete list.

Nevertheless it is instructive to see at least a few examples, and there are several methods for constructing them, each probing a different corner of the gigantic space of solutions. Several of these methods are closely related to other ways of constructing strings below ten dimensions, and we will choose our examples so as to clarify the overlaps. A slightly more systematic discussion of other constructions and their relation with covariant lattices may be found in the next subsection. We will mainly consider examples in four space–time dimensions.

D_1 lattices

In constructing lattices for four-dimensional strings we face a new technical difficulty, namely the constraint vectors for the supercurrent. Consider for example the choice (6.14). The first thing one should observe is that the form of this supercurrent forces all vectors in the nine internal dimensions to have as their components integers or half-integers. This is true because by addition of constraint vectors one gets, among others, vectors of the form

$$(0,\ldots,0,2,0,\ldots,0)$$

in the nine-dimensional internal space, and zero elsewhere. The structure of the lattice in this nine-dimensional space can then be summarized in a nice way by saying that all its vectors belong to conjugacy classes of $(\mathrm{D}_1)^9$, where D_1 is the lattice introduced at the end of appendix A.3.

Now the entire right lattice is of the form $\mathrm{D}_5 \times (\mathrm{D}_1)^9$, and for simplicity we will assume that also the left lattice can be written as a product of D_n factors. Any D_n factor on the left can always be decomposed into a product of D_1 factors, so one could try to enumerate all the even self-dual lattices based on

$$[(\mathrm{D}_1)^{22}]_{\mathrm{L}}[(\mathrm{D}_1)^9\mathrm{D}_5]_{\mathrm{R}} \tag{7.1}$$

conjugacy classes. One can easily write a computer program that does this. It would start with a lattice containing the conjugacy classes of the constraint vectors

$$[(0)^{22}, (v), (v), (v), (0), (0), (0), (0), (0), (0), (v)]$$

$$[(0)^{22}, (0), (0), (0), (v), (v), (v), (0), (0), (0), (v)]$$

$$[(0)^{22}, (0), (0), (0), (0), (0), (0), (v), (v), (v), (v)]\,,$$

and then it would go through the list of conjugacy classes of (7.1) to find one of even Lorentzian length, and with even inner product with all the vectors that are already on the lattice. This procedure can be repeated to find a next conjugacy class, and when no new conjugacy class can be found, the lattice that one has is self-dual by definition.

Although the number of solutions is certainly finite, one quickly realizes that it must be big because the total number of conjugacy classes one has to search through is gigantic: 4^{32}. Therefore a self-dual lattice based on eq. (7.1) contains 2^{32} conjugacy classes. The magnitude of the problem is not unlike that of classifying all 32-dimensional even self-dual lattices using D_1^{32} conjugacy classes, which has at least 8×10^7 solutions*).

To specify a lattice of the form (7.1) completely one has to provide a list of conjugacy classes, which can be specified more concisely by giving a list of generators, as in table 10. Very often one finds root vectors of the form $(0, \ldots, 0, v, 0, \ldots, 0, v, 0, \ldots, 0)$ among the conjugacy classes, and it is then convenient to group several D_1 factors into larger D_n's.

A useful starting point is the lattice $(\mathrm{D}_{22})_{\mathrm{L}} \times (\mathrm{D}_{14})_{\mathrm{R}}$. To the root lattice one has to add four conjugacy classes to obtain a unimodular lattice, and if one furthermore requires it to be even, there is not much choice: The conjugacy classes must be $(0, 0) + (v, v) + (s, s) + (c, c)$ (this is just an Englert–Neveu lattice with the left factor stretched by 8 dimensions). The phenomenology is extremely simple: The first term gives a graviton (plus $B_{\mu\nu}$ and a dilaton), and the gauge bosons of SO(44) and the second one gives a tachyon in the vector representation of SO(44). The spinors are all massive and non-chiral, so that this example is certainly not of much interest. Note that the D_{14} lattice on the right contains the constraint vectors of world sheet supersymmetry. In fact there are many possible sets of constraint vectors that one could choose.

*) Estimates of this kind include all non-supersymmetric theories, which below ten dimensions appear to be far more numerous than supersymmetric ones. An exhaustive computer search for supersymmetric four-dimensional theories with this supercurrent yields (only!) a few thousand solutions [65]. In eight dimensions the total number of solutions for the supercurrent (6.14) is roughly 600 to 800, of which 13 are supersymmetric and 2 tachyon-free but not supersymmetric [66].

This lattice can be modified by adding conjugacy classes of sub-algebras of $(D_{22})_L(D_{14})_R$. This works in the same way as adding shift vectors. To preserve world sheet supersymmetry one has to make sure that the shift vectors have integral inner product with one set of constraint vectors, so that these constraint vectors are on the invariant sublattice Λ_0 (see appendix A.4).

Let us now make the model supersymmetric. This is easily achieved by adding a conjugacy class containing a gravitino. To do this, split D_{14} into $D_6 \times D_8$ and add the conjugacy class $(0, 0, s)$ of $(D_{22})_L \times (D_6 \times D_8)_R$, where we embed the D_5 space–time lattice into D_8 (in the following we adopt the convention of embedding the space–time lattice in the last factor).

Using the shift vector method explained in appendix (A.4) one can determine the conjugacy classes that appear. Since we have three D_n factors, there are in fact $2^3 = 8$ of them, and they are

$$(0,0,0) + (v,v,0) + (s,s,s) + (c,c,s) + (0,0,s) + (v,v,s) + (s,s,0) + (c,c,0)\,.$$

What we have now obtained is a ten-dimensional heterotic string, compactified on a lattice $\Gamma_{22;6} = D_{22} \times D_6$. The D_8 factor of the covariant lattice has been embedded in E_8 because of the addition of the gravitino. This example has in fact more supersymmetry than we bargained for, namely $N = 4$. It is of course not chiral.

There are two obvious obstacles to chirality in our example. The first is that the D_5 root system is part of a larger D_n root system (that of D_8 in this case), which always inhibits chirality as explained in section 7.2. The second is that the SO(44) gauge group simply does not have any complex representations, which is necessary for chirality in four dimensions. The only SO(N) groups with complex representations are SO($4k + 2$), $k \in \mathbb{Z}$. Their chiral representations are the spinors, and the fundamental spinor representations have weight length $(2k + 1)/4$. Clearly only the fundamental spinors of SO(2), SO(6), SO(10) and SO(14) can occur for massless fermions, since all others would have too high a mass. Furthermore a fermion cannot be chiral due to spinors of SO(6), because the SO(6) spinor representations are anomalous, and modular invariant string theories are anomaly free, as we will discuss in section 9. (Note that SO(2) anomalies can be canceled by the four-dimensional analog of the Green–Schwarz mechanism [2, 67].)

These chiral representations can only appear in combinations with total weight length equal to two. This still leaves many possibilities, and one of the simplest is the SO(10) × SO(6) representation $(16, 4) + (\overline{16}, \bar{4})$. The necessity to cancel the SO(6) anomaly implies that the number of generations is a multiple of eight in this case.

Breaking SO(44) into such small groups cannot be achieved by adding just one spinor shift vector, but two turn out to be sufficient. First decompose

$$(D_{22})_L(D_6D_8)_R \rightarrow (D_{14}D_8)_L(D_4D_2D_2D_6)_R$$

by adding a shift vector $(0, s; 0, s, 0, c)$. This shift vector projects out half of the gravitinos, so that we can get an $N = 2$ model with an SO(16) × SO(28) gauge group. It is still not chiral, but we can now break the lattice root system further down to

$$(D_7D_7D_5D_3)_L(D_2D_2D_2D_1D_1D_1D_5)_R$$

by adding the shift vector $(s, 0, 0, s; s, 0, c, s, 0, 0, c)$. This again removes one gravitino, so that we have

$N = 1$ supersymmetry. The remaining lattice vector giving a gravitino is $(0,0,0,0;0,0,0,s,s,s,s)$. The constraint vectors appear in the conjugacy classes

$$(0,0,0,0;v,0,v,v,0,0,v)\,, \qquad (0,0,0,0;0,v,v,0,v,0,v)\,, \qquad (0,0,0,0;v,v,0,0,0,v,v)\,.$$

To check if these vectors are on the lattice, it suffices to verify that they have integer inner product with the shift vectors we used.

The model we just constructed turns out to be chiral, and its complete fermion spectrum consists of the $SO(14)\times SO(14)\times SO(10)\times SO(6)$ representations

$$\begin{aligned}&2(1,1,16,4)+2(1,1,16,\bar{4})+(14,14,1,1)+(1,1,10,6)\\&\quad+(14,1,10,1)+(1,14,10,1)+(14,1,1,6)+(1,14,1,6)\,.\end{aligned}$$

Only the fermions in the first two terms are chiral, and give 16 generations of (16)'s of SO(10). These fermions originate from the lattice vectors

$$(0,0,s,s,0,0,c,0,0,s,c) \quad \text{and} \quad (0,0,s,c,0,0,s,0,0,s,c)\,.$$

Notice that the $(D_2)^3(D_1)^3$ factor of the right lattice only determines the multiplicities of the states in the theory, not their gauge quantum numbers. Since the model is supersymmetric, there are scalars in the same representations as the fermions.

This model is a well-known one, and appears in most of the papers on four-dimensional strings because it is one of the simplest to obtain. The construction given here closely resembles the methods used in the fermionic constructions.

One can continue from here, and add more shift vectors to slice the lattice into still smaller D_n chunks. It should be clear from the above that the reason for the rather large number of generations is the fact that their multiplicity is due to a spinor of SO(6) and one of SO(4). Thus the key to lowering the number of generations is to break these factors. This works indeed as expected, but we do not think it would be illuminating to show such models in detail. The model given above is just the tip of a gigantic iceberg. If one splits the lattice into more and more factors, there are more and more possibilities for doing this. Clearly the bulk of the models correspond to lattices which are split as much as possible. Unfortunately this "base of the iceberg" is the hardest part to explore, because of the huge number of conjugacy classes one has to deal with.

The shift vector method

The method described above provides many examples for one choice of supercurrent. To get examples of string theories with one of the other possible realizations of world sheet supersymmetry one can apply supersymmetry-preserving shifts to torus-compactified ten-dimensional strings. Hence we start with a lattice $\Gamma_{22;14} = \Gamma_{22;6}\times(E_8)_R$, where $\Gamma_{22;6}$ has the root system of some rank-6 Lie algebra $\mathcal{G}$ on the right lattice. We then apply a shift of the form $s = (s_L; s_R)$, where $s_R = (\boldsymbol{\sigma}, \theta_1, \theta_2, \theta_3, 0,0,0,0,0)$. Here $\boldsymbol{\sigma}$ is the shift vector representing some non-degenerate element w of the Weyl group of $\mathcal{G}$, and θ_i are the phases that parametrize its torus eigenvalues (see section 6). By construction this shift vector respects the constraint vectors of the supercurrent defined by w.

There are no restrictions on $\boldsymbol{w}_{\mathrm{L}}$ other than "level-matching", which is essentially the requirement that $s^2 = 0 \bmod 2$*). The large number of possibilities can be reduced drastically by requiring also a supersymmetry in the left sector, for practical rather than physical reasons.

The notion of a world sheet supersymmetry for the left-movers can be understood most easily by using the bosonic string map described in section 5.3. In four dimensions, this map takes a left- or right-moving superstring partition into a bosonic string partition function by replacing the D_5 space–time factor by a $\mathrm{D}_5 \times \mathrm{E}_8$ gauge factor. In this way, covariant lattices for type-II strings can be mapped to covariant lattices for heterotic strings. Only a small subset of the heterotic strings is obtained in this way and they are usually called (1; 1) theories (where $(p; q)$ indicates p left-moving and q right-moving supersymmetries). If in addition the type-II theory has space–time supersymmetry in both sectors one usually speaks of (2; 2) theories**).

String theories with (at least) (1; 1) supersymmetry are obtained from lattices of the form $\Gamma_{22;14} = (\mathrm{E}_8')_{\mathrm{L}}(\mathrm{E}_8)_{\mathrm{L}}\Gamma_{6;6}(\mathrm{E}_8)_{\mathrm{R}}$, where $\Gamma_{6;6}$ has a rank-6 Lie algebra in its left and right sector. (Actually for rank 6 (or less) the left and right Lie algebra are always indentical, and in all but one cases $\Gamma_{6;6}$ is an Englert–Neveu lattice.) We ignore the "hidden sector" E_8', and choose $\boldsymbol{s}_{\mathrm{L}} = (0, 0, 0, 0, 0, \phi_1, \phi_2, \phi_3, \boldsymbol{\rho})$, where $\boldsymbol{\rho}$ and ϕ_i represent a Weyl group element w', which need not be the same as w.

Even if w and w' are different, level matching is automatically satisfied [18]. This procedure enables us to construct chiral theories for all supercurrents obtained in section 6, and supersymmetric ones for about half of them.

This method is very similar to the (asymmetric) orbifold construction, and indeed all lattice theories obtained in this way can also be obtained by means of asymmetric orbifold constructions employing Weyl twists. We will discuss the relation between orbifolds and covariant lattices in more detail in the next subsection.

Conjugacy class pairing

A second method for constructing (1; 1) theories for each of the supercurrents of section (6.3) is left–right pairing of the conjugacy classes of the dual of the constraint vector lattice Δ, defined in section (6.5). More precisely, in eq. (6.17), take

$$\Delta_p = (\boldsymbol{\delta}_p + \Delta) \times \mathrm{E}_8 .$$

It is obvious that such a lattice is integral and unimodular, and hence self-dual. Such a theory has always a D_{8-n} gauge group in $10 - 2n$ dimensions (i.e. SO(10) in four dimensions) and is always chiral. It is also always tachyonic, because such a lattice contains in any case the vector representations of the two D_{8-n}'s, paired between left and right. This yields a tachyon in the vector representation of D_{8-n}.

It is easy to get around this problem. Instead of the lattice Δ generated by the constraint vectors, one starts with a lattice Δ', which is obtained by adding a "gravitino" to Δ. That is, one adds a vector s which has norm-2, integral inner product with Δ and whose $\mathrm{D}_{8-n}^{\text{space-time}}$ components are in the spinor conjugacy class (for orbifold supercurrents such vectors always exist). Then one constructs the dual of Δ' and goes through exactly the same procedure as explained for Δ above. The resulting lattice is certain

*) The level-matching conditions of e.g. [15] seem to be weaker, allowing also $s^2 = 2k/N$, $k \in \mathbb{Z}$, where N is the order of the shift. However, it is not difficult to see that there exists then an equivalent shift vector of even norm.

**) This notation is somewhat inadequate, since one can have $N = 2$ world sheet supersymmetry without having space–time supersymmetry. In fact *all* covariant lattice theories, even the ones without space–time supersymmetry, possess an $N = 2$ world sheet supersymmetry in their right-moving sector [68]. To get space–time supersymmetry, one has to satisfy an additional quantization condition on the U(1) charge of the $N = 2$ algebra. In the following, (2; *) or (*; 2) should always be understood as "$N = 2$ world sheet supersymmetry plus charge integrality".

to contain the vector s on left and right. On the right this vector leads to a space–time gravitino; on the left it leads to an extra gauge boson, extending $D_{8-n}^{\text{space-time}}$ to an exceptional group. Even though we added just one gravitino, we might still end up with more than one; in other words, there is now no guarantee that one does not end up with a non-chiral extended supergravity theory. But there are many examples where that is not the case, and we will discuss such an example below.

Tensoring of $N=2$ building blocks

Another way of satisfying the world sheet supersymmetry conditions is to use supersymmetric building blocks. The simplest of these is the one-dimensional "root"-lattice

$$\mathcal{U} = \{2k\sqrt{3} | k \in \mathbb{Z}\} \tag{7.2}$$

and its dual, or "weight"-lattice

$$\mathcal{U}^* = \{\tfrac{1}{6}k\sqrt{3} | k \in \mathbb{Z}\} . \tag{7.3}$$

This lattice $\mathcal{U}^*$ contains $\mathcal{U}$ as a sublattice, and has 12 cosets (which, by analogy with Lie-algebra lattices we shall call "conjugacy classes") with respect to it.

This lattice admits two complex conjugate supercurrents of the form

$$T_{\mathrm{F}}^{\pm}(z) = \mathrm{e}^{\pm \mathrm{i}\sqrt{3}X(z)} .$$

We have seen in section 6.4 that three copies of this factor yield the SU(3) Coxeter supercurrent. The lattice $\mathcal{U}$ can also be interpreted as a lattice realization of the $k=1$, $c=1$ element of the $N=2$ discrete series, and its bosonic realization was first discussed in ref. [69]. Lattices built out of $\mathcal{U}$ building blocks are thus special cases of $N=2$ tensoring, studied in ref. [16].

Perhaps the simplest examples are the ones obtained by conjugacy class pairing. The lattice Δ is generated by the constraint vectors, which have the form

$$\begin{gathered}(\sqrt{3}, 0, 0, \dots, 0, (v)) \\ (0, \sqrt{3}, 0, \dots, 0, (v)) \\ \vdots \\ (0, 0, \dots, 0, \sqrt{3}, (v)) .\end{gathered}$$

To get a supersymmetric theory, we should add a norm-2 spinor to Δ, turning it into the lattice Δ' discussed above. All "gravitini" s_{g} have the form

$$s_{\mathrm{g}} = (\pm \tfrac{1}{6}\sqrt{3}, \dots, \pm \tfrac{1}{6}\sqrt{3}, (F)) ,$$

where (F) stands for (s) or (c). Clearly any choice is as good as any other, so that to define Δ' we can choose s_{g} to have positive signs in all components.

This forces any spinor in the dual of Δ' to have n $+$-signs and m $-$-signs, with $|n-m| = 0 \bmod 3$. The construction yields in any case all left–right pairings of such spinors, and it is not hard to see that no other pairs of norm-2 spinors occur (this could happen if two such norm-2 spinors belonged to the same coset). Hence we can immediately read off the number of generations. For n copies of SU(3) we will get a $3n$-dimensional space spanned by vectors $\boldsymbol{t}$. The number of ways of choosing 3 negative signs for the $3n$ spinor components is of course $\binom{3n}{3}$, which is equal to 20 for $n = 2$ (6 space–time dimensions) and 84 for $n = 3$ (4 space–time dimensions)*). Thus we get 20 generations of (56)'s of E_7 in 6 dimensions, and 84 generations of (27)'s of E_6 in $d = 4$.

This is the simplest lattice one can make out of these building blocks, but not the only one. In four dimensions there are at least six solutions:
- The 84 generation model, constructed above and in ref. [16].
- The Coxeter-twisted $(A_2)^3$ lattice. This gives 36 generations and a gauge group $E_8 \times E_6 \times A_2 \times U(1)^6$. This is the "$Z_3$-orbifold", and has already been constructed in refs. [15] and [55].
- The E_6-lattice, twisted by the $(A_2)^3$ Coxeter element. This yields also 36 generations, but the gauge group is now $E_8 \times E_6 \times (A_2)^4$.
- The 6-dimensional 20 generation model, compactified on the A_2 torus. This has $N = 2$ space–time supersymmetry.
- The (untwisted) $(A_2)^3$ lattice, yielding an $N = 4$ theory.
- The (untwisted) E_6-lattice, also an $N = 4$ theory.

Here we mean by "$\mathcal{G}$-lattice" the Englert–Neveu lattice belonging to the Lie algebra $\mathcal{G}$. Similar models can be obtained by means of standard symmetric orbifold constructions without B_{ij} background fields (see e.g. [70]). These have rank-16 gauge groups.

A different construction of this kind is the one of ref. [71]. These authors build lattices for the supercurrent (6.14) out of $c = 3$ building blocks, each containing a set of norm-3 vectors of the form (6.14). In four dimensions, three of these blocks must be tensored together to get a realization of world supersymmetry on Γ^{int}_{3n}. It can be shown that the left lattice can then be obtained from the odd-self-dual lattices of dimension 22, and in particular that all gauge algebras are sub-algebras of the algebras associated with those lattices.

Modifying Niemeier lattices

It is possible to construct lattices for four-dimensional strings from other self-dual lattices, in particular from the Niemeier lattices. This method was used in ref. [12], but seems to have as a general rule the disadvantage that the number of SO(10) generations is either 16 or 0. This can presumably be attributed to the fact that the resulting covariant lattices are not split into a sufficiently large number of components. For more details we refer to ref. [12].

7.6. Overlaps between different constructions

We have discussed the relation with other constructions already several times before, and this is a good point to make a comparison between various constructions, and discuss their overlaps, as far as we know them. We consider only heterotic strings here, and for definiteness we work in four space–time dimensions.

First of all, lattice constructions give string theories with maximal rank $16 + 2n$, i.e. 22 in four dimensions. Furthermore, any maximal rank theory that we know of can be written in terms of lattices, suggesting that perhaps the correspondence between maximal rank theories and lattice theories is

*) The argument is a bit more dimension dependent than the notation would suggest, but the result is correct for $d = 4$ and $d = 6$, with the usual count of CPT-conjugates in those dimensions.

one-to-one. This would be quite elegant if true, but we do not know a general proof of this. A promising approach in the direction of a proof would be to map the heterotic string to a bosonic string as explained in section 5.3. It should not be too hard to prove that a bosonic string with maximal rank in both left- and right-moving sectors can be described by compactification on an even self-dual lattice. The missing link in the proof is however that we do not know the rank in the right-moving sector.

Fermionic constructions

Free fermionic constructions [11, 13] start with 44 left-moving fermions χ^i and 18 right-moving internal fermions ψ^a. The most general boundary conditions on the torus are specified by two orthogonal matrices M_{ab} and M_{ij} relating the fermions at $\sigma_1 = 0$ and $\sigma_1 = \pi$:

$$\psi^a(\pi) = M_{ab}\psi^b(0) \,, \qquad \chi^i(\pi) = M_{ij}\chi^j(0) \,.$$

In addition the space–time fermions ψ^μ can be periodic or anti-periodic. The complete partition function is a sum of many terms, each with a different choice for the boundary condition matrices and for the (anti-)-periodicity of ψ^μ. This sum is severely restricted by the conditions for modular invariance, and furthermore in each term the matrices M_{ab} are restricted by world sheet supersymmetry. The requirement is that every term in the supercurrent

$$T_{\mathrm{F}} = \mathrm{i}\psi^\mu \partial X_\mu + \tfrac{1}{6} f_{abc}\psi^a\psi^b\psi^c$$

has the same periodicity on the cylinder (and hence, by modular transformations, on any Riemann surface). This means that $\pm M_{ab}$ must be an element of $\mathrm{Ad}(\mathscr{G}_{\mathrm{SC}})$ (see section 6.4), or must be a permutation of isomorphic factors in $\mathscr{G}_{\mathrm{SC}}$, where the sign in front of M depends on the periodicity of ψ^μ.

In the worst possible case the matrices M in different terms in the sum over fermion boundary conditions do not commute. In that case the conditions for modular invariance are not yet fully understood at present. Let us therefore assume that all matrices do commute. In that case they can be simultaneously diagonalized. To do this we must go to a complex basis, and all complex eigenvalues appear then in complex conjugate pairs. Furthermore complex conjugate spinors have complex conjugate eigenvalues. If *all* eigenvalues are complex, we can replace the complex pair by exponentiated bosons, $\exp(\pm \mathrm{i}X)$. A term in the fermion sum corresponds to the bosons having a lattice periodicity $X(\pi) = X(0) + 2\pi L$, where L is a component of a lattice vector. Clearly such a periodicity leads always to complex conjugate eigenvalues. The modes of the bosonized fermions produce, when acting on the vacuum, states with momenta that lie on a lattice. Thus the fermionic theory admits a quantum-equivalent interpretation as a lattice theory.

If $L = 0$ or $L = \frac{1}{2}$ the eigenvalues are 1 and -1. A bosonized pair of complex fermions must have the same *real* eigenvalues, but the diagonalization of the orthogonal matrix M does not guarantee that the real eigenvalues appear with equal sign for complex conjugate fermions. Hence this bosonization is only possible if such real fermions are not present at all, or if the basis can be chosen in such a way that they pair up properly. In particular, an odd number of eigenvalues -1 inhibits a lattice interpretation.

The diagonalization of M_{ab} also rotates the structure constants f_{abc} in the supercurrent precisely as discussed in section 6.4. This produces the set of constraint vectors for the resulting lattice theory.

By inverting this procedure, it is always possible to write a lattice theory in terms of fermions with complex boundary conditions. However, as explained in section 6.4, it is not always possible to write the supercurrent in terms of free fermion trilinears.

Orbifolds

In orbifold constructions one starts with a covariant lattice theory with $N=4$ supersymmetry, which is specified by a lattice $\Gamma_{22;14}=\Gamma_{22;6}\times E_8$. This is nothing but a generalized torus compactification [9] of a ten-dimensional theory. One then modifies the theory by "modding out" by one of the symmetries of this torus, i.e. one regards points on the torus which are related by the action of some automorphism of the lattice $\Gamma_{22;6}$ as identical. This symmetry group may act differently on the left- and right-moving lattice, in which case one speaks of *asymmetric* orbifolds [15]. The symmetry operations that one can consider are discrete rotations of the left and/or right sector ("twists"), or discrete translations ("shifts"). States that are not invariant under these transformations are projected out. They are replaced by twisted sectors, which contain closed strings whose endpoints do not coincide, but connect points on the torus that are identified.

A special subclass are the *symmetric* orbifolds [8]. Here one considers lattices $\Gamma_{22;6}$ of the form $E_8\times E_8\times\Gamma_{6;6}$, where $\Gamma_{6,6}$ is related to a six-dimensional torus $\mathbb{R}^6/\Lambda_6$, exactly in the way discussed in section 2, usually (but not necessarily) without B_{IJ} background fields. One mods out by symmetries of the six-dimensional lattice Λ_6, which of course are also symmetries of the lattice $\Gamma_{6;6}$.

The orbifold procedure is completely analogous to torus compactification, which can in this language by described as uncompactified space-time $\mathbb{R}^N$ modded out by an N-dimensional lattice, where the basis vectors of the lattice are discrete translations. Roughly speaking asymmetric orbifolds relate to symmetric orbifolds in the same way as genuine torus compactifications (with lattice Λ_6) relate to generalized torus compactifications (with lattices $\Gamma_{22;6}$).

To preserve modular invariance of the torus-compactified theory, the norm of the shift vector and the eigenvalues of the discrete rotation must satisfy "level matching" conditions, to ensure invariance under the transformation $\tau\to\tau+1$. These conditions are the orbifold equivalent of the requirement that covariant lattices should be Lorentzian even. Invariance under the transformation $\tau\to-1/\tau$ is satisfied by including the twisted sectors. The lattice equivalent of this is the requirement of Lorentzian self-duality. In orbifold constructions world sheet supersymmetry of the original, untwisted $N=4$ theory is manifest, and it is preserved in the construction by making the same twists on the fermions ψ^μ and the bosons X^μ.

The relation between covariant lattices and orbifolds follows essentially from the discussion in section 6.3. Any asymmetric orbifold constructed with Weyl twists on left- and right-moving sector can be written in terms of a covariant lattice. The relation is made explicit by representing the Weyl twist in terms of shift vectors, which can be used to shift the lattice $\Gamma_{22;6}\times E_8$ to a non-trivial covariant lattice $\Gamma_{22;14}$. In the right-moving sector these shift vectors are nothing but the vectors $(\theta_1,\theta_2,\theta_3,\boldsymbol{\sigma})$ of section 6.3, with θ_i in the E_8 factor. Since these vectors have integral inner product with the constraint vectors of world sheet supersymmetry, the latter is preserved. Furthermore orbifolds constructed with additional shift vectors can be written in terms of lattices, provided that the shifts lie in the invariant subspaces of the twists (i.e. the twists should not act on the shift vectors; in general this would lower the rank of the gauge group). Orbifolds constructed with outer automorphisms, or with any other transformation that is known to lower the rank of the gauge group (e.g. the class of theories constructed in ref. [14]) can of course not be written in terms of lattices.

Symmetric orbifolds can only be written in terms of lattices if the torus that is being twisted can be described by an even self-dual lattice $\Gamma_{6;6}$ with left and right roots. This requires a very special choice of the radius of the compactification manifold, as well as background B_{IJ} fields (except for SU(2) groups). Indeed, a generic symmetric orbifold compactification leads to a gauge group of rank 16 or less.

A minor caveat in this equivalence are the "mod-2" conditions that are part of the level matching conditions of ref. [15]. It turns out that certain Weyl twists that are explicitly forbidden by these conditions lead to perfectly acceptable covariant lattice theories, provided that one chooses the order of the shift vector on the lattice twice as large as the order of the Weyl group element from which it originates.

Conversely, any chiral lattice theory we know can be written in terms of asymmetric orbifolds. The proof assumes that the supercurrent satisfies the conditions specified at the end of section 6.3. This is a non-trivial assumption, and we cannot exclude the possibility that counterexamples exist, but we do not know any. Given this assumption, it is easy to show that the covariant lattice theory can be turned into a torus compactification of a ten-dimensional theory by adding shift vectors, and vice-versa, and that the shift vectors can be interpreted as corresponding to Weyl twists.

For more details about the relation between orbifolds and covariant lattices we refer to refs. [58, 72].

Discrete series tensor products

In tensor product constructions one writes the $c = (22, 9)$ conformal field theory describing the integral degrees of freedom in terms of tensor products of discrete series models [16, 73]. By choosing the $N = 1$ or $N = 2$ discrete series one can make sure that world sheet supersymmetry is built in from the beginning. The advantage of $N = 2$ tensor products is that it is rather easy to build space–time supersymmetric models, since this property can be understood in terms of $N = 2$ world sheet supersymmetry plus a certain charge integrality condition (see section 8.1). In the class of models considered so far one chooses for the left-moving degrees of freedom an $SO(10) \times E_8$ Kac–Moody level-1 Kac–Moody algebra, and constructs the remaining $c = 9$ system out of $N = 2$ building blocks in a left–right symmetric way, so as to facilitate satisfying the modular invariance constraints [16]. More general constructions are undoubtedly possible as well.

The central charge of the kth element of the $N = 2$ discrete series is $3k/(k + 2)$, $k = 1, \ldots, \infty$, and ranges between 1 and 3, accumulating at the latter value. Hence to get $c = 9$ one needs at least 4 and at most 9 building blocks. Since each $N = 2$ building block contributes a U(1) factor to the gauge group, there are at least as many as there are building blocks. If these were the only U(1) factors, only tensor products of $c = 1$ models would have maximal rank. We have discussed this case in the previous subsection, and found that the $c = 1$ element of the discrete series has a lattice description. In rare cases there turn out to be additional U(1)'s. The only maximal rank examples we know appear to be equivalent to products of $c = 1$ models. Thus the overlap between this class and covariant lattice models exists, but is very small.

There exist now many other four-dimensional string constructions or compactifications (see for example ref. [74–76] for various other ideas), but we have neither the space nor the knowledge to compare all of these to lattice constructions.

8. Lattices and space–time supersymmetry

8.1. Supersymmetry and exceptional groups

It was observed in ref. [12] that there is an intimate relation between exceptional groups and space–time supersymmetry in string theory. Although this observation was made initially in the context of covariant lattice theories, it has become clear that it applies to all space–time supersymmetric string theories. This follows indirectly from the observation [77–80] that the vertex operator algebra of space–time supersymmetry is the same for any string theory, no matter how it is constructed. We will present here a simple but generally valid argument establishing this relationship [81], by employing the bosonic string map.

Under the bosonic string map in even space–time dimensions ($d = 10 - 2n$), $D_{5-n,1}^{\text{space-time}}$ is replaced by $D_{8-n}^{\text{space-time}}$. It is important to realize that this map replaces the indefinite "Hilbert" space associated with the Lorentzian lattice by a positive definite Hilbert space. As the space–time NSR-fermions generate a level-one $\hat{D}_{5-n}$ Kac–Moody algebra, it follows that $\hat{D}_{8-n}$ in the bosonic string formulation is also a level-one Kac–Moody algebra. Under the map, the supercharge is replaced by a set of operators which transform as spinors of D_{8-n}. Since these operators have conformal weight one, it follows from the discussion in section 4.1 that they extend $\hat{D}_{8-n}$ to a larger level-one Kac–Moody algebra. As the level of a sub-algebra is equal to the level of the embedding algebra times the embedding index, the possibilities for such extensions are completely determined by the classification of Lie algebras: we have to look for all index-one embeddings of D_{8-n} in algebras $\mathcal{G}$ such that the adjoint of $\mathcal{G}$ contains at least one D_{8-n} spinor. The possibilities are simply E_6, E_7 and E_8 (and Spin(N) in two dimensions). One can extend this also to odd dimensions. (There is no covariant lattice construction for odd-dimensional strings, but the bosonic string map is nevertheless valid, as emphasized in section 5.3.) In odd dimensions one must find a Lie algebra containing $SO(d+6)$, and a root in the spinor representation of $SO(d+6)$. For four or more dimensions this leads again to the exceptional groups E_n, suggesting that many of these theories can be viewed as torus compactifications of even dimensional theories. An exception occurs in three dimensions where minimal $N = 1$ supergravity corresponds to the embedding $B_4 \subset F_4$. The results are summarized in table 6*).

In even dimensions $d \geq 4$ we are dealing with level-one, simply laced Kac–Moody algebras, which can be realized completely in terms of free bosons. This implies that when $D_{8-n}^{\text{space-time}}$ is embedded into a group of larger rank, part of the internal sector of the theory can be described in terms of free, untwisted bosons. Thus, this part – together with the space–time sector D_{8-n} – can always be described in lattice language. This observation is very useful, as it allows one to obtain information about part of the compact sector, no matter how complicated the entire internal sector is. For example, this sector might even correspond to a Calabi-Yau compactification.

Consider for example a theory in four dimensions, and in particular, the E_6 ($N = 1$) case. From the foregoing it follows that every vertex operator can be written in the form

Table 6
Complete classification of extended supergravity in heterotic string theories. Type Spin(N) corresponds to triality rotated embeddings

	space–time dimension								
Type	10	9	8	7	6	5	4	3	2
E_8	1	1	1	2	2	4	4	8	(8, 8)
E_7	–	–	–	–	1	2	2	4	(4, 4)
E_6	–	–	–	–	–	–	1	2	(2, 2)
F_4	–	–	–	–	–	–	–	1	(1, 1)
Spin(N) ($N > 8$)	–	–	–	–	–	–	–	–	(N − 8, 0) or (0, N − 8)

*) For type-II theories one can get supersymmetries from both the left and right-moving sectors, yielding $N = 1, 2, 3, 4, 5, 6$ and 8 as the possible supersymmetries in four dimensions. Note that there are several ways to obtain $N = 2$ and $N = 4$ supersymmetry.

$$V(z,\bar z)=\tilde V(z,\bar z)\,\mathrm{e}^{\lambda\cdot\Phi}(z)\,, \tag{8.1}$$

where Φ denotes the vector $(H^{\mathrm{int}}, H_1, H_2|\phi)$ and $\tilde V(z,\bar z)$ corresponds to the other parts of the theory. The bosons $H_{1,2}$ are related to the space–time fermions ψ^μ, $\mu=1,4$ via bosonization, and H^{int} is a free boson related to an internal Kac–Moody algebra $\hat{\mathcal{U}}$, corresponding to $\mathrm{E}_6\supset \mathcal{U}^{\mathrm{int}}\mathrm{D}_5^{\text{space–time}}$. Here, $\mathcal{U}$ denotes a U(1) factor with charge unit $\frac{1}{6}\sqrt{3}$, as defined in eqs. (7.2) and (7.3) (the normalization is such that the norm of $\mathcal{U}^{\mathrm{int}}\mathrm{D}_5^{\text{space–time}}$ "weights" is equal to the norm of the corresponding E_6 weights). If the ghost ϕ is replaced by three positive metric bosons in the manner prescribed by the bosonic string map, then the vector $\boldsymbol{\lambda}$ becomes a weight of E_6. In particular, the supercharge (in the $-\frac{1}{2}$ picture) is realized by $\tilde V(z,\bar z)=1$ and $\boldsymbol{\lambda}^{\mathrm{Q}}=(\frac{1}{2}\sqrt{3}, \pm\frac{1}{2}, \mp\frac{1}{2}|-\frac{1}{2})$ (where one may choose either all upper signs or all lower ones), and becomes a spinor root of E_6 under the bosonic string map.

Similarly, $N=2$ supersymmetry in four dimensions is characterized by two free bosons in the internal sector, one associated with an $\hat{\mathrm{A}}_1$, the other associated with a $\hat{\mathrm{D}}_1$ Kac–Moody algebra*). This follows from the decomposition $\mathrm{E}_7\to[\mathrm{A}_1\mathrm{D}_1]^{\mathrm{int}}\mathrm{D}_5^{\text{space–time}}$. For $N=4$ one has three internal free bosons that form a $\hat{\mathrm{D}}_3$ Kac–Moody algebra. This is a consequence of $\mathrm{E}_8\to\mathrm{D}_3^{\mathrm{int}}\mathrm{D}_5^{\text{space–time}}$, and means that the right-moving sector is a torus compactification of a $N=1$, ten-dimensional theory. It is clear that all this works in the other dimensions in an analogous way.

Actually, the bosonic string map is a map between exceptional groups $\mathrm{E}_6\equiv(\mathcal{U}\mathrm{D}_5)^{(0)+(s)}$, $\mathrm{E}_7\equiv(\mathrm{A}_1\mathrm{D}_6)^{(0)+(s)}$, $\mathrm{E}_8\equiv\mathrm{D}_8^{(0)+(s)}$ and certain Lorentzian superalgebras $\mathrm{E}_{3,1}\equiv(\mathcal{U}\mathrm{D}_{2,1})^{(0)+(s)}$, $\mathrm{E}_{4,1}\equiv(\mathrm{A}_1\mathrm{D}_{3,1})^{(0)+(s)}$ and $\mathrm{E}_{5,1}\equiv\mathrm{D}_{5,1}^{(0)+(s)}$. Here, the superscript $(0)+(s)$ indicates that the algebra associated with the root lattice (0) is enlarged by an extra operator that is a spinor of the D-factor. Except for $\mathrm{E}_{5,1}$, the supercharge acts as a "glue vector" linking together a priori separate algebras. The hyperbolic superalgebra $\mathrm{E}_{5,1}$ has been discussed in ref. [49]. It would certainly be interesting to investigate the representation theory of these superalgebras in more detail. They can be viewed as string extensions of the conventional space–time supersymmetry algebras.

Thus any supersymmetric string theory has a structure related to exceptional groups. Presence of space–time supersymmetry in a heterotic (or type-II) string theory and presence of exceptional groups in the corresponding bosonic string are equivalent statements. It seems that everything that is known about supersymmetry and supermultiplets (for the kinds of supersymmetry that can appear in string theory) can be found back in the representation theory of the exceptional groups. For example, we can derive the structure of the simplest $N=1$ supermultiplets in four dimensions from the E_6 weight lattice. To obtain the light-cone Lorentz quantum numbers, decompose $\mathrm{E}_6\to\mathcal{U}^{\mathrm{int}}\mathrm{D}_1^{\text{light-cone}}\mathrm{D}_4^{\text{ghost}}$ and apply the physical state selection rules: only states which contain (8_v) or (8_s) of D_4 are physical. We first consider the smallest representations: the singlet (1), the (27) and the (78) (the adjoint). They are related to states with smallest masses and spins. Omitting the $\mathcal{U}^{\mathrm{int}}$ charges, we find for their $\mathrm{D}_1^{\text{light-cone}}\mathrm{D}_4^{\text{ghost}}$ content

$$\begin{aligned}
&(1)\to(0,1)\,,\\
&(27)\to(0,8_v)\oplus(\tfrac{1}{2},8_s)\oplus(-\tfrac{1}{2},8_c)\oplus(\pm1,1)\oplus(0,1)\,,\\
&(78)\to(\pm1,8_v)\oplus(\pm\tfrac{1}{2},8_s)\oplus(\pm\tfrac{1}{2},8_c)\oplus(0,28)\oplus2(0,1)\,,
\end{aligned} \tag{8.2}$$

*) Actually, the $\hat{\mathrm{A}}_1$ algebra is part of a global $N=4$ extended world sheet superalgebra [82], and $\hat{\mathcal{U}}$ above is the $\hat{\mathrm{U}}(1)$ part of a global $N=2$ algebra [78, 80, 83]. Note however that the characterization of space–time supersymmetry described in this section does not use extended world sheet supersymmetries at all.

where the first entry denotes helicity. According to the physical state selection rules, the D_4^{ghost} singlets do not contain physical states, while $(0, 8_v)$ yields a scalar and $(\frac{1}{2}, 8_s)$ a fermion. Thus the (27) contains the scalar multiplet. (The representation (27), which has weights of norm $\frac{4}{3}$, must appear in combination with a $\tilde{V}(z, \bar{z})$ with $(\bar{h}, h) = (1, \frac{1}{3})$ so as to give a massless state.) Similarly, (78) contains the vector multiplet: $(\pm 1, 8_v)$ gives the gauge boson and $(\frac{1}{2}, 8_s)$ the gaugino. Note that to count the spinors we should consider only $(\frac{1}{2}, 8_s)$ because $(-\frac{1}{2}, 8_s)$ is just a CPT conjugate. Note also that for every (27) the lattice necessarily contains a CPT conjugate $(\overline{27})$, which we will not discuss separately in the following. It is also easy to show that (351) and (650) of E_6 generate states at the first massive level.

In fact, one can push this further, as it is possible to relate some of the "unphysical" left-over singlet components to auxiliary fields. (Vertex operators for auxiliary fields have been investigated in refs. [79, 84–88]. It turns out that the natural choice for the auxiliary fields is the zero ghost picture. That is, the selection rule for auxiliary fields is that they are associated with D_4^{ghost} singlets. The results for chiral and vector $N = 1$ supermultiplets are summarized in table 7. Via combination with appropriate left-moving fields, one can also get the supermultiplets of the supergravity sector [85, 87]. It is conceivable that this procedure works for arbitrary spin and mass levels.

On any lattice, as soon as there exists a vector belonging to a particular conjugacy class, there exist also all other vectors belonging to that class. This implies that given, for example, a massless chiral supermultiplet coming from (27), also a tower of further, massive states with arbitrary high spins with the same gauge quantum numbers has to exist, all belonging to the same E_6 conjugacy class as the (27), i.e. the class (1) (see table 8). These states constitute an entire E_6 conjugacy class, and are not all merely descendant excitations. Although the fact that the spectrum has this structure can be seen most easily in the bosonic image of the theory, one may, if one prefers, also derive it without using the bosonic string map. In the odd-self-dual covariant lattice language, this structure can be attributed to the string extension $E_{3,1}$ of the supersymmetry algebra.

Each such E_6 tower separately can be decomposed according to $E_6 \to (\mathcal{U}^{\text{int}} D_1^{\text{light-cone}}, D_4^{\text{ghost}})$ to determine its light-cone state content. The E_6 conjugacy classes (y), $(y) = (0)$, (1) or $(\bar{1})$ decompose as follows

$$(y) \to (\Sigma_0^y, 0) \oplus (\Sigma_v^y, v) \oplus (\Sigma_s^y, s) \oplus (\Sigma_c^y, c)\,, \tag{8.3}$$

where obviously Σ_x^y denotes the conjugacy class of $\mathcal{U} D_1$, associated with the D_4^{ghost} conjugacy class (x) in the decomposition of the E_6 conjugacy class (y). According to the physical state selection rule only Σ_v^y and Σ_s^y contain physical states. It is not hard to determine then the physical state content of the conjugacy classes (0) and (1):

Table 7
Chiral and vector supermultiplets of $d = 4$, $N = 1$ supersymmetry from E_6 representations. The third column gives the canonical vertex operators in terms of vectors $\boldsymbol{\lambda}$ on $\mathcal{U}^{\text{int}} D_{2,1}^{\text{space-time}}$ (except for the D-auxiliary field), as in eq. (8.1). Curly brackets indicate that permutations should also be included. In the spinor entries in column 3 one may have either only upper, or only lower indices.

E_6-rep.	$\mathcal{U}^{\text{int}} D_1^{\text{light-cone}} D_4^{\text{ghost}}$	Cov. vertex operator	Field
(27)	$(\frac{1}{3}\sqrt{3}, 0, 8_v)$	$(\frac{1}{3}\sqrt{3}, 0, 0 \mid -1)$	ϕ
	$(-\frac{1}{6}\sqrt{3}, \frac{1}{2}, 8_s)$	$(-\frac{1}{6}\sqrt{3}, \pm\frac{1}{2}, \mp\frac{1}{2} \mid -\frac{1}{2})$	ψ
	$(-\frac{2}{3}\sqrt{3}, 0, 1)$	$(-\frac{2}{3}\sqrt{3}, 0, 0 \mid 0)$	F
(78)	$(0, \pm 1, 8_v)$	$(0, \{\pm 1, 0\} \mid -1)$	A_μ
	$(\frac{1}{2}\sqrt{3}, \frac{1}{2}, 8_s)$	$(\frac{1}{2}\sqrt{3}, \pm\frac{1}{2}, \mp\frac{1}{2} \mid -\frac{1}{2})$	λ
	$(-\frac{1}{2}\sqrt{3}, -\frac{1}{2}, 8_s)$	$(-\frac{1}{2}\sqrt{3}, \mp\frac{1}{2}, \mp\frac{1}{2} \mid -\frac{1}{2})$	$\bar{\lambda}$
	$(0, 0, 1)$	$\mathrm{i}\sqrt{3}\, \partial H^{\text{int}}$	D

$$
\begin{aligned}
&(0) \to (0, (v)), (\tfrac{1}{2}\sqrt{3}, (s)), (-\tfrac{1}{2}\sqrt{3}, (c)), (\sqrt{3}, (0)), \\
&(1) \to (\tfrac{1}{3}\sqrt{3}, (0)), (-\tfrac{1}{6}\sqrt{3}, (s)), (\tfrac{5}{6}\sqrt{3}, (c)), (-\tfrac{2}{3}\sqrt{3}, (v)),
\end{aligned}
\tag{8.4}
$$

where on the right-hand sides one finds $\mathcal{U}\mathrm{D}_1$ conjugacy classes. Note that the $\mathcal{U}$ entry in (8.4) should be regarded as a conjugacy class label for one of the twelve conjugacy classes of the lattice $\mathcal{U}$. Thus it is defined mod $2k\sqrt{3}$, $k \in \mathbb{Z}$. The light-cone structure of $N=1$ supermultiplets at every mass level can be directly read off from eq. (8.4). The upper entries in every sector include the potentially massless physical states of table 7. The lower entries are related to towers of states that start at non-zero mass. They are important for supersymmetry at higher mass and spin levels. In particular, $(\sqrt{3}, (0))$ describes the analogue of the constant anti-holomorphic three-form field of Calabi–Yau compactified theories [7, 83]*).

Note that because of the exceptional group structure, the infinite tower of supersymmetry multiplets can be characterized by (level-one) Kac–Moody characters of $\mathrm{Ch}^{\mathcal{K}}$ that have well-defined modular properties. These functions are defined in the next section.

All states in a given class ((0) or (1)) differ from each other by vectors $\in (0)$. One such vector is $\boldsymbol{\lambda}^Q$ associated with the supersymmetry charge, so that the classes (0) and (1) map into themselves under a shift by $\boldsymbol{\lambda}^Q$. This is intimately related to the approach that describes space–time supersymmetry via extended world sheet supersymmetries [78, 16]. In this approach, the mapping between various conformal fields (8.1) within one class is called "spectral flow". The spectral flow is described by continuous lattice shifts generated by the $\hat{\mathrm{U}}(1)$ current

$$
J^Q(z) = \mathrm{i}\lambda^Q \cdot \partial\Phi(z) . \tag{8.5}
$$

The condition for space–time supersymmetry is that the zero-mode of J^Q has integral eigenvalues on all states of the theory. In our language this means that $\boldsymbol{\lambda}^Q$ has integral inner product with all lattice vectors, and thus, by self-duality, that it lies on the lattice itself.

Most of the foregoing discussion remains true for extended supersymmetry and also in higher dimensions, though an, as yet, unresolved point is the structure of the auxiliary fields. More specifically, although it was shown that the auxiliary fields of the neutral sector of four-dimensional $N=2$ supersymmetry can be obtained from (0) of E_7 [85, 88, 89], the procedure appears to be problematical in the matter sector [88]. We would like to point out, however, that the no-go theorems for auxiliary fields do not apply to the kind of extended supergravity theories that we are considering here.

8.2. *Ghost triality and generalized Riemann identities*

We have seen in the last section that the structure of supersymmetric string theories is encoded in the representation theory of the exceptional algebras (we will not consider Spin(N) supersymmetries here). One might thus expect that other likely consequences of space–time supersymmetry, such as vanishing of the cosmological constant and nonrenormalization theorems, are also clarified by the exceptional group structure.

A first hint of this can be seen at one loop. As discussed before, the boson and fermion representations at any string level can be obtained from the exceptional group representations by decomposing them with respect to $\mathrm{D}_4^{\mathrm{ghost}}$. Fermions have weights of the spinor representation (8_s) as their $\mathrm{D}_4^{\mathrm{ghost}}$ components, whereas bosons have weights of the vector representation (8_v). The lattice D_4

*) Note that the presence of this field is a necessary, but not a sufficient condition for the existence of a supercharge [68].

has an outer automorphism which cyclically permutes (8_s), (8_c) and (8_v), known as triality (see also appendix C.4). Since we will apply triality to the D_4 lattice related to the ghost-sector of the theory, we will refer to this as *ghost triality* in the following.

It is a special property of the regular embedding of D_4 in the exceptional groups that this outer automorphism becomes an inner automorphism of the exceptional groups. Thus the rotation that takes (8_s) into (8_v) acts *within* an exceptional group representation. This means in particular that for every (8_v) of D_4 contained in some representation of an exceptional group, there is also an (8_s) (as well as an (8_c)). This is nicely illustrated by eq. (8.2) above. For the physical state spectrum that statement translates into having a fermion for every boson, and hence having supersymmetry at all levels, at least as far as multiplicities are concerned.

Translated in terms of partition functions this means the following. The multi-loop partition function of any (not necessarily supersymmetric) string theory has the form (cf. eqs. (5.41) and (5.44))

$$\mathscr{P}_H(\Omega, \bar{\Omega}) = \sum_\alpha \mathscr{P}_\alpha(\Omega, \bar{\Omega}) \tilde{Y}_\alpha(\Omega) , \tag{8.6}$$

where $\tilde{Y}_\alpha$ is equal to Y_α, defined in (5.40), but including also the supercurrent correlators omitted in section 5.3, and $\mathscr{P}_\alpha$ represents all other contributions to the partition function. The sum in eq. (8.6) is over all spin structures at genus γ, but it will be rather illuminating to convert the expression to conjugacy class basis. The relation between these two bases is explained in appendix C.2, and the result is

$$\mathscr{P}_H(\Omega, \bar{\Omega}) = 2^{2\gamma} \sum_{[x]} \mathscr{P}^{D_4}_{[x]}(\Omega, \bar{\Omega}) \tilde{Y}_{[x]}(\Omega) , \tag{8.7}$$

where $[x]$ denotes a collection of γ conjugacy classes of D_4 (the expression (8.7) is of course completely analogous to eq. (5.44), except for a superscript D_4 to emphasize that $\mathscr{P}_{[x]}$ it is written in D_4 conjugacy class basis). At one loop, the sum is simply over the four conjugacy classes (0), (v), (s) and (c).

Ghost-triality gives us information about the functions $\mathscr{P}^{D_4}_{[x]}$ appearing in the partition function. We know that if we replace $Y_{[x]}$ (or equivalently $\tilde{Y}_{[x]}$) by the factor $X_{[x]}$ defined in section 5.3, then the D_4^{ghost} factor contained in $X_{[x]}$ becomes part of an exceptional algebra $\mathscr{E}$. In terms of characters this implies that after the bosonic string map the partition function reads

$$\sum_{[x]} \mathscr{P}^{D_4}_{[x]}(\Omega, \bar{\Omega}) \, \mathrm{Ch}^{D_4}_{[x]}(\Omega) = \sum_{[y]} \mathscr{P}^{\mathscr{E}}_{[y]}(\Omega, \bar{\Omega}) \, \mathrm{Ch}^{\mathscr{E}}_{[y]}(\Omega) , \tag{8.8}$$

where $\mathrm{Ch}^{\mathscr{E}}_{[y]}$ denotes the character of the collection of conjugacy classes $[y]$ of the exceptional algebra $\mathscr{E}$. It can be written in terms of characters of the D_4^{ghost} sub-algebra, and the remainder, denoted $\mathscr{K}$:

$$\mathrm{Ch}^{\mathscr{E}}_{[y]}(\Omega) = \sum_{[x]} \mathrm{Ch}^{\mathscr{K}\,[x]}_{[y]}(\Omega) \, \mathrm{Ch}^{D_4}_{[x]}(\Omega) . \tag{8.9}$$

The characters $\mathrm{Ch}^{\mathscr{K}}$ are defined by this equation; they describe the supersymmetry multiplet structure at every string level. If $\mathscr{E} = E_n$, they are equal to the conjugacy class characters of the lattices K_{n-4} defined in appendix C.4. Comparing eq. (8.8) with (8.9) we find

$$\mathscr{P}^{D_4}_{[x]}(\Omega, \bar{\Omega}) = \sum_{[y]} \mathscr{P}^{\mathscr{E}}_{[y]}(\Omega, \bar{\Omega}) \, \mathrm{Ch}^{\mathscr{K}\,[x]}_{[y]}(\Omega) . \tag{8.10}$$

The point is now that the $\mathscr{K}$ characters satisfy generalized Riemann identities which can be derived using ghost triality rotations. These identities are derived for conjugacy class φ-functions in appendix C.4, and are also satisfied by the conjugacy class characters, which are proportional to the ϑ-functions (see appendix C.6). Thus the character identity reads (cf. (C.20))

$$\mathrm{Ch}^{\mathscr{K}\,[x]}_{[y]}(z|\Omega) = \mathrm{Ch}^{\mathscr{K}\,t[x]}_{[y]}(Tz|\Omega)\,, \tag{8.11}$$

where T is a matrix representing the inner automorphism of $\mathscr{E}$ within $\mathscr{K}$ and t is the corresponding (ghost)-triality rotation on the D_4 conjugacy classes. The character parameters z are equal to zero for the functions $\mathscr{P}^{D_4}_{[x]}$ defined above, but we give the more general formula because it is important in diagrams with external lines. If $\mathscr{E} = E_n$ there are $n-4$ such parameters for each handle of the Riemann surface and no parameters if $\mathscr{E} = F_4$. The matrices T are given explicitly in appendix C, eq. (C.21).

As an example, consider a four-dimensional, $N=1$ theory at one loop. Corresponding to the three conjugacy classes (0), (1) and $(\bar{1})$ of E_6, there are three identities (8.11), which describe the contributions of the gauge and matter sectors. For the triality rotation that interchanges $[v]$ and $[s]$ one gets

$$\mathrm{Ch}^{K_2\,[v]}_{[y]}(0|\tau) - \mathrm{Ch}^{K_2\,[s]}_{[y]}(0|\tau) = 0\,, \quad y = (0), (1), (\bar{1})\,. \tag{8.12}$$

The conjugacy classes Σ^y_x of $K_2 = \mathscr{U}D_1$ can be directly read off from eq. (8.4). Defining

$$\mathrm{Ch}^{\mathscr{U}D_1}_{[Q,x]}(\nu_1, \nu_2|\tau) = \mathrm{Ch}^{\mathscr{U}}_{[Q]}(\nu_1|\tau)\cdot \mathrm{Ch}^{D_1}_{[x]}(\nu_2|\tau)\,, \qquad \mathrm{Ch}^{\mathscr{U}}_{[Q]}(\nu|\tau) = \frac{1}{\eta(\tau)} \sum_{m\in(2\mathbb{Z}+Q)} q^{3m^2/2}\, e^{2\pi i\sqrt{3}m\nu}\,,$$

one has explicitly

$$\begin{aligned}
&\mathrm{Ch}^{\mathscr{U}D_1}_{[0,v]}(0,0|\tau) + \mathrm{Ch}^{\mathscr{U}D_1}_{[1,0]}(0,0|\tau) = \mathrm{Ch}^{\mathscr{U}D_1}_{[1/2,s]}(0,0|\tau) + \mathrm{Ch}^{\mathscr{U}D_1}_{[-1/2,c]}(0,0|\tau)\\
&\mathrm{Ch}^{\mathscr{U}D_1}_{[1/3,0]}(0,0|\tau) + \mathrm{Ch}^{\mathscr{U}D_1}_{[-2/3,v]}(0,0|\tau) = \mathrm{Ch}^{\mathscr{U}D_1}_{[-1/6,s]}(0,0|\tau) + \mathrm{Ch}^{\mathscr{U}D_1}_{[5/6,c]}(0,0|\tau)\,,
\end{aligned} \tag{8.13}$$

where the first identity belongs to the class (0) and the second one to (1).

At one loop, a four-dimensional superstring partition function is some linear combination of the three E_6 characters, as shown in eq. (8.10). The coefficient functions $\mathscr{P}^{\mathscr{E}}_{[y]}(\Omega, \bar{\Omega})$ can be very different from theory to theory, but this does not matter since the identity (8.12) is satisfied separately for each E_6 conjugacy class. To see the relevance of this identity, one should note that, by virtue of eqs. (5.44), (5.48) and (5.49), the one-loop partition function becomes simply $\mathscr{P}_H = \mathscr{P}_{[v]} - \mathscr{P}_{[s]}$, which vanishes because of eq. (8.11), if one chooses for t the transformation that interchanges the (v) and (s) conjugacy classes. For the ten-dimensional superstrings (which have "E_8-type" supersymmetry) one can write this in terms of standard (spin-structure) ϑ-functions, and the partition function is then proportional to the well-known identity $\vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4 \equiv 0$. It is very difficult to derive such identities directly for four-dimensional $N=1$ theories, starting with some explicit construction of the partition function. Ghost triality, on the other hand, provides a very simple derivation, that does not require knowledge of the details of various constructions.

At one loop, this is simply a more complicated way of arriving at an already obvious result: of course the vanishing of the partition function follows already from the fact that the number of bosons is equal to the number of fermions at each level, as we proved earlier in this section. The validity of the identity (8.11) is however by no means limited to one loop, and the question arises whether it might also be useful beyond one loop. At higher genus, matters are a bit more subtle, because the functions $\tilde{Y}$ are rather complicated objects, and are certainly not constant. The authors of ref. [90] have proposed a higher genus generalization of eq. (5.48), which becomes remarkably simple when written in terms of conjugacy classes. Observe that (5.49) implies

$$\tilde{Y}_{[0]} = \tilde{Y}_{[v]} + \tilde{Y}_{[s]} + \tilde{Y}_{[c]} = 0$$

i.e. the triality-singlet combinations vanish. The proposal of ref. [90] can be shown [81] to imply exactly that all triality singlets vanish also at higher genus. For example at genus 2 one would have, according to this conjecture, the following 5 identities

$$\begin{aligned} &\tilde{Y}_{[00]} = 0\,, && \tilde{Y}_{[0v]} + \tilde{Y}_{[0s]} + \tilde{Y}_{[0c]} = 0\,,\\ &\tilde{Y}_{[v0]} + \tilde{Y}_{[s0]} + \tilde{Y}_{[c0]} = 0\,, && \tilde{Y}_{[vv]} + \tilde{Y}_{[ss]} + \tilde{Y}_{[cc]} = 0\,,\\ &\tilde{Y}_{[vs]} + \tilde{Y}_{[vc]} + \tilde{Y}_{[sv]} + \tilde{Y}_{[sc]} + \tilde{Y}_{[cv]} + \tilde{Y}_{[cs]} = 0\,. \end{aligned} \tag{8.14}$$

Note that these identities have nothing whatsoever to do with supersymmetry, since – if they are valid – they hold even for non-supersymmetric theories. The supersymmetry identities are the ones one derives for eq. (8.10) using (8.11), and those identities hold irrespective of any subtleties in the superghost determinant and the supermoduli integration. One can use these supersymmetry identities to pick a subset $\{\mathcal{P}_{[r]}\}$ of $\{\mathcal{P}^{\mathrm{D}_4}_{[x]}\}$, such that one $\mathcal{P}_{[r]}$ belongs to each of the triality orbits in $\{\mathcal{P}^{\mathrm{D}_4}_{[x]}\}$. Then, using eqs. (8.11), (8.10) and (5.44), the partition function $\mathcal{P}_H$ can be reduced to the form

$$\mathcal{P} = \sum_r \mathcal{P}_{[r]} \cdot \{\text{triality singlets of } \tilde{Y}_{[x]}\} \equiv 0\,.$$

For example, the two-loop partition function is a sum of 16 terms, among which one finds $\mathcal{P}_{[00]}\tilde{Y}_{[00]} + \mathcal{P}_{[0v]}\tilde{Y}_{[0v]} + \mathcal{P}_{[0s]}\tilde{Y}_{[0c]} + \mathcal{P}_{[0c]}\tilde{Y}_{[0c]} + \cdots$. Using eq. (8.11) (with a triality rotation that interchanges (s) and (c) and one that interchanges (s) and (v)) and (8.10) one finds that the functions $\mathcal{P}_{[y]}$ satisfy $\mathcal{P}_{[0v]} = \mathcal{P}_{[0s]} = \mathcal{P}_{[0c]}$, so that the four terms in the partition function can be written as $\mathcal{P}_{[00]}\tilde{Y}_{[00]} + \mathcal{P}_{[0v]}(\tilde{Y}_{[0v]} + \tilde{Y}_{[0s]} + \tilde{Y}_{[0c]}) + \cdots$. Now using the conjectured identities (8.14) for the functions $\tilde{Y}$ one sees immediately that these terms (as well as the remaining 12) add up to zero. If the assumption in ref. [90] were correct, the foregoing would constitute a proof that the cosmological constant vanishes to all loops for any space–time supersymmetric string theory, independent of any particular construction.

It is however questionable that the identities for $\tilde{Y}$ proposed in ref. [90] hold for arbitrary genus. Since it is known that the multi-loop partition functions are gauge-dependent (before integration over the moduli), the issue is whether a gauge exists for which they vanish everywhere in moduli space. Whether this is true or not, any property of ten-dimensional superstring theories that relies on Riemann identities has a potential generalization to the entire class of lower-dimensional superstrings, by means of eq. (8.11). Of course, not every non-renormalization theorem valid in ten dimensions is automatically valid below ten dimensions as well. It is clear that the various types of non-renormalization theorems depend on the exceptional group structure. For amplitudes with external lines, it is important that the number of character parameters z of $\mathrm{Ch}^{\mathcal{K}}(z|\tau)$ is smaller in theories with less supersymmetry: their number is directly related to the number of external lines of amplitudes for which non-renormalization theorems hold.

9. The anomaly generating function and the elliptic genus

The anomaly generating function, $\mathcal{A}$, plays a central rôle in the discussion of the one-loop string anomalies. There are several different ways of obtaining it, each arising from the different perspectives that one can take. The first is by considering the field theory limit of the string and then assembling into a generating function the various contributions to the anomaly of the limiting field theories. This was historically the first method by which $\mathcal{A}$ was obtained [91], and we will describe this approach in the next sub-section. The second method of obtaining it is to realize that the anomaly of the string is related to the family index of the Dirac–Ramond operator, and that this can be calculated by the high temperature expansions of the appropriate supersymmetric sigma models. This was done in refs. [92, 93], and we will only indirectly use this approach here. The third method is by doing the honest loop calculation in the string, in which case the correlation functions of the appropriate vertex operators that appear in the string amplitude can be reduced to an expression in terms of the moments of essentially the anomaly generating function of the theory [67, 94].

The anomaly generating function turns out to be a holomorphic function which, apart from a crucial anomalous phase, is a modular function of weight $-p$ for a string in $2p+2$ dimensions. (One should note that in this section we will be concentrating on partition functions associated with the left-moving part of the string, and so for convenience we will reverse our earlier convention, and the modular parameter τ will now be associated with the left-movers, while $\bar{\tau}$ will be associated with the right-movers.) It is interesting to observe that in each of the approaches described above, the holomorphicity comes about in different ways. In the first it is holomorphic by fiat, in the second it is a consequence of the independence of the index from the massive states, and in the third it follows from observing that only the zero-modes of the right-moving vertex operators can contribute to the anomalous amplitude. These observations are obviously closely related, but each perspective gives a different insight. For example the last was instrumental in showing how, by triality rotation, the anomaly generating function could be related to the one-loop effective action of the string [89].

9.1. The anomaly generating function assembled

In a string theory the space–time fermion fields are generically obtained in representations of the form $g \otimes r \otimes s^{\pm}$ where g is some gauge representation, r is some Lorentz tensor representation and s^{+} (respectively, s^{-}) denotes the fundamental right-handed (respectively, left-handed) space–time spinor field. Denote such a field by $\Psi^{\pm}(g, r)$. Let R_r denote the Riemann two-form acting on a representation r, and let F_g denote the Yang–Mills two-form acting on the representation g. That is

$$\begin{aligned} F_g &= \tfrac{1}{2} F^a_{\mu\nu}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}\, \Lambda^a_g \\ R_r &= \tfrac{1}{2} R_{\alpha\beta\mu\nu}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}\, \Lambda^{\alpha\beta}_r\,, \end{aligned} \tag{9.1}$$

where Λ^a_g and $\Lambda^{\alpha\beta}_r$ are matrices of the gauge group in the representation g and the Lorentz group in the tensor representation r. As a notational convenience, R, without a subscript, will denote the Riemann two-form acting on the Lorentz vector representation. It is also useful to choose a basis in which R is diagonalized and introduce its two-form eigenvalues x_α, i.e.

$$\frac{R}{2\pi} \equiv \mathrm{diag}(\mathrm{i}x_1, -\mathrm{i}x_1, \mathrm{i}x_2, -\mathrm{i}x_2, \ldots, \mathrm{i}x_{p+1}, -\mathrm{i}x_{p+1}), \tag{9.2}$$

where we are taking the space–time dimension to be $2p+2$.

Using this, the Dirac genus, $\hat{A}(R)$ is defined by taking the formal power expansion of

$$\hat{A}(R) \equiv \prod_{\alpha=1}^{p+1} \frac{x_\alpha/2}{\sinh(x_\alpha/2)}. \tag{9.3}$$

The contribution of a massless fermion field $\Psi^{\pm}(g, r)$ to the anomaly is obtained by extracting the $(2p+4)$-form from refs. [95, 96]:

$$\pm\hat{A}(R)\,\mathrm{Ch}(F_g)\,\mathrm{Ch}(R_r) + \text{ghost contributions}, \tag{9.4}$$

where

$$\begin{aligned} \mathrm{Ch}(F_g) &\equiv \mathrm{Tr}[\exp(\mathrm{i}F_g/2\pi)], \\ \mathrm{Ch}(R_r) &\equiv \mathrm{Tr}[\exp(\mathrm{i}R_r/2\pi)], \end{aligned} \tag{9.5}$$

and the traces are, of course, evaluated over the representations g and r respectively. For a general higher spin field the calculation of the ghost contributions in eq. (9.4) appears to be a miserable proposition. However there is a simple saving grace. If one passes to the physical, light-cone gauge then one has no ghosts or gauge fixing to worry about, and moreover we may obtain the correct anomaly contribution for the field $\Psi^{\pm}(g, r)$ (with no γ-traces removed) by dropping all ghost contributions in eq. (9.4) and interpreting r as the representation of the transverse rotation group, and restricting the matrices R and R_r in (9.4) to the $2p$-dimensional transverse space. Thus, for the moment, we restrict to the light-cone states and only consider anomalies of the transverse rotation group. Even though we truncate the curvature matrices down to the directions of the polarization tensors one should note that we do *not* truncate the 2-form part, that is, in $R^{\alpha}_{\beta\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ the indices μ and ν still run from 1 to $2p+2$.

To see why this prescription is correct, we only really need to verify that it works for the gravitino field as there are no string theories with massless fermions of higher spin than this. The gravitino appears as part of a field $\Psi^{\pm}(g, r)$ where r is the vector representation of the Lorentz group. Indeed the physical content of $r \otimes s^{+}$ is a right-handed gravitino and a left-handed spinor, the latter coming from the γ-trace. (For $r \otimes s^{-}$ the chiralities are interchanged.) This gravitino has two right-handed ghosts and one left-handed ghost, and so the Lorentz covariant anomaly contribution has a gravitational curvature term [96] of $\hat{A}(R)(\mathrm{Ch}(R) - 2 + 1)$, while the contribution of the left-handed spinor is simply $-\hat{A}(R)$. Now one observes that if the curvature components vanish in the non-transverse directions, i.e. $x_{p+1} \equiv 0$, then $\mathrm{Ch}(R_{\mathrm{Lorentz}}) - 2 = \mathrm{Ch}(R_{\mathrm{transverse}})$ and the result reduces to the prescription above. As far as the field theory limit is concerned, the contribution to the anomaly of fields with spin 5/2 or more is irrelevant, and so, when we come to make the anomaly generating function, $\mathscr{A}$, the higher spin part of the function can be chosen for convenience. Having said this, one should note that when the anomaly generating function appears in the calculation of the loop amplitudes for the string one can no longer appeal to this field theoretical argument, and indeed the higher spin part is by no means irrelevant. As we will see shortly, it is possible to Lorentz covariantize our prescription for the transverse, rotation group anomaly. This prescription is not only covenient but also properly takes into account all the higher spin gauge invariances. Moreover it is this covariantized function that appears in the loop calculations, and can be thought of as the index of the Dirac–Ramond operator.

Consider now all the states in the Hilbert space of a heterotic string prior to imposing the $m_L = m_R$ constraint. Since we ultimately want to isolate the massless fermions, we fix $m_R = 0$ and require that the right-moving state be either s^+ or s^- in the space–time sector. Such states are, in general, associated with many gauge and transverse rotation group representations (g, r) at each mass level, m_L. Therefore, define the transverse anomaly generating function by:

$$\mathscr{A}^{t}(q, F, R) \equiv \hat{A}(R) \sum_{(g,r)} \varepsilon(g, r) q^{m_L} \mathrm{Ch}(F_g) \mathrm{Ch}(R_r) , \tag{9.6}$$

where $\varepsilon(g, r) = +1$ or -1 if (g, r) is associated with the right-moving state s^+ or s^- respectively, and m_L is the left-moving "mass", normalized so that the bosonic string tachyon has mass -1 (i.e. m_L is the eigenvalue of $L_0 - c/24$ cf. sections 3.2 and 4.4). The sum is over all representations at each mass level, and over all mass levels. The anomaly of the field theory limit of this string is then obtained by taking the coefficient of q^0 and extracting the $(2p+4)$-form.

It is extremely instructive to isolate the dependence of eq. (9.6) on the Riemann tensor. First observe that all the representations r are created only by the left-moving oscillators α^{μ}_{-n}, and that the action of such an oscillator on a state, $|\phi\rangle$, is to add on a vector index that will be automatically symmetrized relative to vector indices belonging to all other α_{-n}'s that make up the state $|\phi\rangle$. It is then a moment's thought to realize that all these bosonic oscillators give rise to the following multiplicative factor in $\mathscr{A}(q, F, R)$:

$$\prod_{\alpha=1}^{p} \left[\prod_{n=1}^{\infty} (1 - q^n\, \mathrm{e}^{2\pi \mathrm{i} \mu_\alpha})(1 - q^n\, \mathrm{e}^{-2\pi \mathrm{i} \mu_\alpha}) \right]^{-1} , \tag{9.7}$$

where $\mu_\alpha \equiv \mathrm{i} x_\alpha / 2\pi$, and the product over α ranges from 1 to p since we are restricted, at present, to the transverse modes. Similarly, for the Dirac genus we have, in terms of μ_α

$$\hat{A}(R) = \prod_{\alpha=1}^{p} \left(\frac{\pi \mu_\alpha}{\sin(\pi \mu_\alpha)} \right) , \tag{9.8}$$

and hence we have

$$\mathscr{A}^{t}(q, F, R) = \left[\prod_{\alpha=1}^{p} \frac{2\pi \mu_\alpha\, \eta(\tau)}{\vartheta_1(\mu_\alpha | \tau)} \right] \left(\sum_{g} \varepsilon\, \mathrm{Ch}(F_g) q^{m_L} \right) . \tag{9.9}$$

Now the appropriate Lorentz covariantization is clear, and so we define the complete anomaly generating function by

$$\mathcal{A}(q, F, R) \equiv \eta^2(\tau)\left[\prod_{\alpha=1}^{p+1} \frac{2\pi\mu_\alpha \eta(\tau)}{\vartheta_1(\mu_\alpha|\tau)}\right]\left(\sum_g \varepsilon \operatorname{Ch}(F_g) q^{m_L}\right). \tag{9.10}$$

Observe that as $\mu_{p+1} \to 0$,

$$\frac{\vartheta_1(\mu_{p+1}|\tau)}{2\pi\mu_{p+1}\eta(\tau)} \to \eta^2(\tau)$$

and so eq. (9.10) reduces to (9.9). Moreover, we can write $\mathcal{A}(q, F, R)$ in a manifestly Lorentz covariant form by using the Eisenstein expansion of $\vartheta_1(\mu|\tau)$. That is, remembering that in the vector representation:

$$\operatorname{Tr}\left[\left(\frac{R}{2\pi}\right)^{2k}\right] = 2\sum_{\alpha=1}^{p+1} (x_\alpha)^2 = 2\sum_{\alpha=1}^{p+1} (2\pi i\mu_\alpha)^2,$$

one obtains from eq. (C.27)

$$\eta^2(\tau)\left[\prod_{\alpha=1}^{p+1} \frac{2\pi\mu_\alpha \eta(\tau)}{\vartheta_1(\mu_\alpha|\tau)}\right] \equiv (2\pi)^{p+1}[\eta(\tau)]^{-2p} \exp\left[\sum_{k=1}^{\infty} \frac{1}{4k} \frac{1}{(2\pi i)^{2k}} \operatorname{Tr}\left(\frac{iR}{2\pi}\right)^{2k} G_{2k}(\tau)\right], \tag{9.11}$$

where the trace in eq. (9.11) is over all Lorentz indices.

It turns out that (9.10) (or (9.11)) represents the Lorentz covariant anomaly generating function in the sense that this function is what would be obtained if one had assembled all the anomaly contributions from the Lorentz invariant fields and their corresponding ghost systems. We have already demonstrated this for the massless gravitino; however it is also true for the unphysical "massive states" with $m_L \neq 0$ but $m_R = 0$. The proof of this is another consequence of the work of ref. [92] in which it was shown how $\mathcal{A}(q, F, R)$, as defined by eq. (9.10), can be interpreted as the index density of the Lorentz covariant Dirac–Ramond operator, and how the gauge and gravitational anomalies can be obtained from the family index of this operator. It is interesting to note that in ref. [92] the extra factor of $\eta^2(\tau)$ in $\mathcal{A}(q, F, R)$ comes from the left-moving reparametrization ghost determinant. These ghosts do not couple to the background F and R fields, but do play an essential rôle in the construction of the BRST operators and in defining gauge transformations of the space–time fields. Thus this factor of $\eta^2(\tau)$ can be viewed as generating all the anomaly contributions from the space–time ghosts coming from the space–time gauge fixing.

We could proceed from here to isolate the dependence of $\mathcal{A}(q, F, R)$ on the gauge fields, and go on to discuss the modular properties, but before doing this we will go back and re-express $\mathcal{A}(q, F, R)$ in terms of character-valued partition functions.

9.2. *The anomaly generating function as a chiral, character-valued partition function*

Consider a character-valued partition function of a string theory:

$$P(\boldsymbol{\lambda}, \tau, \bar\tau) = \operatorname{Tr}(e^{2\pi i \boldsymbol{\lambda}\cdot \boldsymbol{J}_0} q^{H_L} \bar q^{H_R}), \tag{9.12}$$

where the trace is taken over all states in the theory*), and $\boldsymbol{J}_0$ is a vector of charges of left-moving and

*) We also include traces over the ghost Hilbert spaces in the usual manner.

right-moving currents of the theory. In particular suppose that J_0 decomposes into $(\boldsymbol{H}_0, \boldsymbol{K}_0; \boldsymbol{L}_0, \boldsymbol{M}_0)$ where (i) $\boldsymbol{H}_0$ are a maximal commuting subset of zero-modes of the left-moving Kac–Moody currents that define the Yang–Mills gauge symmetry; (ii) $\boldsymbol{K}_0$ are a maximal commuting subset of zero-modes of the left-moving space–time Lorentz currents, and (iii) $\boldsymbol{L}_0$ and $\boldsymbol{M}_0$ are both maximal commuting subsets of zero-modes of the right-moving Lorentz currents, but, $\boldsymbol{L}_0$ is made purely out of bosonic oscillators with space–time indices and $\boldsymbol{M}_0$ is made out of NSR-fermion bilinears, or their equivalent bosonized form. Indeed, in the covariant lattice formulation, the set of currents corresponding to $\boldsymbol{M}_0$ is given by $\bar{\partial}\boldsymbol{H}(\bar{z})$, where $\boldsymbol{H}$ is the set of right-moving bosons in terms of which the fermions are expressed (see section 4.3). We introduce character parameters $\boldsymbol{\lambda} = (\boldsymbol{\nu}, \boldsymbol{\mu}; \boldsymbol{\rho}, \boldsymbol{\sigma})$ corresponding to the foregoing decomposition of J_0, and denote the components of this vector of parameters by ν_a, μ_α, ρ_α and σ_α where $\alpha = 1, 2, \ldots, p+1$ and $a = 1, 2, \ldots, N$, where N is the total rank of the gauge group, and $d = 2p+2$ is the dimension of space–time.

For $\boldsymbol{\lambda} \equiv 0$ P is nothing but the ordinary partition function of the theory, discussed in section 3.2. The rôle of the extra factor $\exp(2\pi i \boldsymbol{\lambda} \cdot \boldsymbol{J}_0)$ is to record information about the gauge and Lorentz representations of all the states in the theory. In this way one generalizes the partition function to an object that will not only give us the multiplicities of all states at a given mass level, but also the characters of their representations (such character-valued partition functions have already been considered many years ago by Nahm [97], although for a different purpose). Of course this kind of function is very similar to the anomaly generating function of the previous subsection. We will now make this similarity more precise.

One can separate out the contribution to P of the bosons and fermions with space–time indices. Indeed one has

$$P(\boldsymbol{\lambda}, \tau, \bar{\tau}) = \left\{ \prod_{\alpha=1}^{p+1} \left[\frac{4\pi^2 \mu_\alpha \rho_\alpha |\eta(\tau)|^2}{\vartheta_1(\mu_\alpha|\tau)\vartheta_1(\rho_\alpha|\bar{\tau})} \right] \right\} \sum_{j=1}^{4} P_j^{\text{ghost}}(\tau, \bar{\tau}) P_j(\boldsymbol{\nu}, \tau, \bar{\tau}) \left[\prod_{\alpha=1}^{p+1} \left(\frac{\vartheta_j(\sigma_\alpha|\bar{\tau})}{\eta(\bar{\tau})} \right) \right], \tag{9.13}$$

where the first factor represents the contribution of the bosons X^μ, the last factor is the contribution of the NSR-fermions ψ^μ, P_j^{ghost} is the contribution of all the ghosts and $P_j(\boldsymbol{\nu}, \tau, \bar{\tau})$ represents the internal degrees of freedom. It is only $P_j(\boldsymbol{\nu}, \tau, \bar{\tau})$ that differs from theory to theory; for a fixed space–time dimension the other factors are always the same. The sum in eq. (9.13) is over the four boundary conditions $(--)$, $(-+)$, $(+-)$ and $(++)$ (corresponding, respectively, to $j = 3, 4, 2, 1$) that an NSR-fermion can have along the two non-contractible loops of the world sheet torus (here + means period and − anti-periodic). The term of most interest to us is $j = 1$ ($(++)$ boundary condition), since only this sector contains chiral fermions.

Define, therefore, the chiral, Ramond component, P_{++}, of P to be the part of P that is proportional to $\Pi_{\alpha=1}^{p+1} \vartheta_1(\sigma_\alpha|\bar{\tau})$, i.e.

$$P_{++}(\boldsymbol{\rho}, \boldsymbol{\mu}, \boldsymbol{\nu}, \tau, \bar{\tau}) \equiv \eta^2(\tau) \left\{ \prod_{\alpha=1}^{p+1} \frac{4\pi^2 \mu_\alpha \rho_\alpha \eta(\tau) \vartheta_1(\sigma_\alpha|\bar{\tau})}{\vartheta_1(\mu_\alpha|\tau)\vartheta_1(\rho_\alpha|\bar{\tau})} \right\} P_1(\boldsymbol{\nu}, \tau, \bar{\tau}) . \tag{9.14}$$

Here we have used the fact that in the $(++)$ sector the right-moving reparametrization and superghost contributions cancel exactly, so that we are left with only the contribution of the left-moving reparametrization ghosts: $P_1^{\text{ghost}}(\tau, \bar{\tau}) = \eta^2(\tau)$.

Suppose that one now identifies all the Lorentz character parameters with the skew eigenvalues of the Riemann tensor, i.e. one takes

$$\rho_\alpha = \sigma_\alpha = \mu_\alpha = \mathrm{i}x_\alpha/2\pi\,, \tag{9.15}$$

where the x_α are defined in eq. (9.2). The function P_{++} then reduces to

$$P_{++}(\boldsymbol{\mu}, \boldsymbol{\nu}, \tau, \bar{\tau}) = \eta^2(\tau)\left\{\prod_{\alpha=1}^{p+1}\left[\frac{4\pi^2\mu_\alpha^2\,\eta(\tau)}{\cancel{\vartheta_1(\mu_\alpha|\tau)}}\right]\right\} P_1(\boldsymbol{\nu}, \tau, \bar{\tau})\,. \tag{9.16}$$

This cancellation of the ϑ_1-factors in going from eq. (9.14) to (9.16) represents the fact that the right-moving fermionic and bosonic oscillators create states in the same Lorentz representation, but with opposite chirality. This cancellation has a simple interpretation: it can be viewed as a manifestation of the space–time part of the world sheet supersymmetry. Physically, this cancellation, and a similar cancellation in P_1 (see below), had to take place because if they did not the theory would have massive chiral fermions, and these would violate Lorentz invariance. String theory avoids this potential catastrophe by ensuring that the function P_{++} is holomorphic. Thus the only states in the theory that can contribute to P_{++} and which satisfy $m_\mathrm{L} = m_\mathrm{R}$, are massless and chiral.

This concludes the discussion of dependence of P_{++} on the Lorentz character parameters, and we turn now to the dependence on the gauge character parameters $\boldsymbol{\nu}$. One can always conjugate in the Lie algebra of the gauge group so as to bring the background Yang–Mills field to the form

$$F/2\pi = \mathrm{i}y \cdot \boldsymbol{H}_0\,, \tag{9.17}$$

where $\boldsymbol{H}_0$ lies in the Cartan sub-algebra of the gauge group, and can of course be identified with the charges, $\boldsymbol{H}_0$, introduced above. If one therefore identifies the character parameters $\boldsymbol{\nu}$ with the skew two-forms $\boldsymbol{y}$, according to

$$\boldsymbol{\nu} = \mathrm{i}\boldsymbol{y}/2\pi\,, \tag{9.18}$$

then one obtains something very reminiscent of the anomaly generating function. Indeed, we will show that

$$P_{++}(\boldsymbol{\mu}, \boldsymbol{\nu}, \tau, \bar{\tau}) = \left(\prod_{\alpha=1}^{p+1} 2\pi\mu_\alpha\right)\mathcal{A}(q, F, R)\,. \tag{9.19}$$

Comparing eq. (9.16) with (9.10) one sees that P_1 equals the last factor (9.10). First of all we will show that P_1 is holomorphic. Let Ω denote the Hilbert space of states of the string theory, and let Ω' denote the Hilbert space of states of a new theory in which the chirality of all the fermions is inverted relative to Ω. That is, one obtains the states of Ω' from the states of Ω merely by interchanging the (s) and (c) conjugacy classes of the $\mathrm{SO}(d)$ Lorentz group generators corresponding to $\boldsymbol{M}_0$ defined above. In the same way that the Dirac operator flips chirality, the Dirac–Ramond operator, G_0, maps the states of the Ramond sector of Ω to the Ramond sector of Ω', and vice-versa. Moreover $G_0^2 = H_\mathrm{R}$ and G_0 commutes with $\boldsymbol{\lambda}\cdot\boldsymbol{J}_0$ provided that we take $\boldsymbol{\rho} = \boldsymbol{\sigma} = \boldsymbol{\mu}$. Thus the restriction of $P(\boldsymbol{\lambda}, \tau, \bar{\tau})$ to the Ramond sector is identical on both theories, with the exception of those states annihilated by G_0 and hence by H_R. Since the $\bar{\tau}$-dependence of the partition function enters through $\bar{q}^{H_\mathrm{R}}$, it follows that

$$\tfrac{1}{2}\left(P_\Omega^{\mathrm{Ram}}(\boldsymbol{\lambda}, \tau, \bar{\tau}) - P_{\Omega'}^{\mathrm{Ram}}(\boldsymbol{\lambda}, \tau, \bar{\tau})\right) \tag{9.20}$$

is holomorphic. However, this difference merely picks out the piece of $P(\boldsymbol{\lambda}, \tau, \bar{\tau})$ that is proportional to $\Pi_\alpha\, \vartheta_1(\sigma_\alpha|\bar{\tau})$. Hence P_{++} is holomorphic. It is an immediate consequence of this that the function $P_1(\boldsymbol{\nu}, \tau, \bar{\tau})$ is holomorphic. Therefore the function $P_1(\boldsymbol{\nu}, \tau)$ depends entirely on the left-moving gauge sector. It manifestly represents a sum over Chern characters, and thus all that remains to be checked in order to establish (9.19) is that $P_1(\boldsymbol{\nu}, \tau)$ reproduces the correct sign, ε, in eq. (9.10). This, however, is obvious when one bears in mind that P_1 is the coefficient of $\Pi_\alpha\, \vartheta_1(\sigma_\alpha|\tau)$. The latter function represents all the chiral excitations of the chiral ground state of the theory. This function, by definition, gives oscillator excitations of s^+ a weight of $+1$ and oscillator excitations of s^- a weight of -1.

Putting the foregoing together we can obtain the following simple recipe for the anomaly generating function. Let $P(\boldsymbol{\lambda}, \tau, \bar{\tau})$ be the partition function of the entire theory, as defined by (9.12). Take $\boldsymbol{\lambda} = (\boldsymbol{\nu}; \boldsymbol{\sigma})$ where $\boldsymbol{\nu} \cdot \boldsymbol{H}_0$ represents the Cartan sub-algebra of all the gauge groups, and $\boldsymbol{\sigma} \cdot \boldsymbol{M}_0$ is merely the Lorentz current of the fermion bilinears. The function P will contain a term of the form $F(\boldsymbol{\nu}, \tau, \bar{\tau})\, \Pi_{\alpha=1}^{p+1} (\vartheta_1(\sigma_\alpha|\bar{\tau})/\eta(\bar{\tau}))$. Take this term and differentiate once with respect to each of the σ_α, then set $\sigma_\alpha \equiv 0$, and use $\vartheta_1'(0|\bar{\tau}) = 2\eta^3(\bar{\tau})$. The result will be a holomorphic function $Q(\boldsymbol{\nu}, \tau)$. Then

$$\mathscr{A}(q, F, R) \equiv \prod_{\alpha=1}^{p+1} \left[\frac{2\pi\mu_\alpha\, \eta^3(\tau)}{\vartheta_1(\mu_\alpha|\tau)}\right] Q(\boldsymbol{\nu}, \tau)\,. \tag{9.21}$$

It is elementary to check that this recipe does indeed generate $\mathscr{A}(q, F, R)$ as given by eq. (9.19).

We have now given a definition of $\mathscr{A}(q, F, R)$ for an arbitrary string theory. If the string theory in question is described by an even lattice, one can obtain a more explicit expression for $\mathscr{A}(q, F, R)$. To derive $\mathscr{A}$ for covariant lattice theories (in the even lattice formulation) requires a relatively trivial modification in order to take into account the superghost (D_4) part of the lattice. Suppose that $\Gamma = \Gamma_{24-2p,16-2p}$ is a self-dual even lattice describing a string theory in $2p+2$ dimensions. Consider the character-valued lattice partition function (cf. (3.13)):

$$\mathscr{L}(\boldsymbol{\nu}, \boldsymbol{\sigma}, \tau, \bar{\tau}) = \sum_{(\boldsymbol{v}_{\rm L}; \boldsymbol{v}_{\rm R}) \in \Gamma} q^{(\boldsymbol{v}_{\rm L})^2/2}\, \bar{q}^{(\boldsymbol{v}_{\rm R})^2/2}\, e^{2\pi i(\boldsymbol{\nu}\cdot\boldsymbol{v}_{\rm L} - \boldsymbol{\sigma}\cdot\boldsymbol{v}_{\rm R})}\,, \tag{9.22}$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\sigma}$ are arbitrary vectors in $\mathbb{R}^{24-2p,16-2p}$. Because the "fermionic" space–time Lorentz generators are now embedded in a D_{p+4} factor of the lattice, when one applies the physical state projection rule, the chiral part of the partition function now appears as the coefficient of $\Pi_{\alpha=1}^{p+4}\, \vartheta_1(\sigma_\alpha|\bar{\tau})$ where σ_α are the components of $\boldsymbol{\sigma}$ in the D_{p+4} directions. Suppose that all other components of $\boldsymbol{\sigma}$ vanish, then we may write

$$\mathscr{L}(\boldsymbol{\nu}, \boldsymbol{\sigma}, \tau, \bar{\tau}) = \sum_{j=1}^{4} \mathscr{L}_j(\boldsymbol{\nu}, \tau, \bar{\tau}) \left[\prod_{\alpha=1}^{p+4} \vartheta_j(\sigma_\alpha|\bar{\tau})\right]. \tag{9.23}$$

One now divides $\mathscr{L}_1(\boldsymbol{\nu}, \tau, \bar{\tau})$ by the appropriate number of η-functions to obtain the oscillator contributions to the corresponding sector of the "internal" lattice (that is, we exclude both the oscillators with space–time indices, as well as the oscillators corresponding to the D_{p+4} part of the lattice). The result is the function $P_1(\boldsymbol{\nu}, \tau)$ defined above:

$$P_1(\boldsymbol{\nu}, \tau) = \frac{\mathscr{L}_1(\boldsymbol{\nu}, \tau, \bar{\tau})}{(\eta(\tau))^{24-2p}(\eta(\bar{\tau}))^{12-3p}}\,. \tag{9.24}$$

Thus from eqs. (9.16), (9.17) and (9.19) we get

$$\mathscr{A}(q, F, R) = \eta^2(\tau)\left\{\prod_{\alpha=1}^{p+1}\left[\frac{2\pi\mu_\alpha\,\eta(\tau)}{\vartheta_1(\mu_\alpha|\tau)}\right]\right\}\frac{\mathscr{L}_1(\boldsymbol{\nu}, \tau, \bar{\tau})}{(\eta(\tau))^{24-2p}(\eta(\bar{\tau}))^{12-3p}}\,. \tag{9.25}$$

To deduce the modular properties of $\mathscr{A}$ for lattice theories we use the transformation (3.14) of the lattice partition function $\mathscr{L}$. The transformation properties of θ_1 and η are well-known (see appendix C.6), and one finds

$$\begin{aligned}&\mathscr{A}\left(\frac{a\tau+b}{c\tau+d}, \frac{F}{c\tau+d}, \frac{R}{c\tau+d}\right)\\ &\quad = (c\tau+d)^{-p}\exp\left[\frac{\mathrm{i}c}{32\pi^3(c\tau+d)}\left(\mathrm{Tr}(F^2)-\mathrm{Tr}(R^2)\right)\right]\mathscr{A}(\tau, F, R)\,,\end{aligned} \tag{9.26}$$

(where $\mathrm{Tr}(F^2)$ is to be evaluated on any representation, but with the matrices, H_i, of the Cartan sub-algebra normalized so that $\mathrm{Tr}(H_iH_j) = 2\delta_{ij}$). The phase factor is usually referred to as the *modular anomaly*.

Although we have established (9.26) for lattice theories, there is a natural generalization to theories where Yang–Mills symmetries arise from arbitrary Kac–Moody algebras of arbitrary level*⁾ (see, for example ref. [98]). Observe first of all that the Lorentz part of $\mathscr{A}$ is universal in a given dimension (this part of the anomaly generating function has become known as the *elliptic genus* and has received a lot of attention in the mathematics literature recently [99]). The gauge part of the anomaly generating function of a lattice theory is nothing but a combination of characters of level-1 Kac–Moody algebras associated with the left sector. However, the formal definition of such characters is not restricted to level-1 Kac-Moody algebras: the generalization is known as the Weyl–Kac character formula (see e.g. [60]). The function Q in eq. (9.21) is in general a combination of such character functions.

The modular transformation properties of the Weyl–Kac characters is also known. Without going into details, we merely mention here that these transformations are similar to those of ϑ-functions [see eqs. (C.23, C.24)] in the sense that the characters transform into each other with a certain phase factor, and with a modular anomaly factor as in eq. (9.26). Modular invariance of the theory requires that the combination of characters appearing in the character-valued partition function $\mathscr{A}$ is such that all phases except the modular anomaly factors cancel. Hence the anomaly generating function of any string theory must transform like (9.26) (for level $k > 1$ the term $\mathrm{Tr}\,F^2$ in the modular anomaly differs however by a factor k in comparison with (9.26)).

The presence of a modular anomaly factor in $\mathscr{A}$ is in itself not a problem for the consistency of the theory, since $\mathscr{A}$ does not directly represent an amplitude of a physical process. As we will see in the next subsection, functions similar to $\mathscr{A}$ *do* appear in physical amplitudes, but always in a modified, modular invariant form.

9.3. Anomaly cancellation

It is very convenient to define two functions $\mathscr{A}_0$ and $\tilde{\mathscr{A}}$ that are naturally associated with the anomaly generating function:

*⁾ This includes all known heterotic string constructions, and in general any construction where the left-moving part of a vector boson vertex operator is a current of conformal weight one in a unitary conformal field theory. Strictly speaking, U(1) factors require a separate discussion, but their character-valued partition function can always be described by lattices.

$$\mathcal{A}_0(\tau, F, R) = \exp\left[\frac{1}{64\pi^4}\, G_2(\tau)(\mathrm{Tr}(F^2) - \mathrm{Tr}(R^2))\right]\mathcal{A}(\tau, F, R)\,, \tag{9.27}$$

$$\tilde{\mathcal{A}}(\tau, \bar\tau, F, R) = \exp\left[\frac{1}{64\pi^3}\,\frac{1}{\mathrm{Im}\,\tau}\,(\mathrm{Tr}(F^2) - \mathrm{Tr}(R^2))\right]\mathcal{A}(\tau, F, R)\,. \tag{9.28}$$

These two functions transform with modular weight $-p$, and have no modular anomaly. Indeed

$$\mathcal{A}_0\left(\frac{a\tau+b}{c\tau+d}, \frac{F}{c\tau+d}, \frac{R}{c\tau+d}\right) = (c\tau+d)^{-p}\mathcal{A}_0(\tau, F, R)\,, \tag{9.29}$$

$$\tilde{\mathcal{A}}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar\tau+b}{c\bar\tau+d}, \frac{F}{c\tau+d}, \frac{R}{c\tau+d}\right) = (c\tau+d)^{-p}\tilde{\mathcal{A}}(\tau, \bar\tau, F, R)\,. \tag{9.30}$$

The functions $G_2(\tau)$ and $1/\mathrm{Im}\,\tau$ introduced here both have precisely the right modular transformations so as to cancel the modular anomaly of $\mathcal{A}$.

This can also be seen as follows. Write $\mathcal{A}_0(\tau, F, R) = \mathcal{A}_0(\tau, 0, 0) \times \mathcal{W}(\tau, F, R)$, and then expand $\mathcal{W}$ into a power series in skew eigenvalues of F and R. The coefficient functions of terms of degree k in this power series are modular functions of weight k that are holomorphic for all τ in the complex upper half plane, including $\tau = \mathrm{i}\infty$. Hence it follows from the classification theorem of modular functions that all these coefficients can be written as polynomials in the two Eisenstein functions G_4 and G_6 (see appendix C.5). Thus we see from eq. (9.29) that $\mathcal{A}$ does contain factors of G_2, but that these are exactly canceled in $\mathcal{A}_0$ by the prefactor. In (9.30) G_2 is not canceled, but is uniformly replaced by the modular (but not holomorphic) function $\hat{G}_2(\tau) = G_2(\tau) - \pi/\mathrm{Im}\,\tau$. It is, in fact, the function $\hat{G}_2$ that appears naturally in the perturbation theory calculation of anomalies [67]. The factors of $\mathrm{Im}\,\tau$ appear essentially because of the zero-modes of the uncompactified bosonic coordinates.

To isolate the contribution of the space–time fermions to the gauge and gravitational anomalies in the field theory limit, we must expand $\mathcal{A}(q, F, R)$ in powers of q, and isolate the $(2p+4)$-forms in the coefficient of q^0. Consider, instead, any particular $(2p+4)$-form, for example $\mathrm{Tr}(R^{p+2})$, in the expansion of $\mathcal{A}_0(q, F, R)$. Its coefficient will be some power series in q. Considered as a function of τ, this coefficient function, $C(\tau)$, is analytic in the upper half-plane, except possibly for poles at $\tau = \mathrm{i}\infty$. One can also easily see from eq. (9.29) that $C(\tau)$ must be a modular function of weight 2. Let $j = G_4^3/\eta^{24}$ be the absolute modular invariant (See appendix C.5). The function j has a simple pole ($\sim q^{-1}$) at $\tau = \mathrm{i}\infty$. It is clear that one can find some polynomial, P, in j such that $C(\tau) - \mathrm{d}P(j(\tau))/\mathrm{d}\tau$ is modular of weight 2 and analytic everywhere including $\tau = \mathrm{i}\infty$. It follows from the classification theorem of modular functions (see, for example, ref. [100]) that such a function is identically zero, and hence if $P(j) = \sum_{m\geq -M}^{\infty} b_m q^m$ for some integer M, then

$$C(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\, P(j(\tau)) = 2\pi\mathrm{i} \sum_{m\geq -M}^{\infty} m b_m q^m\,. \tag{9.31}$$

From this it is clear that the coefficient of q^0 in the Fourier series of $C(\tau)$ is zero. Therefore there are no $(2p+4)$-forms appearing in the coefficient of q^0 in the q-expansion of $\mathcal{A}_0$, and consequently the $(2p+4)$-forms in the coefficient of q^0 in $\mathcal{A}$ must have the factorized form:

$$(1/2\pi)(\mathrm{Tr}(F^2) - \mathrm{Tr}(R^2)) \wedge X_{2p}\,, \tag{9.32}$$

for some $2p$-form X_{2p}.

This is precisely the form of the anomaly that can be canceled by the Green–Schwarz mechanism [2]. That is, in the field theory one introduces a 2-form, $B_{\mu\nu}$, that transforms under gauge and Lorentz transformations according to

$$\delta B = (1/4g)(\mathrm{Tr}(\mathrm{d}\Lambda_g \wedge A) - \mathrm{Tr}(\mathrm{d}\Lambda_l \wedge \omega)) , \tag{9.33}$$

where Λ_g and Λ_l are infinitesimal gauge and Lorentz parameters, while A and ω are the background gauge and spin connections, and g is the gauge coupling constant. The field theory anomaly can then be canceled by adding a local counterterm of the form

$$S = -4g \int \mathrm{d}^{2p+2}x \, B \wedge X_{2p} . \tag{9.34}$$

This, however, is not sufficient for the string theory. One knows that the spectrum of the theory contains such a field B, and one knows that at tree level in the string this field B couples to the gauge and spin connections through a Chern–Simons term as described in refs. [2, 101]. Requiring gauge and local Lorentz invariance therefore implies that B must transform according to eq. (9.33). However, one must still show that the string effective action generates a term of the form (9.34) and that this term has the correct coefficient.

This calculation has been done at one loop in string perturbation theory, and only to leading order in the external momenta [67] (so that the result is valid for energies far below the Planck scale). The relevant string diagram consists of a torus with one external B-field and another p external lines consisting of some combination of gauge fields and gravitons. As usual the amplitude can be written as an integral over all the insertion points, ν_i, of the external lines, and a final integral over τ taken over the fundamental modular region (in ten dimensions the integrand has also been obtained in ref. [102]). By expanding the world sheet propagators into Fourier series on the torus, one can explicitly perform the ν-integrals. The result of this calculation is that the string effective action contains a term of the form

$$S = \int \mathrm{d}^{2p+2}x \, (-4gB) \wedge \left(-\frac{1}{64\pi^2} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2} \, \tilde{\mathscr{A}}(\tau, \bar{\tau}, F, R) \Big|_{2p\text{-forms}} \right) . \tag{9.35}$$

To establish that this yields (9.34) we must describe how one performs the τ-integration, and show how the $2p$-form structure of $\tilde{\mathscr{A}}$ is essentially responsible for the appearance of X_{2p} in eq. (9.32).

If one wishes to calculate

$$I_k = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2} \, \hat{G}_2^k(\tau) F(\tau) , \tag{9.36}$$

where $\hat{G}_2(\tau)$ is given by (C.22), and $F(\tau)$ is a meromorphic modular function of weight $-2k$, then one observes that

$$I_k = -\frac{4\mathrm{i}}{\pi(k+1)} \int_{\mathcal{F}} \mathrm{d}(\mathrm{Re}\,\tau) \, \mathrm{d}(\mathrm{Im}\,\tau) \, \frac{\mathrm{d}}{\mathrm{d}\bar{\tau}} \left(\hat{G}_2^{k+1}(\tau) F(\tau) \right) , \tag{9.37}$$

and then applies Stokes' theorem to reduce it to a line integral around the boundary of $\mathcal{F}$. Since the differential $\hat{G}_2^{k+1}(\tau)F(\tau)\,\mathrm{d}\tau$ is modular invariant, the boundary terms cancel except at $\tau = \mathrm{i}\infty$, and since $\hat{G}_2(\tau) \to G_2(\tau)$ as $\tau \to \mathrm{i}\infty$, the result reduces to

$$I_k = \frac{2}{\pi(k+1)} \, G_2^{k+1}(\tau)F(\tau)\Big|_{\text{Coeff. of } q^0} . \tag{9.38}$$

Consider expanding $\mathscr{A}_0(\tau, F, R)$ into independent $2l$-form components, that is:

$$\mathscr{A}_0(\tau, F, R) = \sum_{l=0}^{\infty} \Big(\sum_{\alpha} C_\alpha^l(\tau) \omega_{2l}^\alpha \Big) , \tag{9.39}$$

where α indexes all the $2l$-forms, ω_{2l}^α, that can be obtained from traces of powers of F and R. The coefficient functions $C_\alpha^l(\tau)$ are modular functions of weight $l - n$. From eqs. (9.27), (9.28) and (9.39) one sees that $\tilde{\mathscr{A}}(\tau, \bar{\tau}, F, R)$ has a $2p$-form part of the form

$$\sum_{k=0}^{[p/2]} \sum_{\alpha} \frac{(-1)^k}{k!} \Big(\frac{1}{64\pi^4} \hat{G}_2(\tau) \Big)^k C_\alpha^{p-2k}(\tau) [(\mathrm{Tr}(F^2) - \mathrm{Tr}(R^2))^k \wedge \omega_{2p-4k}^\alpha] , \tag{9.40}$$

where $[p/2]$ denotes the largest integer less than or equal to $p/2$. Integrating this expression over $\mathscr{F}$ yields

$$-128\pi^3 \sum_{k=0}^{[p/2]} \sum_{\alpha} \frac{(-1)^{k+1}}{(k+1)!} \Big(\frac{1}{64\pi^4} \hat{G}_2(\tau) \Big)^{k+1} C_\alpha^{p-2k}(\tau)\Big|_{\text{Coeff. of } q^0}$$
$$\times [(\mathrm{Tr}(F^2) - \mathrm{Tr}(R^2))^k \wedge \omega_{2p-4k}^\alpha] . \tag{9.41}$$

However, using (9.27) and (9.39) we see that the coefficient of q^0 in the $(2p+4)$-form part of $\mathscr{A}(q, F, R)$ is simply

$$\sum_{k=1}^{[p/2]+1} \sum_{\alpha} \frac{(-1)^k}{k!} \Big(\frac{1}{64\pi^4} \hat{G}_2(\tau) \Big)^k C_\alpha^{p+2-2k}(\tau)\Big|_{\text{Coeff. of } q^0}$$
$$\times [(\mathrm{Tr}(F^2) - \mathrm{Tr}(R^2))^k \wedge \omega_{2p+4-4k}^\alpha] . \tag{9.42}$$

The anomaly factorization (9.32) established above shows that the foregoing sum starts at $k = 1$ and not $k = 0$. Apart from an overall factor of $-128\pi^3$ the expressions in eqs. (9.41) and (9.42) are the same, and remembering the factor of 2π in (9.32) we have therefore established that the term (9.35) appearing in the string theory effective action is precisely the required field theory counterterm (9.34).

One can of course calculate the anomaly graph purely in string perturbation theory [103], rather than what we have done here, which is to isolate the relevant contributions to the gauge transformations of the field theory effective action. In the string calculation one finds that the anomaly calculation requires one to integrate a total derivative over the moduli space of a torus with $p + 2$ punctures. While the result is zero, it is a rather non-trivial zero in that it requires one to have done the entire calculation described above before one can conclude that it is zero. More precisely, it has been shown in ref. [94] that there are two mutually canceling contributions from two distinct boundaries of moduli space. The first boundary term arises at $\tau = \mathrm{i}\infty$ and yields an expression proportional to the field theory anomaly term that can be obtained by applying the method of descent [95, 96] to the expression (9.32). The second boundary term arises in the limit where the insertion points of two of the external gauge bosons, or two gravitons, coincide. The coincidence limit of the two corresponding vertex operators can be factorized into a B-field vertex operator with a tree level coupling to the gauge or gravitational fields. In this way it was shown in ref. [94] that, at one loop, the careful treatment of the anomaly calculation reduces, at least in the low energy limit, to the foregoing discussion of the effective section. It is still an open question as to what happens at higher loops in string theory.

10. Outlook

During the last four years there has been progress in string theory along two lines, the construction of new theories, and the formulation of perturbation theory. Both these developments have been highly successful, and when put together they provide us with a stunningly large set of four-dimensional theories, each of which (at least if it is supersymmetric) is likely to be a perturbatively finite theory of gravity and all other interactions. In fact the two lines of development have been somewhat orthogonal, and work on perturbation theory has concentrated on ten-dimensional strings. The proof of finiteness in ten dimensions appears to be approaching completion, and it seems likely that the proof for four dimensional supersymmetric strings will not be far behind.

In relation to this we should emphasize that all string theories we have discussed are rooted in superconformal field theory, just like the ten-dimensional ones, and that there is absolutely no fundamental obstacle to writing down any correlation function at arbitrary genus. The most difficult part of the partition function, the β–γ ghost system, is universal, and we expect it to be no more problematic in four dimensions that in ten. Furthermore there is also an appealing universality in that part of the theory that distinguishes different theories, namely the internal sector. Despite their differences, the partition functions of all supersymmetric string theories satisfy universal identities very similar to Riemann identities, a fact that might be helpful in generalizing ten-dimensional results to four dimensions. Of course one should check carefully that there are no surprises, such as the Fayet–Iliopoulos supersymmetry breaking that was found to exist in four-dimensional string theories with anomalous U(1)'s, but nonetheless it seems unlikely that perturbative effects will significantly reduce the number of consistent four-dimensional superstrings.

Thus string theory appears to have succeeded too well. One can debate whether this abundance of consistent theories is good or bad in principle, but it certainly poses horrendous practical problems. With regard to matters of principle, several schools of thought exist. Some people hope to find a criterion which favors one four-dimensional theory, hopefully containing the standard model, over all others. The most obvious place to look for such a criterion is in non-perturbative effects, but unfortunately next to nothing is known about such effects. Others hope for some universality theorem stating that if one considers some ensemble of all vacua, the dominant contribution will come from those theories that lead to the standard model. Still others speculate that nothing but our own existence selects the vacuum we live in (this is also known as the anthropic principle). If one takes this last point of view, the main worry is that the number of consistent string theories is too small rather than too large.

These points of view can make sense only if the four-dimensional string theories are not really different theories, but different vacua of the same theory. This is generally believed to be likely (circumstantial evidence exists in the form of non-trivial connections between different theories, for example via torus compactification), but it would be interesting to know of what exactly they are ground states. The recent progress in string theory has taught us a lot about possible ground states, and perturbations to arbitrary order about them, but we still know disappointingly little about effects that go beyond perturbation theory. Such effects must certainly be there: string theory contains Yang–Mills theory, which does have non-perturbative effects that are unlikely to result from string perturbation theory.

The behavior of string theory beyond perturbation theory could in principle jeopardize the very idea that string ground states are simply conformal field theories satisfying some additional conditions. Non-perturbative tunneling effects destabilizing the would-be ground states are by no means excluded. Even if that were the case, experience with similar situations in field theory suggests that understanding the "false" vacua is an important first step towards a more exact solution to the problem.

The study of perturbative string ground states appears to be a nearly closed subject for lattice theories. A few open problems still exist, though. We do not yet have a proof that the list of lattice realizations of world sheet supercurrents for chiral theories is exhausted by the orbifold supercurrents of section 6.3. It would also be nice to have a proof of (or a counterexample to) the statement that all maximal rank theories can be described by lattices. Given this, one could in principle enumerate all the lattice theories, and with a restriction to "interesting" (supersymmetric, chiral, ...) theories, the list might even be manageable.

It is however clear that lattice theories are by no means the complete answer. Many other string constructions exist, but rather than forever extending the set of constructions for special solutions, one would like to find ways to solve directly the string consistency conditions formulated in section 1.1. Very interesting progress in that direction is made in recent work on the classification of rational conformal field theories, although one still has a long way to go before anything relevant to four-dimensional string theory can be classified. Nevertheless, one may expect that in the near future we will gain much more insight into general features of four-dimensional string theory. As we have often emphasized, the properties of lattice theories provide useful clues about strings in general, which is why we believe they are worth studying.

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Appendix A. Self-dual lattices

In this section we collect most of the known facts about self-dual lattices. Some of the papers containing these facts are [100, 104, 105], and readers interested in proofs not provided below should consult these papers. We also recommend the reviews by Goddard and Olive [3].

A.1. Definitions

A lattice Λ is defined as a set of points in a vector space V of dimension N, and has the form

$$\Lambda = \left\{ \sum_{a=1}^{N} n_a \boldsymbol{e}_a | n_a \in \mathbb{Z} \right\} , \tag{A.1}$$

where $\boldsymbol{e}_a$ $(a = 1, \ldots, N)$ is a set of basis vectors of V. The vector spaces we will consider are either $\mathbb{R}^N$, with a Euclidean inner product, or $\mathbb{R}^{p,q}$, $(p + q = N)$, with a "Lorentzian" inner product. In terms of the components v^I of vectors $\boldsymbol{v} \in \mathrm{V}$ we define these inner products as $\boldsymbol{v} \cdot \boldsymbol{w} = v^I w^J d_{IJ}$, with $d_{IJ} = \delta_{IJ}$ for $\mathbb{R}^N$, and $d_{IJ} = \varepsilon(I)\delta_{IJ}$, $\varepsilon(I) = 1$ for $1 \leq I \leq p$, $\varepsilon(I) = -1$ otherwise, for $\mathbb{R}^{p,q}$. In the following we will denote these inner products as $(+)^N$ and $(+)^p(-)^q$ respectively, and we will refer to d_{IJ} as the signature.

The *unit cell* of a lattice is the set of points $\{\boldsymbol{x} = \Sigma_a x_a \boldsymbol{e}_a | 0 \leq x_a < 1\}$. It contains precisely one lattice point (i.e. $\boldsymbol{x} = 0$), and every vector of V has precisely one "image" in the unit cell, obtained by adding a lattice vector to it. The *volume* $\mathrm{vol}(\Lambda)$ of the unit cell of Λ is obtained by defining a matrix

$$g_{ab} = \boldsymbol{e}_a \cdot \boldsymbol{e}_b , \tag{A.2}$$

in terms of which

$$\mathrm{vol}(\Lambda) = \sqrt{|\det g|} . \tag{A.3}$$

The matrix g_{ab} is called the *metric* of the lattice. It contains all information regarding the angles of the basis vectors and their lengths. Notice that if we change the signature from Lorentzian to Euclidean (by changing $(+)^p(-)^q$ to $(+)^{p+q}$) then $\det g$ changes at most by a sign, so that the volume of the unit cell remains the same.

The *dual* Λ^* of Λ is defined as

$$\Lambda^* = \{\boldsymbol{w} \in V | \boldsymbol{w} \cdot \boldsymbol{v} \in \mathbb{Z}, \forall \boldsymbol{v} \in \Lambda\} . \tag{A.4}$$

A basis for Λ^* is the set of vectors $\boldsymbol{e}^*_a$, with $\boldsymbol{e}^*_a \cdot \boldsymbol{e}_b = \delta_{ab}$. It is easy to see that this is indeed a basis for Λ^*, i.e.

$$\Lambda^* = \left\{ \sum_{a=1}^{N} m_a \boldsymbol{e}^*_a | m_a \in \mathbb{Z} \right\} . \tag{A.5}$$

To calculate the volume of Λ^* we define

$$g^*_{ab} = \boldsymbol{e}^*_a \cdot \boldsymbol{e}^*_b , \tag{A.6}$$

which is the metric of the dual lattice. Using the definition of the basis vectors one can easily show that $\Sigma_a e^{*I}_a e^J_a = d^{IJ}$, and that therefore g^* is the inverse of g. Thus we have

$$\mathrm{vol}(\Lambda) = (\mathrm{vol}(\Lambda^*))^{-1} . \tag{A.7}$$

A lattice Λ is called
- *unimodular* if $\mathrm{vol}(\Lambda) = 1$,
- *integral* if $\boldsymbol{v} \cdot \boldsymbol{w} \in \mathbb{Z}$, $\forall \boldsymbol{v}$, $\boldsymbol{w} \in \Lambda$,
- *even* if Λ is integral, and $\boldsymbol{v}^2 = 0 \bmod 2$, $\forall \boldsymbol{v} \in \Lambda$,
- *odd* if Λ is integral, but not even,
- *self-dual* if $\Lambda = \Lambda^*$.

Obviously Λ is integral if and only if $\Lambda \subset \Lambda^*$.

A.2. Cosets

If Λ_s is a sublattice of equal dimension of Λ, then we can decompose Λ in *cosets* with respect to Λ_s. To do so, choose a set of *coset representatives* $\boldsymbol{\beta}_\alpha$, so that

$$\begin{aligned} &\boldsymbol{\beta}_\alpha \in \Lambda\,, \qquad \boldsymbol{\beta}_\alpha \not\in \Lambda_s\,, \\ &\boldsymbol{\beta}_\alpha - \boldsymbol{\beta}_{\alpha'} \not\in \Lambda_s \quad \text{if} \quad \alpha \neq \alpha'\,. \end{aligned} \tag{A.8}$$

If we extend this set with the vector $\boldsymbol{\beta} = 0$ to represent Λ_s itself, we may write Λ as follows

$$\Lambda = \biguplus_\alpha (\boldsymbol{\beta}_\alpha + \Lambda_s)\,. \tag{A.9}$$

This notation means that every vector in Λ can be written as $\boldsymbol{\beta}_\alpha + \boldsymbol{v}_s$, $\boldsymbol{v}_s \in \Lambda_s$. Every coset $\boldsymbol{\beta}_\alpha + \Lambda_s$ has precisely one point in the unit cell of Λ_s. Therefore, if there are N_s cosets (including Λ_s), we find

$$\mathrm{vol}(\Lambda) = (1/N_s)\,\mathrm{vol}(\Lambda_s)\,. \tag{A.10}$$

This is a fairly trivial, but very useful observation. A direct consequence is that a lattice is self-dual if and only if it is both integral and unimodular.

A.3. Lie algebra lattices

It is often convenient to express a self-dual lattice in terms of Lie algebra lattices. In the following when we speak of a Lie algebra lattice we mean any sublattice of the weight lattice of a semi-simple Lie algebra (for a discussion of Lie algebras, roots and weights we refer to ref. [106]). In particular we need the *simply laced* lattices, belonging to semi-simple Lie algebras that are products of D_n, A_n and E_6, E_7, E_8. They have the property that all their roots have the same length. If one normalizes them so that they have all norm 2, the weight lattice Λ_W is the dual of the root lattice Λ_R, which is itself a sublattice of the weight lattice (the *norm* of a vector is its squared length)

$$\Lambda_\mathrm{W} = \Lambda_\mathrm{R}^*\,; \qquad \Lambda_\mathrm{W} \supset \Lambda_\mathrm{R}\,. \tag{A.11}$$

Therefore we can decompose Λ_W into cosets with respect to Λ_R. In the case of Lie algebra lattices these cosets are called *conjugacy classes*. In general cosets form a group under addition, with the sublattice Λ_s as the unit element. For Lie algebra lattices this group is isomorphic to the *center* of the Lie algebra.

We will make extensive use of D_n lattices, for which we denote the conjugacy classes as follows

$$\begin{aligned}
(0) &= \Lambda_R = \left\{ (k_1, \ldots, k_n),\ k_i \in \mathbb{Z},\ \sum_{i=1}^{n} k_i = 0 \bmod 2 \right\}, \\
(v) &= \Lambda_R + (1, 0, 0, \ldots, 0), \\
(s) &= \Lambda_R + (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}), \\
(c) &= \Lambda_R + (-\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}).
\end{aligned} \tag{A.12}$$

One may check that the center is isomorphic to $\mathbb{Z}_4$ if n is odd, and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even.

The irreducible representations of the semi-simple Lie algebras are characterized by their weight vectors, which for a given representation all belong to the same conjugacy class of the lattice. Hence we can specify conjugacy classes also by giving representations that belong to them. In the case of D_n, (0) contains all even rank tensor products of the vector representation (which includes the singlet and the adjoint), (v) contains all odd-rank tensors, and (s) and (c) contain spinors of opposite chirality.

A useful property of conjugacy classes is that their mutual inner products modulo integers depend only on the class, and not on the individual vectors. If $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \Lambda_W$, and $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \Lambda_R$, we have

$$(\boldsymbol{\beta}_1 + \boldsymbol{\lambda}_1) \cdot (\boldsymbol{\beta}_2 + \boldsymbol{\lambda}_2) = \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2 \bmod 1,$$

because $\boldsymbol{\beta}_i \cdot \boldsymbol{\lambda}_j \in \mathbb{Z}$, and because the root lattice of simply laced algebras is integral (this follows from the fact that it is contained in its own dual, see (A.11)). In fact Λ_R is an even lattice, because its basis vectors are roots, which have norm 2. Using this property one can show by a similar argument that the norms of two vectors in the same conjugacy class are always equal modulo 2. These properties are clearly of great help in constructing even integral lattices, because one only needs to check the lengths and inner products of conjugacy class representatives. For this reason we have collected some information concerning the conjugacy classes of the (simple) simply laced algebras in table 8.

The volume of the unit cells of Λ_R and Λ_W are easy to obtain. Since they are each other's dual

$$\mathrm{vol}(\Lambda_R) = \mathrm{vol}(\Lambda_W)^{-1}.$$

If we denote the order of the center (i.e. the number of conjugacy classes) by N_c, we learn from (A.10) that

$$\mathrm{vol}(\Lambda_W) = (1/N_c)\, \mathrm{vol}(\Lambda_R),$$

so that

$$\mathrm{vol}(\Lambda_R) = \sqrt{N_c}, \qquad \mathrm{vol}(\Lambda_W) = \frac{1}{\sqrt{N_c}}.$$

All these results are valid for simple as well as semi-simple Lie algebras. The order of the center of a semi-simple Lie algebra is simply the product of the order of the centers of its simple factors. The same is true for the volume of the unit cell.

Table 8
Conjugacy classes of simply laced simple Lie algebras. The third column gives the representation of smallest dimension in each class (except for A_n)

Lie algebra	conjugacy class	smallest representation	weight norm (mod 2)
D_n	(0)	singlet	0
	(v)	vector	1
	(s)	spinor	$\frac{1}{4}n$
	(c)	conjugate spinor	$\frac{1}{4}n$
A_n	(p)	rank-p tensor	$\frac{p(n+1-p)}{n+1}$
E_6	(0)	singlet	0
	(1)	(27)	$\frac{4}{3}$
	(2)	$(\overline{27})$	$\frac{4}{3}$
E_7	(0)	singlet	0
	(1)	(56)	$\frac{3}{2}$
E_8	(0)	singlet	0

To specify a Lie algebra lattice completely, we have to supply two pieces of information: the semi-simple Lie algebra, and a list of the conjugacy classes that appear. For a given semi-simple Lie algebra there are in general many possible choices of conjugacy classes. This choice is restricted only by the requirement that the list must close under addition, which in particular means that the root lattice must always be present. The possible sets of conjugacy classes are thus in one-to-one correspondence with the subgroups of the center. Instead of giving a full list of conjugacy classes, one can give a shorter list of *generators*, which is a subset of conjugacy classes from which all others can be obtained by addition.

Since the volume of the unit cell of the root lattice is $\sqrt{N_c}$, it follows immediately that a Lie algebra lattice is unimodular if and only if it contains precisely $\sqrt{N_c}$ conjugacy classes. This is only possible if we choose a Lie algebra for which N_c is a perfect square. The lattice will be integral if and only if all conjugacy class generators have integral norm and integral mutual inner product. It will be even if and only if they have even norm and integral inner products.

Using table 8 one can now easily construct many examples of Euclidean self-dual Lie algebra lattices. The reader is invited to check that

- D_n with conjugacy classes (0) and (v) is odd and self-dual (this lattice is also known as $\mathbb{Z}^n$);
- D_n with (0) and (s) is odd self-dual for $n = 4 \bmod 8$, and even self-dual for $n = 0 \bmod 8$.
- A_{n^2-1} with (0) and (n) is odd self-dual for n even and even self-dual for n odd.

In general we call vectors of norm 2 on Euclidean lattices *roots*. The roots include in any case those of the Lie algebras explicitly used in the construction of the lattice (i.e. those in the conjugacy class (0)), but in some cases there may be additional norm 2 vectors in the other conjugacy classes. In that case we can always write the lattice in a different way, using larger Lie algebras and fewer conjugacy classes, although this is not always convenient. For example, the lattices D_8 with $(0) + (s)$ and A_8 with $(0) + (3) + (6)$ defined above are both equal (up to rotations) to E_8, which has only one conjugacy class, namely (0) (and which is therefore automatically self-dual!).

Although many of the known Euclidean self-dual lattices are Lie algebra lattices, this is not true in general. We can regard a general Euclidean self-dual lattice in $\mathbb{R}^N$ as a Lie algebra lattice if the set of roots spans $\mathbb{R}^N$. The lattice of smallest dimension that does not have this property is the famous Leech lattice, an even self-dual lattice of dimension 24 that does not have any roots at all, and that will be discussed later in this appendix.

The notion of a Lie algebra lattice can be extended to Lorentzian lattices in the following way. Choose a basis so that the Lorentzian metric has the form $(+)^p(-)^q$. Now every vector $\boldsymbol{v}$ on the lattice can be written $\boldsymbol{v} = (\boldsymbol{v}_L, \boldsymbol{v}_R)$, where $\boldsymbol{v}_L$ denotes the first p components, and $\boldsymbol{v}_R$ the last q (the subscripts L and R denote "left" and "right" respectively, a notation which will be natural in applications to string theory). We define left and right roots of such a lattice to be vectors of the form $(\boldsymbol{r}, 0)$ respectively $(0, \boldsymbol{r})$, with $\boldsymbol{r}^2 = 2$. We will call such a Lorentzian lattice a Lie algebra lattice if the left and right roots together span $\mathbb{R}^{p,q}$.

A large class of even Lorentzian self-dual lattices in $\mathbb{R}^{p,p}$ can be constructed as follows. Take a semi-simple Lie algebra $\mathcal{G}$ of rank p. Consider the set of all vectors in $\mathbb{R}^{p,p}$ of the form $(\boldsymbol{v}_1, \boldsymbol{v}_2)$, so that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ belong to the *same* conjugacy class of the Lie algebra. With respect to the Lorentzian metric any such vector has integral inner product with any other such vector, and furthermore they all have even length. It is also easy to see that the resulting Lorentzian lattice is unimodular, so that it is a self-dual lattice. Lattices of this type were first described by Englert and Neveu [107], and will henceforth be called Englert–Neveu lattices. Such lattices have the roots of $\mathcal{G}$ as their left and right roots.

To conclude this discussion of Lie algebra lattices, we want to introduce a minor, but useful extension of this set, namely the lattice D_1. Although it is not part of Cartan's classification of simple Lie algebra's, it is easy enough to define by simply extending the definition of D_n to $n = 1$. The resulting one-dimensional root lattice consists simply of the even integers. The shortest non-trivial vector has norm four, so there are no roots on this root lattice. The vector conjugacy class consists of the odd integers, and the two spinor conjugacy classes are obtained by shifting the root lattice by $+\frac{1}{2}$ and $-\frac{1}{2}$.

A.4. Lattice engineering

There are several tricks for making new self-dual lattices out of already known ones. These tricks are all variations on the same basic theme: one adds new conjugacy classes and removes old ones, in such a way that their total number stays the same (i.e. the lattice remains unimodular), and that all inner products remain integral.

Shift vectors

A very general method is the following. Let Λ be a self-dual lattice. Choose a vector $\boldsymbol{s}$ so that $k\boldsymbol{s} \not\in \Lambda$, $1 \le k \le p-1$, but $p\boldsymbol{s} \in \Lambda$. We can define a lattice Λ_0 as the sublattice of Λ which has integer inner product with $\boldsymbol{s}$. Since $\boldsymbol{s} \not\in \Lambda$ there must be at least one vector $\boldsymbol{t} \in \Lambda$ such that $\boldsymbol{s} \cdot \boldsymbol{t}$ is not integer. We can decompose Λ in cosets

$$\Lambda = \biguplus_{k=0}^{\tilde{p}-1} (\boldsymbol{t}_k + \Lambda_0)$$

Here the vectors $\boldsymbol{t}_k$ are coset representatives, chosen in such a way that $\boldsymbol{t}_0 = 0$. The representatives must satisfy

$$\boldsymbol{t}_k \cdot \boldsymbol{s} = m_k/p \mod 1 ,$$

where $m_k - m_l \neq 0 \bmod p$ if $m \neq l$, so that each of them represents a different coset. Let $\tilde{p}$ be the total number of cosets. Since the cosets must form a group under addition it follows that m_k must take the values $n(p/\tilde{p})$, $n = 1, \ldots, \tilde{p} - 1$. But then $\tilde{p}s$ has integer inner product with all vectors on the lattice, and hence must lie on the lattice itself. Since p is the smallest positive number with $ps \in \Lambda$, it follows that $\tilde{p} = p$. This allows us to conclude that there exists a vector t with $t \cdot s = 1/p \bmod 1$, so that we can write the coset decomposition as

$$\Lambda = \biguplus_{k=0}^{p-1} (kt + \Lambda_0)\,.$$

Consider now the new lattice

$$\Lambda' = \biguplus_{k=0}^{p-1} (ks + \Lambda_0)\,.$$

Clearly Λ' is unimodular, because its unit cell has the same volume as that of Λ. If furthermore s has integral norm, it is self-dual. If Λ is even and s has even length, Λ' is even self-dual.

It is sometimes useful to extend this shift vector method by allowing the shift vector to satisfy

$$s^2 = 2k/p\,, \quad k \in \mathbb{Z} \tag{A.13}$$

instead of having even norm (notice that in general we know that $s^2 = 2k/p^2$). In this case the shift vector itself cannot be on the new lattice, but we can define a new shift vector $s' = s - kt$. The new vector s' can now be used in the same way as s above; notice in particular that s' has even norm. Of course we might as well have started with s' instead of s. However, shift vectors satisfying the (seemingly) weaker condition often appear in orbifold (-inspired) constructions. For purists we add that the condition can be weakened even more if p is odd. In that case it is sufficient that $s^2 = k/p$, $k \in \mathbb{Z}$. If k is even, this is the case already discussed. If k is odd, we use $s' = (p-1)s$ as the shift vector, which does satisfy (A.13).

Replacement of Lie algebra factors

Another useful trick is the following. Consider a Lie algebra lattice containing, among others, a D_n factor. Now replace this D_n factor by D_m, where $m = n + 8l$, $l \in \mathbb{Z}$. Since the unit cells of D_n and D_m have the same volume for *any* n and m, the new lattice is unimodular. Furthermore one can easily check that if $n = m \bmod 8$ all inner products and norms of the four conjugacy classes remain unchanged modulo two. Hence the new lattice will be (even) self-dual if the original lattice was (even) self-dual.

This transformation is of course meaningless if it results in $m = 0$. But one can give a meaning to it if m becomes negative. Consider for example a Euclidean even self-dual lattice $\Lambda_p \times D_n$ with inner product with signature $(+)^{p+n}$. For $m < 0$ (with m defined as above) we can consider the Lorentzian lattice $\Lambda_p \times D_m$ with signature $(+)^p(-)^{|m|}$. This lattice is unimodular for the reasons explained above. Furthermore the Lorentzian inner products of the new lattice are equal to the Euclidean ones of the original one modulo two. Thus we see that shifting n by multiples of eight to negative values has the effect of changing a Euclidean self-dual lattice to a Lorentzian one.

This trick is especially useful for the classification of the self-dual Lorentzian Lie algebra lattices. As we will see in a moment, in general the classification of self-dual Lorentzian lattices is either trivial or impossible, depending on what one hopes to achieve. If one is interested in classifying them up to Lorentz transformations, the result is trivial: there is just one (odd self-dual) lattice for every non-zero value of p and q. If one the other hand one is interested in a problem where the structure of the left of the right part of the lattice is relevant (such as in torus compactification of bosonic strings) classification is impossible, since Lorentz transformations (which respect the metric and hence the self-duality conditions) change the left and right parts, thus creating a continuous infinity of inequivalent lattices.

In some cases one knows however that either the left or the right lattice must have a definite structure, e.g. that it must be a Lie algebra lattice. Then classification is possible. For example, if the right part of the lattice consists entirely of D_n factors, one can use the map described above to map it to a Euclidean lattice by subtracting multiples of 8 of the dimensions of the D_n factors.

For this purpose it is useful to extend this "Euclideanization" trick to other Lie algebra lattices, which also have A_n or E_n factors. Denote such a factor as $\mathcal{G}_R$, and let (c_R) be one of its conjugacy classes. What we are looking for is a Lie algebra lattice $\mathcal{G}_R$ with the following properties:

- The number of conjugacy classes of $\mathcal{G}_R$ and $\mathcal{G}_L$ is the same, i.e. the two root lattices have unit cells of the same volume.
- There exists a map ϕ from the conjugacy classes (c_R) of $\mathcal{G}_R$ to the conjugacy classes (c_L) of $\mathcal{G}_L$ so that the sum of the norms of (c_R) and its ϕ-image (c_L) is even.

It is easy to see that if in a Lorentzian (even) self-dual lattice $\Gamma^{p,q}$ we replace a right factor $\mathcal{G}_R$ by a left factor $\mathcal{G}_L$ and map the conjugacy classes using ϕ, then the resulting lattice is again (even) self-dual, but with respect to a metric $(+)^{p'}(-)^{q'}$, where $p' = p + \text{rank } \mathcal{G}_L$ and $q' = q - \text{rank } \mathcal{G}_R$.

In table 9 we give the simplest maps having the properties listed above.

Of course ϕ respects the addition rules for conjugacy classes, so that is sufficient to specify it for the conjugacy class generators. The reader may observe that the last five entries can be obtained by considering the regular embedding of $\mathcal{G}_R \times \mathcal{G}_L$ in E_8. It is easy to see that this generalizes as follows. Take any (even) self-dual Euclidean Lie algebra lattice with Lie algebra $\mathcal{G}$, and split it into two factors $\mathcal{G}_R$ and $\mathcal{G}_L$ (which need not be simple) so that $\mathcal{G}_R \times \mathcal{G}_L$ is a regular sub-algebra of $\mathcal{G}$ of equal rank. Then $\mathcal{G}_L$ and $\mathcal{G}_R$ satisfy the conditions given above, with a conjugacy class map ϕ which can be read off from the embedding in the obvious way. Using the known Euclidean self-dual lattice one can extend table 9 to some of the larger Lie algebras. Clearly one can classify all Lorentzian self-dual Lie algebra lattices using the enumeration of the Euclidean ones. The practical use of this is limited because of the fact that the latter are only known for dimensions less than or equal to 24, as we will discuss now.

Table 9
Euclideanization of Lie algebra factors

$\mathcal{G}_R$	(c_R)	$\mathcal{G}_L$	(c_L)
D_n	(s)	D_{8l-n} $(8l > n)$	(s)
	(v)		(v)
A_1	(1)	E_7	(1)
A_2	(1)	E_6	(1)
A_3	(1)	D_5	(s)
A_4	(1)	A_4	(2)
A_5	(1)	A_2A_1	(1, 1)

A.5. Lattice classification theorems

A lot is known about the possibilities for constructing self-dual lattices. One should keep in mind here that Euclidean lattices can only be classified up to rotations, because any rotation respects the self-duality conditions. Likewise, Lorentzian lattices can only be classified modulo Lorentz transformations. The known results are as follows:

(1) *Lorentzian, odd*. All such lattices are Lorentz transformations of the lattice $\mathbb{Z}^{p,q}$, which is the set of points in $\mathbb{R}^{p,q}$ defined as $\{(n_1, \ldots, n_p, m_1, \ldots, m_q),\ n_i, m_j \in \mathbb{Z}\}$.

(2) *Lorentzian, even*. Lattices of this type exist only for $|p-q| = 0 \bmod 8$. For $p = q + 8n$ they are Lorentz transformations of the lattice $n\mathrm{E}_8 \times q\mathrm{P}^{1,1}$. The E_8 lattice is most easily described as D_8 with conjugacy classes (0) and (s), while $\mathrm{P}^{1,1}$ is a Lorentzian lattice of signature $(+)(-)$, which may be described as $\mathrm{D}_1 \times \mathrm{D}_1$ with conjugacy classes $(0,0)+(v,v)+(s,s)+(c,c)$ (this is a lattice of the Englert–Neveu type, discussed above).

(3) *Euclidean, odd*. A class of such lattices which exists in any dimension is $\mathbb{Z}^n$. In general, if any odd self-dual lattice has vectors of length one, they can only have mutual inner products $\pm 1, 0$. Thus they form an orthonormal basis for a subspace of $\mathbb{R}^n$, which must be part of the basis of the lattice. Any other basis vector of the lattice must be orthogonal to this subspace. Thus if there are $2p$ length-one vectors, the lattice is of the form $\mathbb{Z}^p \otimes \Lambda^{n-p}$, where the second factor does not contain any vectors of length one.

An odd Euclidean self-dual lattice Λ can be related to an even self-dual one of higher dimension in the following way [104]. Consider the sublattice Δ_0 of even norm vectors. Obviously Δ_0 has only one non-trivial coset in Λ, because the sum of two odd norm vectors is even. Thus

$$\Lambda = \Delta_0 + \Delta_v ,$$

where Δ_v denotes the odd-norm vectors. Since $\mathrm{vol}(\Delta_0)/\mathrm{vol}(\Delta_0^*) = 4$, the dual of Δ_0 can be decomposed into four cosets,

$$\Delta_0^* = \Delta_0 + \Delta_v + \Delta_s + \Delta_c .$$

These four cosets generate an Abelian group of order four, which can only be $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. This allows us to determine the mutual inner products of the four cosets and their norms modulo 2. If the group is $\mathbb{Z}_4$, one finds $\Delta_c = -\Delta_s$ and $2\Delta_s \subset \Delta_v$, so that the norm of all vectors in Δ_s and Δ_c is equal to $\frac{1}{4}k \bmod 2$, k odd. For $\mathbb{Z}_2 \times \mathbb{Z}_2$, $2\Delta_s = 2\Delta_c \subset \Delta_0$, so that the norm of all vectors in Δ_s and Δ_c is half-integer. One can then use $\Delta_s + \Delta_v = \Delta_c$, plus the fact that vectors in Δ_s cannot have integer inner products with those in Δ_v, to show that the norm of vectors in Δ_s and Δ_c must be equal modulo 2. Thus for both possible Abelian groups the vectors in Δ_s and Δ_c have, modulo 2, norms equal to each other and to $\frac{1}{4}k$, $0 \le k < 7$. Consider then the lattice obtained by adding a D_{8-k} factor, so that the entire lattice consists of the vectors

$$(\Delta_0, 0) + (\Delta_v, v) + (\Delta_s, s) + (\Delta_c, c) .$$

This lattice is unimodular, integral and even, so that it is even self-dual. So all odd self-dual lattice can be obtained from the even self-dual ones by inverting this procedure to remove D_n factors. This leaves us with the classification problem of

(4) *Euclidean*, *even* lattices. These exist only in dimensions which are a multiple of 8, and are known explicitly for 8, 16 and 24 dimensions. In 8 dimensions, the only possibility is E_8; in 16 dimensions one can have $E_8 \otimes E_8$ and D_{16} with conjugacy classes (0) and (s). These two lattices cannot be rotated into each other. In 24 dimensions there are 24 even self-dual lattices, which have been classified by Niemeier [105]. A list of their conjugacy classes can be found in the first paper of ref. [104]. For the 23 Niemeier lattices that are Lie algebra lattices, this list is shown in table 10. Here one should include all cyclic permutations of the conjugacy classes within square brackets, and to obtain the complete list of conjugacy classes one should close the set of generators under addition.

One of the 24 lattices is the Leech lattice [108], which differs from the other 23 because it contains no vectors of norm 2. Nevertheless, we can in a certain sense regard it as a Lie algebra lattice, since it is a product of 24 D_1 factors, on which one has 2^{24} conjugacy classes. An example of a list of 12 conjugacy class generators is given below

$$(c, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, v)\,,$$

$$(v, 0, 0, 0, 0, 0, c, s, c, s, 0, 0, v, 0, 0, 0, v, 0, c, s, c, s, v, 0)\,,$$

$$(v, 0, s, s, 0, 0, 0, 0, s, s, s, s, c, s, s, s, c, s, s, s, s, s, v, 0)\,,$$

$$(0, v, v, 0, 0, 0, 0, 0, c, s, c, s, 0, 0, v, 0, 0, 0, v, 0, c, s, c, s)\,,$$

$$(0, v, v, 0, s, s, 0, 0, 0, 0, s, s, s, s, c, s, s, s, c, s, s, s, s, s)\,,$$

$$(c, c, v, 0, v, 0, 0, 0, 0, 0, c, s, c, s, 0, 0, v, 0, 0, 0, v, 0, c, s)\,,$$

$$(v, 0, c, s, v, 0, s, s, 0, 0, 0, 0, s, s, s, s, c, s, s, s, c, s, s, s)\,,$$

$$(c, c, s, s, v, 0, v, 0, 0, 0, 0, 0, c, s, c, s, 0, 0, v, 0, 0, 0, v, 0)\,,$$

$$(0, v, c, s, c, s, v, 0, s, s, 0, 0, 0, 0, s, s, s, s, c, s, s, s, c, s)\,,$$

$$(v, 0, s, s, s, s, v, 0, v, 0, 0, 0, 0, 0, c, s, c, s, 0, 0, v, 0, 0, 0)\,,$$

$$(c, c, v, 0, c, s, c, s, v, 0, s, s, 0, 0, 0, 0, s, s, s, s, c, s, s, s)\,,$$

$$(s, s, c, s, s, s, s, s, v, 0, v, 0, 0, 0, 0, 0, c, s, c, s, 0, 0, v, 0)\,.$$

To show that this list generates the Leech lattice, one can check that the 12 generators have even norm and integer inner products, that they are independent (i.e. that one gets 2^{24} different conjugacy classes by adding these vectors in all possible ways), and that none of the 2^{24} classes contains vectors of norm 2. These conditions define uniquely the Leech lattice. As a further (though unnecessary) check one may verify that the number of norm 4 vectors is also equal to the known number of such vectors on the Leech lattice.

There is a rather amusing way of estimating the number of lattices in $8n$ dimensions. Every lattice has a discrete automorphism group. This is a discrete subgroup of $O(8n)$ which takes the lattice into

Table 10
The 23 Niemeier lattices that are Lie algebra lattices. Square brackets indicate cyclic permutation

Lie algebra	conjugacy class generators
D_{24}	(s)
$D_{16}E_8$	$(s, 0)$
E_8^3	$(0, 0, 0)$
A_{24}	(5)
D_{12}^2	$(s, v), (v, s)$
$A_{17}E_7$	$(3, 1)$
$D_{10}E_7^2$	$(s, 1, 0), (c, 0, 1)$
$A_{15}D_9$	$(2, s)$
D_8^3	$([s, v, v])$
A_{12}^2	$(1, 5)$
$A_{11}D_7E_6$	$(1, s, 1)$
E_6^4	$(1, [0, 1, 2])$
$A_9^2D_6$	$(2, 4, 0), (5, 0, s), (0, 5, c)$
D_6^4	even permutations of $(0, s, v, c)$
A_8^3	$([1, 1, 4])$
$A_7^2D_5^2$	$(1, 1, s, v), (1, 7, v, s)$
A_6^4	$(1, [2, 1, 6])$
$A_5^4D_4$	$(2, [0, 2, 4], 0), (3, 3, 0, 0, s), (3, 0, 3, 0, v), (3, 0, 0, 3, c)$
D_4^6	$(s, s, s, s, s, s), (0, [0, v, c, c, v])$
A_4^6	$(1, [0, 1, 4, 4, 1])$
A_3^8	$(3, [2, 0, 0, 1, 0, 1, 1])$
A_2^{12}	$(2, [1, 1, 2, 1, 1, 1, 2, 2, 2, 1, 2])$
A_1^{24}	$(1, [0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1])$

itself. It consists of at least two elements, the identity and the reflection through the origin. Denote the total number of elements of this group for a lattice Λ as $g(\Lambda)$. Then the following identity can be proved [100, 109].

$$\sum \frac{1}{g(\Lambda)} = \frac{B_{4n}}{8n} \prod_{j=1}^{4n-1} \frac{B_{2j}}{4j} ,$$

where B_{2j} are the Bernoulli numbers. This formula is called the Minkowski–Siegel mass formula. The sum is over all lattices in $8n$ dimensions, and adds up to a number that is less than half the total number of lattices in this dimension. Thus the right-hand side, multiplied by 2, is a lower limit of the number of lattices. Calculating it for the lowest dimensions, one finds

dimension	lower limit	actual number
8	2.8×10^{-9}	1
16	4.9×10^{-18}	2
24	15.8×10^{-15}	24
32	8.0×10^{7}	?

This explains why the lattices of dimension 32 have not yet been classified. The explosive growth of the number of lattices that one observes here is related to the reason why the number of chiral four-dimensional string theories is so much larger than the number of ten-dimensional ones.

Appendix B. Some properties of the Weyl group

B.1. Generalities

Let G denote any simple, simply-laced Lie group of rank l and dimension d (many of the conclusions of this appendix remain valid for non-simply laced groups, but for our purposes such groups are not needed). Let $\mathscr{G}$ denote the Lie algebra of G and $\mathscr{H}$ a Cartan sub-algebra (CSA), and let T denote the torus generated by $\mathscr{H}$ in G. Choose a set of simple roots $\boldsymbol{\alpha}_i$, $i = 1, 2, \ldots, l$. The Weyl group, W($\mathscr{G}$), of $\mathscr{G}$ is generated by the Weyl reflections

$$r_i(\boldsymbol{\lambda}) = \boldsymbol{\lambda} - \left(\frac{2\boldsymbol{\alpha}_i \cdot \boldsymbol{\lambda}}{\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_i}\right)\boldsymbol{\alpha}_i\,. \tag{B.1}$$

Given a representation of G, there is a natural lift of $\mathrm{W} \equiv \mathrm{W}(\mathscr{G})$ to a finite subgroup $\tilde{\mathrm{W}}$ of G. The generators of $\tilde{\mathrm{W}}$ are defined by

$$\tilde{r}_i = \exp\left(\frac{\mathrm{i}\pi}{2}\,(E_{\alpha_i} + E_{-\alpha_i})\right), \tag{B.2}$$

where $\{E_\alpha\}$ denotes the usual root space basis of $\mathscr{G}$. Let H_i denote an orthonormal basis for $\mathscr{H}$. It is elementary to see that

$$\tilde{r}_i(\boldsymbol{\lambda} \cdot \boldsymbol{H})\tilde{r}_i^{-1} = (r_i(\boldsymbol{\lambda})) \cdot \boldsymbol{H} \tag{B.3}$$

and also

$$\tilde{r}_i E_\beta \tilde{r}_i^{-1} = c_i(\beta) E_{r_i(\beta)}\,, \tag{B.4}$$

where $c_i(\beta)$ is a complex phase. It is important to note that using the lift $\tilde{w}$ defined via eq. (B.2) one cannot always choose the E_β so that these phases are equal to 1.

The group $\tilde{\mathrm{W}}$ has an Abelian normal subgroup D generated by the $(\tilde{r}_i)^2$, such that $\mathrm{W} \cong \tilde{\mathrm{W}}/\mathrm{D}$ [110]. Since $(\tilde{r}_i)^4 = 1$ on all representations of G, and because D is Abelian, it follows that D is isomorphic to some product of Z_2 factors. Moreover, if $w \in \mathrm{W}$ satisfies $w^k = 1$, then any lift $\tilde{w} \in \tilde{\mathrm{W}}$ of w satisfies $\tilde{w}^k \in \mathrm{D}$ and hence $\tilde{w}^{2k} = 1$. In particular one should note that one cannot guarantee that w and $\tilde{w}$ have the same order even upon the adjoint representation of G.

The simplest illustration of this difficulty is the Weyl reflection with respect to a root $\boldsymbol{\alpha}$ of SU(3). As a transformation on weight space it has order 2. However it is easy to check that its lift (B.2) satisfies

$$\tilde{r}_\alpha^2 E_\beta \tilde{r}_\alpha^{-2} = -E_\beta\,, \tag{B.5}$$

where $\boldsymbol{\beta}$ is a root with $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = -1$.

The cover, $\tilde{W}$, of W in G defined by (B.2) is by no means the only possible one. Indeed, since $W \cong N(T)/T$ (where N(T) is the normalizer of T in G), the generators can in general be taken to be

$$\tilde{r}_j' = \tilde{r}_j \exp(2\pi i \boldsymbol{v}_j \cdot \boldsymbol{H}) \tag{B.6}$$

for any set of vectors $\boldsymbol{v}_j$. This manifestly preserves the form of (B.3) and (B.4). One should note that such a lift need not even be of finite order in G, since $\boldsymbol{v}_j$ might be incommensurate with the torus.

From the point of view of string theory the lift $\tilde{w}$ defined by (B.2) is the natural one since the alternative, (B.6), involves the string momentum generators $\boldsymbol{H}$ and thus first translates or shifts the state by $\boldsymbol{v}_j$ before twisting it with $\tilde{r}_j$. The lift $\tilde{w}$ is generated by pure twists, and we will therefore call it the *canonical* or *shiftless* lift.

We will only be interested in the lift of a single element of W, and furthermore only in non-degenerate elements (i.e. those elements, w, for which $\det(1-w) \neq 0$). This apparent restriction greatly reduces the subtleties and ambiguities described above. For the physical application of constructing supercurrents, discussed in section 6.3, the restriction to non-degenerate elements is necessitated by requiring that the corresponding string theory be chiral. From the mathematical point of view one can of course view any degenerate element as a non-degenerate element of a lower rank Lie sub-algebra of $\mathcal{G}$. The ambiguities then arise in passing back to the original algebra, $\mathcal{G}$, but this is usually a straightforward process. For example, the Weyl reflection with respect to the root $\boldsymbol{\alpha}$ of SU(3) described above is degenerate in SU(3) but is a non-degenerate element of the Weyl group of the SU(2) sub-algebra. As we will see in the next sub-section, this observation can be used to account for the doubling of the order that was observed in eq. (B.5).

B.2. The order of lifted Weyl group elements

Suppose that $w \in W$ is a non-degenerate element of order n. Then it can be shown that [58]:
(1) All lifts are conjugate to each other, and hence have the same order.
(2) All lifts have order n on the adjoint representation. Moreover the lifts respect *all* orbits of w, in the following sense: if for some root $\boldsymbol{\alpha}$

$$w^k(\boldsymbol{\alpha}) = \boldsymbol{\alpha} ,$$

then

$$\tilde{w}^k E_\alpha \tilde{w}^{-k} = E_\alpha . \tag{B.7}$$

(3) A lift has order n on a representation characterized by a weight $\boldsymbol{w}$ if and only if

$$\tfrac{1}{2}\boldsymbol{w}^2 = 0 \mod(1/n) \tag{B.8}$$

and has order $2n$ otherwise.

We note that one always has $(\boldsymbol{w})^2 \equiv 0 \mod(1/n)$, or equivalently, n is always divisible by the order of any element of the center of $\mathcal{G}$. This follows from the fact that given any lift, $\tilde{w}$, of w, and any element c in the center of G, then $\tilde{w}c$ is also an equally good lift, and so must be conjugate to $\tilde{w}$ and therefore has the same order. Hence $c^n = 1$. This means that the order, n, of w must be divisible by $k+1$ for A_k, 2 for D_{2k}, 4 for D_{2k+1} and 3, 2 and 1 for E_6, E_7 and E_8 respectively.

Combining the foregoing observation with (3) above, we find that for the simply-laced algebras the possibilities for order doubling of the lift are as follows

– A_n. There is only one conjugacy class of non-degenerate elements, namely the Coxeter class (see the discussion of Weyl conjugacy classes below). Its order is $n+1$, and the norms of the fundamental weights (corresponding to anti-symmetric tensors of rank i) are given by $i(n+1-i)/(n+1)$, $i=0,\ldots,n$. For *even* n, we see that on all representations the Coxeter element has order $n+1$. For *odd* n, the order is $n+1$ on even-rank tensors, and $2(n+1)$ on odd-rank tensors.

– D_{4k}. Non-degenerate Weyl group elements must be of even order and the lifts must have exactly the same order on all representations.

– D_{4k+2}. The order of all non-degenerate elements must be even. If the order is a multiple of four, then the lift has exactly the same order on all representations. Otherwise it doubles on the spinors.

– D_{2k+1}. Every non-degenerate element must have an order divisible by four. If the order is a multiple of eight, then the lift has the same order on all representations. Otherwise it doubles on the spinors.

– E_6. The order is always a multiple of three, and the lift has the same order on all representations.

– E_7. The order is even. If it is a multiple of four, there is no doubling on the (56); otherwise there is.

– E_8. The weight system is the same as the root system and so there is no doubling.

B.3. Shift vectors and conjugacy classes

The importance to covariant lattice string theories of the lift, $\tilde{w}$, of a Weyl element, w, is that it can be used to define a corresponding shift vector. Specifically, there is an element $k \in \mathrm{G}$, and an element, t, in the maximal torus, T, of G such that

$$k\,\tilde{w}\,k^{-1} = t = \exp(2\pi \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{H}) \tag{B.9}$$

for some vector $\boldsymbol{\sigma}$. The vector $\boldsymbol{\sigma}$ is referred to as the shift corresponding to w. Obviously there is considerable ambiguity in the choice of $\boldsymbol{\sigma}$. These ambiguities arise from the choice of $\tilde{w}$ and k, as well as the freedom to add a weight to $\boldsymbol{\sigma}$. If t' is some other element of the torus obtained from a different choice of one or both of $\tilde{w}$ and k, then t' is conjugate to t in G. It then follows [58, 18] that if $t' = \exp(2\pi\mathrm{i}\boldsymbol{\sigma}'\cdot\boldsymbol{H})$, then the shift vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ must be Weyl transformations of each other modulo the *root* lattice. That is:

$$\boldsymbol{\sigma}' = \gamma(\boldsymbol{\sigma}) + \boldsymbol{r}\,, \tag{B.10}$$

where $\boldsymbol{r}$ is a linear combination of roots of $\mathcal{G}$, and γ is an element of the Weyl group, $\mathrm{W}(\mathcal{G})$.

Thus, given one shift vector, all others are obtained by Weyl transformations and addition of roots.

One can tighten up these observations to establish some amusing relations for $\det(1-w)$. For a simply-laced Lie algebra, $\mathcal{A}$, let $\Lambda_r(\mathcal{A})$ and $\Lambda_w(\mathcal{A})$ denote the corresponding root and weight lattices. Let $Z(\mathrm{A})$ denote the center of the group, A, corresponding to the Lie algebra, $\mathcal{A}$. Then one has

$$|Z(\mathrm{A})| = \left|\frac{\Lambda_w(\mathcal{A})}{\Lambda_r(\mathcal{A})}\right|, \tag{B.11}$$

where the vertical bars, $|\ \ |$, denote the order. (Indeed (B.11) holds without the bars if it is viewed as an identity between Abelian groups). Define $\mathcal{G}'$ to be the smallest, regular sub-algebra of $\mathcal{G}$ such that w is an element of $\mathrm{W}(\mathcal{G}')$, the Weyl group of $\mathcal{G}'$. We will call such a sub-algebra, $\mathcal{G}'$, minimal relative to w.

All of the observations made earlier can be applied to w considered as an element of $\mathrm{W}(\mathscr{G}')$. Let $\boldsymbol{\sigma}$ be the shift vector corresponding to w constructed in such a way that $\tilde{w}$, k and t all lie in G'. Then (B.10) applies for $\gamma \in \mathrm{W}(\mathscr{G}')$ and $\boldsymbol{r} \in \Lambda_r(\mathscr{G}')$. Choose $\boldsymbol{\sigma}$ such that $(\boldsymbol{\sigma})^2$ is minimized modulo the root lattice. That is, for all $\boldsymbol{r} \in \Lambda_r(\mathscr{G}')$, $(\boldsymbol{\sigma}+\boldsymbol{r})^2 \geq (\boldsymbol{\sigma})^2$. (There is never a unique choice of such a $\boldsymbol{\sigma}$). One can show [18] that because $\mathscr{G}'$ is minimal relative to w, equality holds in the foregoing expression if and only if $\boldsymbol{r} \equiv \boldsymbol{0}$. It follows from this and (B.10) that there is precisely one weight $\boldsymbol{w}$ in each conjugacy class of $\Lambda_w(\mathscr{G}')/\Lambda_r(\mathscr{G}')$ such that

$$(\boldsymbol{\sigma}+\boldsymbol{w})^2 = (\boldsymbol{\sigma})^2 . \tag{B.12}$$

One can then show [18] that $(1-w)(\Lambda_w(\mathscr{G}')) \equiv \Lambda_r(\mathscr{G}')$ and hence

$$\det(1-w) = \left|\frac{\Lambda_w(\mathscr{G}')}{\Lambda_r(\mathscr{G}')}\right| = |Z(\mathrm{G}')| . \tag{B.13}$$

Finally, by using duality to observe that

$$\frac{\Lambda_r(\mathscr{G})}{\Lambda_r(\mathscr{G}')} \equiv \frac{\Lambda_w(\mathscr{G}')}{\Lambda_w(\mathscr{G})} , \tag{B.14}$$

one sees that

$$\begin{aligned}\left|\frac{\Lambda_r(\mathscr{G})}{\Lambda_r(\mathscr{G}')}\right|^2 &= \left|\frac{\Lambda_r(\mathscr{G}) \times \Lambda_w(\mathscr{G}')}{\Lambda_r(\mathscr{G}') \times \Lambda_w(\mathscr{G})}\right| \\ &\equiv \left|\frac{\Lambda_w(\mathscr{G})}{\Lambda_r(\mathscr{G})}\right|^{-1} \times \left|\frac{\Lambda_w(\mathscr{G}')}{\Lambda_r(\mathscr{G}')}\right| \equiv \frac{\det(1-\sigma)}{|Z(\mathrm{G})|}\end{aligned} \tag{B.15}$$

and so one can derive

$$\left|\frac{\Lambda_r(\mathscr{G})}{\Lambda_r(\mathscr{G}')}\right| = \left|\frac{\Lambda_w(\mathscr{G}')}{\Lambda_w(\mathscr{G})}\right| = \sqrt{\frac{\det(1-\sigma)}{|Z(\mathrm{G})|}} , \tag{B.16}$$

$$\left|\frac{\Lambda_w(\mathscr{G})}{\Lambda_r(\mathscr{G}')}\right| = \sqrt{\det(1-\sigma)|Z(\mathrm{G})|} . \tag{B.17}$$

Returning now to the question of computing the shift vectors, we first observe that we really only need obtain the shift for one element of each conjugacy class of each Weyl group. The shift vectors of other elements of the Weyl conjugacy classes can, once again, be obtained by Weyl transformations of the given shift vector.

A complete description of all conjugacy classes in the Weyl group of any simple Lie algebra has been given by Carter [111], and we will summarize the results here.

Every element of W can be expressed as a product of reflections (B.1). For a given element w, denote by $l(w)$ the minimal number of reflections (not necessarily w.r.t. simple roots) one needs to represent it. Then $l(w)$ is invariant under conjugation, and is less than or equal to the rank l of the Lie algebra. It is not difficult to show that the non-degenerate elements are precisely those for which $l(w) = l$.

An element of order 2 is called an *involution*. One can now prove the following theorems:
- Every involution r can be expressed as a product of $l(r)$ reflections corresponding to mutually orthogonal roots.
- Every element w of W can be expressed as the product of two involutions: $w = w_1 w_2$, $w_1^2 = w_2^2 = 1$, with $l(w) = l(w_1) + l(w_2)$ and $V_{-1}(w_1) \cap V_{-1}(w_2) = 0$, where $V_{-1}(w_i)$ is the set of eigenvectors of w_i with eigenvalue -1.

In a way very similar to Dynkin diagrams, one can associate a graph with such a decomposition. Every root β appearing in the decomposition of $w = w_1 w_2$ in terms of basic reflections r_β is represented by a node. The nodes representing two roots β and γ are connected by an n-fold bond, where $n = 4(\boldsymbol{\beta} \cdot \boldsymbol{\gamma})^2/\boldsymbol{\beta}^2\boldsymbol{\gamma}^2$. In general n can be 0, 1, 2 for 3, but if the Lie algebra is simply-laced (which we will assume to be the case from here on), n is either 0 or 1.

For a Lie algebra of rank l, we must consider graphs with at most l nodes. It follows from the foregoing that these nodes must correspond to *linearly independent* roots (note that they do not have to be simple). Furthermore it must be possible to decompose this set of roots into two sets, so that the roots *within* each set are mutually orthogonal. Such a graph is called an *admissible diagram*. A new feature in comparison with Dynkin diagrams is that a graph may contain *cycles*, i.e. closed loops formed by nodes and bonds. Only diagrams for which each cycle contains an *even* number of nodes are admissible, since otherwise one cannot divide the roots into sets of mutually orthogonal ones.

A subclass of the admissible diagrams consists of all the Dynkin diagrams of all the regular sub-algebras of $\mathscr{G}$. All other diagrams contain at least one cycle. The Dynkin diagram of $\mathscr{G}$ itself can be realized by taking the product of reflections with respect to all the simple roots. This is called the *Coxeter element* of $\mathscr{G}$. Its order is equal to the Coxeter number ($n+1$ for A_n, $2(n-1)$ for D_n and 12, 18, 30 for E_6, E_7, E_8), which in turn is equal to the eigenvalue of the quadratic Casimir operator in the adjoint representation (or half of it, depending on normalization) and to the maximal degree of the Casimir operators of $\mathscr{G}$. The Carter diagrams that are Dynkin diagrams of regular sub-algebras correspond to the Coxeter elements of those sub-algebras (for semisimple sub-algebras one takes the product of the Coxeter elements of the factors.)

In general, to obtain a Weyl group element corresponding to a Carter diagram, one must first find a set of independent roots which "realize" the diagram. One decomposes these roots into two sets $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_m)$, with $\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j = 0$ if $i \neq j$, and similarly for the second set. Then one writes a product of reflections of the form

$$w = (r_{\alpha_1} \cdots r_{\alpha_n})(r_{\beta_1} \cdots r_{\beta_n}) .$$

This is a Weyl group element corresponding to the diagram (the degree of uniqueness of this procedure will be discussed below). Note that in the foregoing expression the ordering of the r_α's amongst themselves is irrelevant, and similarly for the r_β's. Moreover, so is the relative ordering of the two terms in the parentheses. However, beyond this freedom, the ordering of the individual Weyl reflections *is* important, one cannot make arbitrary rearrangements. Coxeter elements form an exception to this rule, since it can be shown that they are conjugate to each other for any ordering.

Since all admissible diagrams of a regular sub-algebra of $\mathscr{G}$ are automatically admissible in $\mathscr{G}$, it is sufficient to classify all diagrams that cannot be realized in a sub-algebra. In particular it can be shown that one only has to consider connected diagrams, because all others can be realized in some sub-algebra.

In ref. [111] a list of all admissible diagrams not belonging to regular sub-algebras is given for all simple Lie algebras. For A_n this list consists only of the Dynkin digram of A_n. For all other simply laced Lie algebras there are additional diagrams with cycles. To obtain the *complete* list of admissible diagrams, one must first of all find all the regular sub-algebras of $\mathscr{G}$, which can be done by means of the extended Dynkin diagram. In the resulting diagrams one must then consider all possible ways of replacing a connected component by a cycle diagram of the corresponding Lie algebra.

The question naturally arises to what extent the admissible diagrams describe the conjugacy classes of $W(\mathscr{G})$. It turns out that the correspondence is not one-to-one, but it is not too far from being so, and the exceptions are listed in ref. [111]. If w_1 is conjugate to w_2 and w_1 can be represented by a graph Γ, then so can w_2. However, it may happen that two different graphs represent the same conjugacy class. Then every element in that class can be written in two essentially different ways as a product of reflections. It may also happen that one graph describes two conjugacy classes, i.e. that it has two non-conjugate realizations. Fortunately the exceptions are (in the latter case) always degenerate elements, so that we will not encounter them.

In figs. 4 and 5 we have collected the Carter diagrams with cycles for all simple, simply-laced Lie algebras (of the non-simply-laced algebras only F_4 has a cycle diagram). Where there are two admissible diagrams for a conjugacy class we give only one of them. This gives in a unique way all the conjugacy classes of elements of the Weyl group that are not Weyl group elements of sub-algebras. Of course the

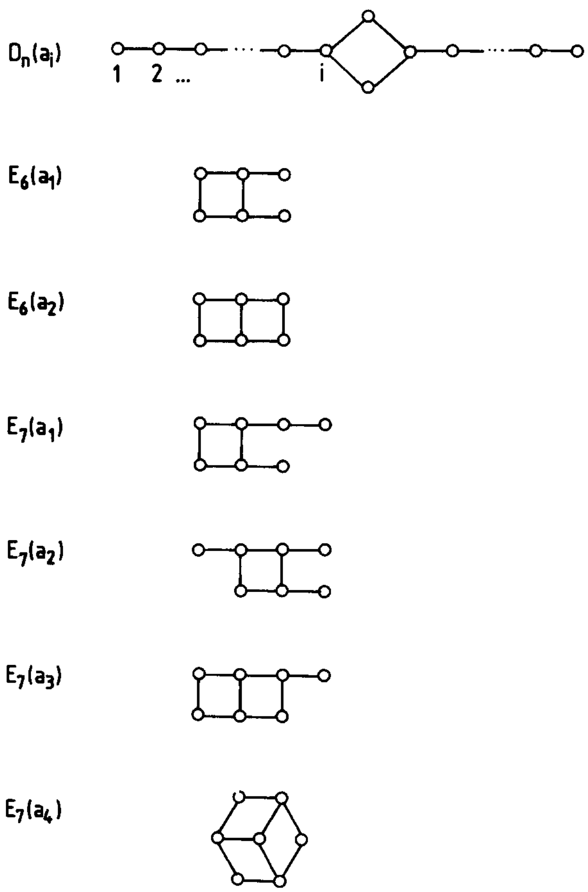

Fig. 4. Carter diagrams for D_n, E_6 and E_7.

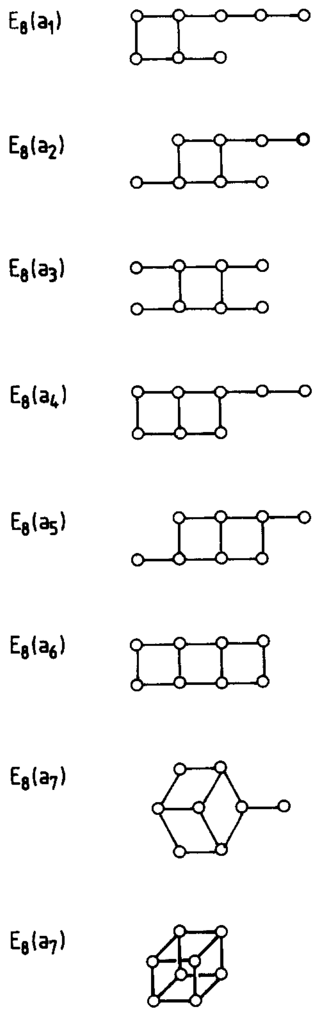

Fig. 5. Carter diagrams for E_8.

complete list of Weyl conjugacy classes can then be obtained by listing all regular sub-algebras, and applying the classification of figs. 4 and 5 to these sub-algebras. Two different diagrams for *sub*-algebras do not necessarily represent different conjugacy classes though. The simplest counter-example is given by the diagrams $D_3 \times D_3$ and $D_4(a1) \times D_2$, which turn out to describe the same conjugacy class of D_6. These cases are very rare, and we will not attempt to enumerate them completely (see ref. [111] for a complete description). In most cases one can easily identify two conjugacy classes as different by checking their eigenvalues on the maximal torus.

To illustrate the use of the diagrams we will consider the cycle diagrams for D_n as an example. They can be realized as follows. First remove one node, so that the cycle disappears, but the diagram is not cut. The resulting string of $n-1$ nodes can be realized by $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n$, where we use the standard basis for D_n. To realize $D_n(a_{i-1})$ one adds to this set the independent root $e_i + e_{i+1}$. For example, $D_4(a_1)$ can be represented by

$$w = r_{e_1-e_2} r_{e_3-e_4} r_{e_2-e_3} r_{e_2+e_3} .$$

The action on the CSA of the representative Weyl element of $D_n(a_{i-1})$ constructed above will be discussed below, where we will also obtain the corresponding shift vector.

Because of the properties of lifts of non-degenerate elements we described earlier, we know all the orbits for all lifts of such elements to G. Furthermore, since we know explicitly the rotation matrix in weight space, we know also all the eigenvalues on the CSA (the characteristic polynomials of these rotation matrices are also listed in ref. [111]). The eigenvalues on the root generators can be obtained in the following way. For an orbit of length k generated from E_α one constructs the following linear combinations of generators

$$V_\alpha(l) = \sum_{i=0}^{k-1} \theta^{il} \tilde{w}^i E_\alpha \tilde{w}^{-i} , \tag{B.18}$$

where $\theta = \exp(2\pi i/k)$ and $\tilde{w}$ is any lift of w. Clearly these linear combinations are twist eigenvectors with eigenvalue θ^l:

$$\tilde{w} V_\alpha(l) \tilde{w}^{-1} = \theta^l V_\alpha(l) .$$

Knowledge of all these eigenvalues, both on the CSA and on the root generators turns out to be sufficient to determine the shift vector $\boldsymbol{\sigma}$ belonging to w. In ref. [18] this observation was used to determine a set of shift vectors of minimal length corresponding to all the non-degenerate Weyl elements for Lie algebras of rank less than or equal to eight. The results, together with other information about non-degenerate elements are given in table 11. (All other equivalent shift vectors can, of course, be obtained from the set in table 11 by adding roots and acting with Weyl rotations.)

It turns out to be an elementary analytic calculation to compute all the shift vectors for Weyl transformations of A_k and D_k. For D_k one introduces the usual orthonormal basis, $\{e_i\}$, in which the roots of D_k are $\{\pm e_i \pm e_j\}$ (see, for example ref. [112]). A typical element, g, of the conjugacy class $D_k(a_{m-1})$ acts on this basis according to:

$$\begin{aligned} &e_1 \to e_2 \to \cdots \to e_m \to -e_1 \to -e_2 \to \cdots \to -e_m \to e_1 \\ &e_{m+1} \to e_{m+2} \to \cdots \to e_k \to -e_{m+1} \to -e_{m+2} \to \cdots \to -e_k \to e_{m+1} . \end{aligned} \tag{B.19}$$

Note that for $m = 1$ this describes the action of a Coxeter element, and thus $D_k(a_0)$ is the Coxeter conjugacy class. The easiest way to determine the corresponding shift vector is to observe that since the order of a non-degenerate Weyl element of D_k does *not* double on the vector representation, the eigenvalues of $\tilde{g}$, a lift of the Weyl element g to the Lie group, can be read off from (B.19) directly. Since (B.19) represents a simple permutation of the vector weight spaces, the eigenvalues of $\tilde{g}$ are:

$$e^{\pi i p/m} , \quad e^{\pi i q/(k-m)} \qquad p = 0, \ldots, 2m-1 ; \quad q = 0, \ldots, 2(k-m)-1 . \tag{B.20}$$

Since $\exp(2\pi i \boldsymbol{\sigma} \cdot \boldsymbol{H})$ must have the same eigenvalues, it is elementary to see that one way to achieve this, and hence one possible choice for the shift vector is

$$\begin{aligned} \boldsymbol{\sigma} = {} & (2m)^{-1}[m e_1 + (m-1)e_2 + \cdots + 2e_{m-1} + e_m] \\ & + [2(k-m)]^{-1}[(k-m-1)e_{m+1} + (k-m-2)e_{m+2} + \cdots + e_{k-1}] . \end{aligned} \tag{B.21}$$

Table 11

Non-degenerate conjugacy classes in the Weyl group for simply-laced algebras of rank not larger than 8. For algebras not appearing in the table the only non-degenerate elements are Coxeter elements, whose shift vectors are given by (B.25). Column 1 defines the Carter diagram, and corresponds to figs. 4 and 5. Column 2 gives the orbit structure, with $(M)^N$ denoting N orbits of length M. The order n is equal to the length of the longest orbit. Column 3 gives the integers k_i so that $\exp(2\pi i(k_i/n))$ are the eigenvalues on the torus. The last column gives the shift vector: $(\ldots, m_i, \ldots)$ indicates that the shift vector has inner product m_i/n with the simple root i. These roots are ordered as in ref. [106]

Graph	orbit structure	torus eigenvalues	shift vector
D_4	$(6)^4$	1, 3, 3, 5	(1, 1, 1, 1)
D_4(a1)	$(4)^6$	1, 1, 3, 3	(1, 0, 1, 1)
D_5	$(8)^5$	1, 3, 4, 5, 7	(1, 1, 1, 1, 1)
D_5(a1)	$(12)^2(6)^2(4)^1$	2, 3, 6, 9, 10	(2, 1, 1, 2, 2)
D_6	$(10)^6$	1, 3, 5, 5, 7, 9	(1, 1, 1, 1, 1, 1)
D_6(a1)	$(8)^7(4)^1$	1, 2, 3, 5, 6, 7	(1, 1, 0, 1, 1, 1)
D_6(a2)	$(6)^{10}$	1, 1, 3, 3, 5, 5	(1, 0, 1, 0, 1, 1)
D_7	$(12)^7$	1, 3, 5, 6, 7, 9, 11	(1, 1, 1, 1, 1, 1, 1)
D_7(a1)	$(20)^2(10)^4(4)^1$	2, 5, 6, 10, 14, 15, 18	(2, 2, 1, 1, 2, 2, 2)
D_7(a2)	$(24)^2(8)^3(6)^2$	3, 4, 9, 12, 15, 20, 21	(3, 1, 2, 2, 1, 3, 3)
D_8	$(14)^8$	1, 3, 5, 7, 7, 9, 11, 13	(1, 1, 1, 1, 1, 1, 1, 1)
D_8(a1)	$(12)^9(4)^1$	1, 3, 3, 5, 7, 9, 9, 11	(1, 1, 1, 0, 1, 1, 1, 1)
D_8(a2)	$(30)^2(10)^4(6)^2$	3, 5, 9, 15, 15, 21, 25, 27	(3, 2, 1, 3, 1, 2, 3, 3)
D_8(a3)	$(8)^{14}$	1, 1, 3, 3, 5, 5, 7, 7	(1, 0, 1, 0, 1, 0, 1, 1)
E_6	$(12)^6$	1, 4, 5, 7, 8, 11	(1, 1, 1, 1, 1, 1)
E_6(a1)	$(9)^8$	1, 2, 4, 5, 7, 8	(1, 1, 0, 1, 1, 1)
E_6(a2)	$(6)^{12}$	1, 1, 2, 4, 5, 5	(1, 0, 1, 0, 1, 0)
E_7	$(18)^7$	1, 5, 7, 9, 11, 13, 17	(1, 1, 1, 1, 1, 1, 1)
E_7(a1)	$(14)^9$	1, 3, 5, 7, 9, 11, 13	(1, 1, 0, 1, 1, 1, 1)
E_7(a2)	$(12)^{10}(6)^1$	1, 2, 5, 6, 7, 10, 11	(1, 0, 1, 0, 1, 1, 1)
E_7(a3)	$(30)^3(10)^3(6)^1$	3, 5, 9, 15, 21, 25, 27	(2, 1, 2, 1, 2, 3, 1)
E_7(a4)	$(6)^{21}$	1, 1, 1, 3, 5, 5, 5	(0, 0, 1, 0, 0, 1, 0)
E_8	$(30)^8$	1, 7, 11, 13, 17, 19, 23, 29	(1, 1, 1, 1, 1, 1, 1, 1)
E_8(a1)	$(24)^{10}$	1, 5, 7, 11, 13, 17, 19, 23	(1, 1, 0, 1, 1, 1, 1, 1)
E_8(a2)	$(20)^{12}$	1, 3, 7, 9, 11, 13, 17, 19	(1, 1, 0, 1, 0, 1, 1, 1)
E_8(a3)	$(12)^{20}$	1, 1, 5, 5, 7, 7, 11, 11	(1, 0, 1, 0, 0, 1, 0, 0)
E_8(a4)	$(18)^{13}(6)^1$	1, 3, 5, 7, 11, 13, 15, 17	(0, 1, 0, 1, 0, 1, 1, 1)
E_8(a5)	$(15)^{16}$	1, 2, 4, 7, 8, 11, 13, 14	(1, 0, 1, 0, 1, 0, 1, 0)
E_8(a6)	$(10)^{24}$	1, 1, 3, 3, 7, 7, 9, 9	(0, 0, 1, 0, 0, 1, 0, 0)
E_8(a7)	$(12)^{18}(6)^4$	1, 2, 2, 5, 7, 10, 10, 11	(0, 1, 0, 1, 0, 0, 1, 0)
E_8(a8)	$(6)^{40}$	1, 1, 1, 1, 5, 5, 5, 5	(0, 0, 0, 1, 0, 0, 0, 0)

The Coxeter shift is obtained by taking $m = 1$.

It is equally straightforward to derive the corresponding result for A_k by using its vector representation. Indeed, using the orthonormal basis $\{e_i,\ i = 1, \ldots, n\}$, the roots of A_{n-1} can be written as $\{\pm(e_i - e_j)\}$. The only non-degenerate conjugacy class is the Coxeter class, whose elements have order n and have eigenvalues $e^{2\pi i m/n}$, $m = 1, \ldots, n-1$ when acting on the CSA. A typical element of this class acts on the basis e_i according to

$$e_1 \to e_2 \to e_3 \to \cdots \to e_n \to e_1 \tag{B.22}$$

and the corresponding shift vector is given by

$$(2n)^{-1}[(n-1)e_1 + (n-3)e_2 + \cdots + (1-n)e_n]\,. \tag{B.23}$$

Table 1 along with eqs. (B.21) and (B.23) therefore provide a complete list of all Weyl shift vectors of simple, simply-laced Lie algebras. The shift vectors for degenerate elements can be obtained from our list for non-degenerate elements by passing to an appropriate sub-algebra in which the given degenerate element becomes non-degenerate. A complete list of shift vectors for degenerate and non-degenerate Weyl elements of simply-laced Lie algebras of rank eight and less has recently appeared in ref. [113], while the shift vectors for E_6, E_7 and E_8 were also obtained in ref. [114].

One can also explicitly check that the shift vectors described here satisfy the identity:

$$\tfrac{1}{2}(\boldsymbol{\sigma})^2 = \frac{1}{4}\sum_{i=1}^{l} \tau_i(1-\tau_i)\,, \tag{B.24}$$

where the τ_i are such that $0<\tau_i<1$ and $e^{2\pi i\tau_j}$ are the eigenvalues of the corresponding Weyl transformation, w, acting on the CSA*). This identity is important in orbifold constructions, since it relates the zero point energy of "twisted" states (right-hand side) to that of "shifted" states (left-hand side). The result is what one might have expected, since it implies that one will get the same answer if one uses a description in terms of Weyl rotations, or a quantum-equivalent description in terms of shifts. Note that eq. (B.24) is an exact equality, not just modulo $1/n$ (where n is the order of w). This is a consequence of the fact that both sides of the equation represent the exact minimum of the energy. Replacing $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma}+\boldsymbol{w}$ leads to a completely equivalent shift vector, provided that $\boldsymbol{w}$ is a weight of the algebra in which w is non-degenerate. For such a shift vector (B.24) holds, in general, only up to multiples of $1/n$. If $\boldsymbol{\sigma}$ is part of a shift vector which satisfies level matching, then adding a weight to it will clearly not upset the matching. The fact that the matching of the zero point energy is no longer exact simply means that there must exist another lattice vector on the same lattice for which eq. (B.24) *is* satisfied. This fact follows directly from eq. (B.10) since it implies that the true vacuum in the twisted/shifted sector can always be obtained by subtracting root vectors from the shift vector.

The identity (B.24) has also been established for all Weyl twists or simply-laced algebras by considering representations of Kac–Moody algebras [59, 60].

As a final remark, it is useful to note that a great deal is known about the Coxeter element of the Weyl group, and that this element is intimately related to the principal three dimensional sub-algebra of the original Lie algebra. There is a simple expression for the shift vector of the Coxeter conjugacy class in terms of the vector $\boldsymbol{\delta}$, which is the sum of all the positive roots:

$$\boldsymbol{\sigma} = (1/h)\boldsymbol{\delta}\,, \tag{B.25}$$

where h is the Coxeter number defined above. The vector $\boldsymbol{\delta}$ has inner product $+1$ with all simple roots, so that in the notation of table 11 the entry in the last column is $(1,1,1,\ldots,1,1)$ for Coxeter elements. For more details on properties of Coxeter elements see refs. [115] and [58].

Appendix C. Modular functions

In this appendix we collect some useful results about functions that have nice properties under modular transformations. Although most of this report concentrates on one-loop modular properties,

*) Note that we count an eigenvalue as well as its complex conjugate, so as to be careful with unpaired -1 eigenvalues.

some of these results can be derived just as easily at higher genus, and therefore it would be a pity to make unnecessary restrictions. We begin therefore with a very brief introduction to multi-loop modular transformations. A proper explanation of this subject would require a discussion of Riemann surface theory, which is beyond the scope of this report; we refer interested readers to refs. [116, 25].

C.1. Modular transformations

Most of the relevant properties of higher genus modular transformations can be understood as a simple generalization of the ones for the torus, discussed in section 3.3. Instead of the two independent homology cycles of the torus, we get now 2γ such cycles, which can be chosen as in fig. 3. They are simply non-contractible loops on the surface, to which one has assigned an orientation, indicated by the arrow in fig. 3. One can choose a set of γ holomorphic 1-forms ω_i on the surface, normalized as follows with respect to integration around the a-cycles:

$$\int_{a_i} \omega_j = \delta_{ij} \,.$$

The integration along the b-cycles defines the *period matrix* Ω_{ij}:

$$\int_{b_i} \omega_j = \Omega_{ij} \,.$$

The period matrix parametrizes the shape of the surface, and is the generalization of τ at genus 1. Indeed, if we define the torus as usual as the complex plane modulo a lattice with basis vectors 1 (for the a-cycle) and τ (for the b-cycle) then the normalized holomorphic 1-form is simply $\mathrm{d}z$, and gives τ when integrated along the b-cycle. It can be shown that Ω_{ij} must be symmetric and have a positive imaginary part, but not all such matrices describe a Riemann surface; at genus γ there are only $3(\gamma-1)$ moduli.

The group of modular transformations (or *mapping class group*) of a genus γ Riemann surface is the set of all diffeomorphisms modulo those diffeomorphisms that can be continuously connected to the identity. It is generated by *Dehn twists* about a non-contractible loop on the surface. To perform such a twist one cuts the surface along the loop, rotates one of the edges by 360° and glues it back to the other edge. In general such a twist will act on the homology cycles. For example, for the cycles depicted in fig. 3, a Dehn-twist along a_i sends b_i into $a_i + b_i$ and leaves all other cycles unaltered. In other words, a loop winding along b_i before the twist, will wind along $b_i + a_i$ after the twist. One can represent such a transformation by a matrix, by writing the cycles as $(b_1, \ldots, b_\gamma, a_1, \ldots, a_\gamma)$. At genus 1, the twist along a_1 is represented by a matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

Dehn-twists send homology cycles into combinations of homology cycles, but they cannot affect the intersections of cycles. The intersection number $I(c_1, c_2)$ of two cycles is the number of points in which they intersect, counting orientation (if the outer product of the orientation vectors of c_1 and c_2 points inwards (outwards) the intersection point counts as $+1$ (-1)). For our basis choice, the intersection matrix giving the intersections of all 2γ cycles is

$$I = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix},$$

where $\mathbf{1}$ is the γ-by-γ unit matrix. The fact that it is preserved by Dehn twists implies that a matrix D representing such twists on the homology basis must satisfy

$$DID^{t} = I,$$

i.e. D is an element of $\mathrm{Sp}_{2\gamma}(\mathbb{Z})$ (here "t" denotes transposition of a matrix).

Not all elements of the mapping class group act non-trivially on the homology basis. The elements that act trivially form a normal subgroup, the Torelli group, whose elements are represented by the identity element of $\mathrm{Sp}_{2\gamma}(\mathbb{Z})$. One has to check modular invariance of superstrings also for these transformations. This has been done in ref. [117] for one ten-dimensional heterotic string theory. Since all other heterotic strings are obtained by changing the periodicities of various fields along the homology cycles, all one really has left to check is invariance under transformations that act non-trivially on the homology cycles.

A symplectic modular transformation $\mathcal{M}$ at genus γ acts in the following way on the period matrix

$$\mathcal{M}\Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2\gamma}(\mathbb{Z}).$$

As for $\gamma = 1$ there is a simple set of generating transformations, which correspond to the following $\mathrm{Sp}_{2\gamma}(\mathbb{Z})$ matrices.

$$T_i \qquad \begin{pmatrix} \mathbf{1} & E_i \\ 0 & \mathbf{1} \end{pmatrix},$$

$$S_i \qquad \begin{pmatrix} \mathbf{1} - E_i & E_i \\ -E_i & \mathbf{1} - E_i \end{pmatrix},$$

$$U_{ij} \qquad \begin{pmatrix} \mathbf{1} & F_{ij} \\ 0 & \mathbf{1} \end{pmatrix},$$

where $(E_i)_{kl} = \delta_{kl}\delta_{ik}$ and $(F_{ij})_{kl} = -\delta_{kl}(\delta_{ik} + \delta_{jk}) + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}$. The first two of these transformations generate the twists around the cycles a_i and b_i*). At genus 1, T_1 reduces to $\tau \to \tau + 1$ and S_1 to $\tau \to -1/\tau$. The transformation U_{ij} generates a twist around a cycle linking handle i with handle j. Checking invariance with respect to T_i and U_{ij} is usually elementary. Their effect is simply to send Ω to respectively $\Omega + E_i$ and $\Omega + F_{ij}$.

To study the effect of S_i take, for convenience, $i = 1$. For the transformed period matrix we find then

$$S_1\Omega_{ij} = \frac{1}{\Omega_{11}} \begin{pmatrix} -1 & -\Omega_{1j} \\ -\Omega_{i1} & \Omega_{ij}\Omega_{11} - \Omega_{i1}\Omega_{1j} \end{pmatrix}, \tag{C.1}$$

where on the right-hand side $2 \le i, j \le \gamma$.

*) More precisely, the twist around b_i is generated by a combination of T_i and S_i, but of course that is irrelevant.

To deal with this modular transformation for lattices we need the *Poisson resummation formula*. This formula relates sums over a lattice $\Lambda \in \mathbb{R}^N$ to sums over its dual Λ^*:

$$\sum_{w\in\Lambda} f(w) = \frac{1}{\mathrm{vol}(\Lambda)} \sum_{v\in\Lambda^*} f^*(v)\,, \tag{C.2}$$

where f is a (sufficiently well-behaved) function and f^* its Fourier transform,

$$f^*(v) = \int_{\mathbb{R}^N} \mathrm{d}w\, \mathrm{e}^{2\pi \mathrm{i} v\cdot w} f(w)\,.$$

Note that this is a Fourier transform over all of $\mathbb{R}^N$ and not over a fundamental cell of the lattice.

This formula is easy to prove. Define

$$F(z) = \sum_{w\in\Lambda} f(w+z)\,. \tag{C.3}$$

Obviously $F(z)$ is periodic in z with the periodicity of Λ. Hence it can be written as a Fourier series

$$F(z) = \sum_{v\in\Lambda^*} \mathrm{e}^{-2\pi \mathrm{i} z\cdot v}\, F^*(v)\,,$$

with

$$F^*(v) = \frac{1}{\mathrm{vol}(\Lambda)} \int_{\text{unit cell of } \Lambda} \mathrm{d}y\, \mathrm{e}^{2\pi \mathrm{i} y\cdot v}\, F(y)\,.$$

Now substitute (C.3) and observe that the sum over all vectors in Λ, or equivalently over all unit cells, combines with the integral over one unit cell to give an integral over the complete vector space $\mathbb{R}^N$. After a trivial shift of integration variables one obtains (C.2).

This formula is usually applied to functions f of the form

$$f(w) = \mathrm{e}^{-(\mathrm{i}\pi/\tau)w^2 + 2\pi \mathrm{i} w\cdot r}\,,$$

where τ is a complex variable with positive imaginary part. The Fourier integral defining f^* is a simple Gaussian and yields

$$f^*(v) = (\sqrt{-\mathrm{i}\tau})^N\, \mathrm{e}^{\mathrm{i}\pi\tau(v+r)^2}\,. \tag{C.4}$$

Here $\sqrt{-\mathrm{i}\tau}$ should be chosen to lie in the right half of the complex plane.

C.2. *Conjugacy class ϑ-functions*

Consider an *even* lattice Λ and its dual Λ^*, which has a coset decomposition with respect to Λ. We will be mainly interested in the case where Λ is the root lattice of a simply laced algebra, and we will therefore refer to these cosets as conjugacy classes, and denote them as (x). Let Ω_{ij} be the period matrix of a genus-γ Riemann surface, and let $[x]=[x_1,\ldots,x_\gamma]$ be a collection of γ conjugacy classes. Thus $[x]$ contains collections of vectors $\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_\gamma$, with $\boldsymbol{\lambda}_i\in(x_i)$. Finally, take a collection of vectors $z_1,\ldots,z_\gamma$ in Λ-weight space. For each collection of conjugacy classes $[x]$ we define the following functions

$$\Theta^{\Lambda}_{[x]}(z|\Omega)=\sum_{\boldsymbol{\lambda}_i\in[x]}\exp(\mathrm{i}\pi\boldsymbol{\lambda}_i\Omega_{ij}\boldsymbol{\lambda}_j+2\pi\mathrm{i}\boldsymbol{\lambda}_i\cdot z_i)\,, \tag{C.5}$$

where the sum is over all collections of vectors in $[x]$. In the exponent there is an implicit sum over the genus-labels i and j, and in the first argument of the function z stands generically for the entire set. We call these functions the multi-loop conjugacy class ϑ-functions for the collection of conjugacy classes $[x]$.

In applications to string theory the zero-mode contribution to multi-loop partition functions can be assembled out of these ϑ-functions. It is also useful to introduce *conjugacy class characters*, which are related to the ϑ-functions by including oscillator contributions. A boson with left- and right-moving components contributes a factor $[\det'(-\Delta)]^{-1/2}$, where $\det'(-\Delta)$ is the determinant of the Laplacian with zero-mode removed. To get the contribution of a *chiral* boson one has to take a holomorphic square root out of this determinant, i.e. find a function $\mathscr{R}(\Omega)$ so that $\mathscr{R}(\Omega)\mathscr{R}(\bar{\Omega})=\det'(-\Delta)$ (this does not yet define $\mathscr{R}$ uniquely, but the ambiguity is not relevant for our purpose). If the lattice Λ has dimension N we define then

$$\mathrm{Ch}^{\Lambda}_{[x]}(z|\Omega)=\Theta^{\Lambda}_{[x]}(z|\Omega)/\mathscr{R}(\Omega)^{N/2}\,. \tag{C.6}$$

At genus 1, $\mathscr{R}(\tau)^{1/2}=\eta(\tau)$, and (if Λ is a Lie algebra lattice), these conjugacy class characters are simply the Kac–Moody characters discussed in section 4.4. Their expansion in powers of $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$ yields the Chern characters of all the Lie algebra representations contained in the Kac–Moody representation, as discussed in section 9.

The conjugacy class ϑ-functions have rather simple modular properties. Their transformation under T_i and U_{ij} is

$$\Theta^{\Lambda}_{[x]}(z|T_i\Omega)=\mathrm{e}^{\mathrm{i}\pi\|x_i\|}\,\Theta^{\Lambda}_{[x]}(z|\Omega)\,,$$

$$\Theta^{\Lambda}_{[x]}(z|U_{ij}\Omega)=\mathrm{e}^{-\mathrm{i}\pi\|x_i-x_j\|}\,\Theta^{\Lambda}_{[x]}(z|\Omega)\,,$$

where $\|x\|$ denotes the norm (modulo 2) of the conjugacy class (x). It is defined by computing the norm of any representative of (x) (note that the phases do not depend on the choice of representative since Λ is even.)

To obtain the transformation under S_1 we use the Poisson resummation formula. By simply substituting (C.1) into (C.5) one obtains an expression of the form

$$\sum_{\boldsymbol{\lambda}_1\in(x_1)}\exp\left[-\frac{\mathrm{i}\pi}{\Omega_{11}}(\boldsymbol{\lambda}_1+\Omega_{1j}\boldsymbol{\lambda}_j-\Omega_{11}z_1)^2\right]\times\{\lambda_1\text{-independent terms}\}\,.$$

This expression has to be summed over all other λ_j, but we concentrate on λ_1, and apply Poisson resummation to this sum alone.

A minor complication is the fact that (x_1) in general is not a lattice, but a shifted lattice. This problem is easily solved by writing the sum over $\boldsymbol{\lambda}_1 \in (x_1)$ as a sum over $\boldsymbol{\mu}_1 + \boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is a fixed representative of (x_1), and μ_1 is summed over all vectors in Λ.

The rest of the calculation is straightforward, and using (C.4) we get

$$\Theta^{\Lambda}_{[x]}(S_1 z|S_1 \Omega) = \frac{(\sqrt{-\mathrm{i}\Omega_{11}})^N}{\mathrm{vol}(\Lambda)}\, \mathrm{e}^{\mathrm{i}\pi\Omega_{11} z_1^2} \sum_{(y_1)\in\Lambda^*/\Lambda} \mathrm{e}^{2\pi\mathrm{i}(x_1, y_1)} \Theta^{\Lambda}_{[y_1, x_2, \dots, x_\gamma]}(z|\Omega)\,. \tag{C.7}$$

Here (x, y) denotes the inner product (modulo 1) between conjugacy classes (x) and (y), and the sum is over all conjugacy classes (y_1). Note that we have chosen a transformed first argument on the left-hand side in order to get simply z on the right-hand side. The transformation of the set of vectors z under S_1 is

$$\begin{aligned} (S_1 z)_1 &= -z_1/\Omega_{11}\,, \\ (S_1 z)_j &= -(\Omega_{1j}/\Omega_{11}) z_j\,, \quad j = 2, \dots, \gamma\,. \end{aligned} \tag{C.8}$$

C.3. *Spin structure ϑ-functions*

The spin structure ϑ functions are close relatives of the conjugacy class ϑ-functions, and can be regarded as D_1 ϑ-functions written in spin structure basis. They are defined as follows

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\Omega) = \sum_{n_i\in\mathbb{Z}^\gamma} \exp[\mathrm{i}\pi(n_i + a_i)\Omega_{ij}(n_j + a_j) + 2\pi\mathrm{i}(n_i + a_i)(z_i + b_i)]\,, \tag{C.9}$$

with implicit summation on i and j. Here z is a collection of γ parameters, and $[{a \atop b}]$ denotes the *spin structure*. Just as conjugacy classes are the natural basis for lattices, spin structures are the natural basis for fermionic constructions. The spin structure is the additional information one has to supply with a Riemann surface to specify the periodicities of fermions along the homology cycles. The spin structure is labeled by 2γ parameters

$$\begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} a_1, \dots, a_\gamma \\ b_1, \dots, b_\gamma \end{bmatrix}.$$

It tells us that a fermion picks up phase $\mathrm{e}^{2\pi\mathrm{i}a_i}$ when transported along the a_i cycle, and similarly for the b-cycles. Untwisted fermions can be periodic or anti-periodic, so that a_i and b_i can be 0 or $\frac{1}{2}$ (mod 1). One distinguishes *even* and *odd* spin structures, depending on whether $\Sigma_i 4a_i b_i$ is even or odd (the terminology stems from the number of zero-modes of the Dirac operator for a given spin-structure).

In string loop computations, ϑ-functions appear in the determinants of world sheet fermions. For example, for two spin-$\frac{1}{2}$ Majorana–Weyl fermions on a Riemann surface with period matrix Ω and spin structure α one gets

$$(\det D_{1/2})_\alpha = \vartheta_\alpha(0|\Omega)/\mathcal{R}(\Omega)^{1/2}\,,$$

where $\mathcal{R}$ is a holomorphic square root of the determinants of the Laplacian, defined above.

If the parameters a_i and b_i are integer or half-integer, the ϑ-functions can be expressed in terms of

D_n conjugacy class ϑ-functions and vice versa. At genus 1, one has

$$\begin{aligned}
\Theta^{D_n}_{[0]} &= \frac{1}{2}\left(\vartheta^n\begin{bmatrix}0\\0\end{bmatrix} + \vartheta^n\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}\right),\\
\Theta^{D_n}_{[v]} &= \frac{1}{2}\left(\vartheta^n\begin{bmatrix}0\\0\end{bmatrix} - \vartheta^n\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}\right),\\
\Theta^{D_n}_{[s]} &= \frac{1}{2}\left(\vartheta^n\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} + \mathrm{i}^n\vartheta^n\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right),\\
\Theta^{D_n}_{[c]} &= \frac{1}{2}\left(\vartheta^n\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} - \mathrm{i}^n\vartheta^n\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right).
\end{aligned} \tag{C.10}$$

Here we have suppressed the argument Ω and have set $z=0$. For non-vanishing z the nth power on the right-hand side is replaced by a product of n ϑ-functions, each having one entry of z as its argument. The factors i^n in front of the odd spin structure ϑ-function are an annoying, but otherwise irrelevant consequence of the standard definition (C.9) of the ϑ-functions.

The generalization of (C.10) to higher genus is straightforward: one simply has such a transformation for each handle. For example

$$\Theta^{D_n}_{[0s]} = \frac{1}{4}\left(\vartheta^n\begin{bmatrix}0\frac{1}{2}\\00\end{bmatrix} + \vartheta^n\begin{bmatrix}0\frac{1}{2}\\\frac{1}{2}0\end{bmatrix} + \mathrm{i}^n\vartheta^n\begin{bmatrix}0\frac{1}{2}\\0\frac{1}{2}\end{bmatrix} + \mathrm{i}^n\vartheta^n\begin{bmatrix}0\frac{1}{2}\\\frac{1}{2}\frac{1}{2}\end{bmatrix}\right).$$

The modular transformations for the spin-structure ϑ-functions can be derived in exactly the same way as those of conjugacy class ϑ-functions. In fact an explicit expression exists for arbitrary elements $\mathcal{M}$ of $\mathrm{Sp}_{2\gamma}(\mathbb{Z})$ [118]:

$$\vartheta\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\mathcal{M}z|\mathcal{M}\Omega) = \varepsilon(\mathcal{M})\,\mathrm{e}^{\mathrm{i}\pi\phi(a,b,\mathcal{M})}\,\mathrm{e}^{\mathrm{i}\pi z(C\Omega+D)^{-1}Cz}\sqrt{\det(C\Omega+D)}\,\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega)\,, \tag{C.11}$$

where

$$\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix} = \begin{bmatrix}a'\\b'\end{bmatrix} + \begin{bmatrix}e\\f\end{bmatrix}, \tag{C.12}$$

$$\begin{bmatrix}a'\\b'\end{bmatrix} = \begin{pmatrix}D & -C\\-B & A\end{pmatrix}\begin{bmatrix}a\\b\end{bmatrix}, \tag{C.13}$$

$$\begin{bmatrix}e\\f\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\mathrm{diag}(CD^{\mathrm{t}})\\\mathrm{diag}(AB^{\mathrm{t}})\end{bmatrix}, \tag{C.14}$$

$$\phi(a,b,\mathcal{M}) = a'\cdot b' - a\cdot b + 2a'\cdot f\,, \tag{C.15}$$

$$\mathcal{M}z = \{(C\Omega+D)^{\mathrm{t}}\}^{-1}z\,, \tag{C.16}$$

and finally,

$$\varepsilon(\mathcal{M})^8 = 1 . \tag{C.17}$$

Here we have suppressed all the genus labels, but one should sum over them in the obvious way. Using the basis transformations between conjugacy classes and spin structures, one may check that these transformations agree with those of the D_n ϑ-functions for T_i, S_i and U_{ij}. In this one can also determine the phase $\varepsilon(\mathcal{M})$, for which apparently no simple general expression exists. Note that (C.16) reduces to (C.8) for $\mathcal{M} = S_1$.

C.4. Generalized Riemann identities

For every automorphism of Λ one can derive an identity for the conjugacy class ϑ-functions. By an automorphism we mean a discrete rotation that takes Λ into itself. Since a rotation preserves inner products, it takes then also Λ^* into itself. This can happen in two ways: either a vector in Λ^* is always rotated to a vector in the same conjugacy class, or some vectors are mapped to different conjugacy classes. In the first case we speak of *inner* automorphisms, in the second of *outer* automorphisms. Either kind of automorphism is specified by some discrete rotation matrix T, which is an $N \times N$ orthogonal matrix. For outer automorphisms, one needs furthermore a conjugacy class map t, which specifies how the conjugacy classes are mapped into each other. Obviously this map can be derived from T, and we can regard inner automorphisms as automorphisms with t equal to the identity.

By applying such an automorphism to the vectors in the sum defining (C.5) we get almost trivially

$$\Theta^{\Lambda}_{[x]}(z|\Omega) = \Theta^{\Lambda}_{t[x]}(Tz|\Omega) . \tag{C.18}$$

This is the generalized Riemann identity. By $t[x]$ we mean of course that t acts simultaneously on all handles.

One automorphism that every lattice has is $T = -\mathbf{1}$. This is an outer automorphism for A_n, $n \geq 2$, E_6 and D_n, n odd, and an inner one for the other simple, simply-laced Lie algebras. In this case (C.18) relates the ϑ-function of a conjugacy class collection $[x]$ to those of its complex conjugate, and it shows that the conjugacy class ϑ-functions of self-conjugate $[x]$'s are even in z.

A more interesting automorphism is triality in SO(8). Triality is a group of outer automorphisms isomorphic to S_3, which acts on the conjugacy classes (v), (s) and (c). For example, consider the following matrix T

$$T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} . \tag{C.19}$$

Acting on the standard D_4 basis (A.12) it interchanges the conjugacy classes (v) and (s), and maps (c) into itself, i.e. $t(v) = (s)$, $t(s) = (v)$, $t(c) = (c)$ and $t(0) = (0)$.

For this choice of T, (C.18) is nothing but the standard Riemann identity. To get these identities in their usual form one starts with any odd or even spin structure, writes it in conjugacy class basis, applies (C.18) and then converts back to spin-structure basis. For the triality rotation given above this procedure corresponds to replacing z by Tz and transforming the spin-structures in each handle according to

$$\begin{bmatrix} 0 \\ \frac{1}{4}(1-\varepsilon) \end{bmatrix} = (0) + \varepsilon(v) \to (0) + \varepsilon(s) \to \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right)$$

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4}(1-\varepsilon) \end{bmatrix} = (s) + \varepsilon(c) \to (v) + \varepsilon(c) \to \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} - \varepsilon \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right),$$

where $\varepsilon = \pm 1$. The result of this transformation of the spin-structures can be summarized by

$$\begin{bmatrix} a \\ b \end{bmatrix} \to \frac{1}{2} \sum_{a',b'=0}^{1/2} \{\exp[4\mathrm{i}\pi(ab' + ba')]\} \begin{bmatrix} a' \\ b' \end{bmatrix}$$

and hence we get at genus γ

$$2^{\gamma} \prod_{p=1}^{4} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z_i^p|\tau) = \sum_{a',b'} \exp\left[4\mathrm{i}\pi \sum_i (a_i b_i' + b_i a_i')\right] \prod_{p=1}^{4} \vartheta\begin{bmatrix} a' \\ b' \end{bmatrix}(T^p_q z_i^q|\tau),$$

where z_i^p are the components of z_i, and a' and b' run over all the spin-structures.

Another interesting application of (C.18) is to Weyl rotations generated by spinor roots of the exceptional algebras E_n. These exceptional algebras contain a regular D_4 sub-algebra, and can be decomposed as $\mathrm{E}_n \supset \mathrm{K}_{n-4} \times \mathrm{D}_4$, where

$$\mathrm{K}_4 = \mathrm{D}_4\,, \qquad \mathrm{K}_3 = (\mathrm{A}_1)^3\,, \qquad \mathrm{K}_2 = \mathrm{D}_1 \mathscr{U}\,.$$

Here $\mathscr{U}$ is not a simple Lie algebra lattice, but it represents a U(1)-factor with charges quantized in units of $\frac{1}{6}\sqrt{3}$. One may think of it as having a root-lattice $\{2m\sqrt{3},\ m \in \mathbb{Z}\}$, and 12 conjugacy classes with respect to that root lattice.

The ϑ-functions of E_n can be decomposed correspondingly,

$$\Theta^{\mathrm{E}_n}_{[y]} = \sum_{[x]} \Theta^{\mathrm{K}_{n-4}[x]}_{[y]} \Theta^{\mathrm{D}_4}_{[x]}\,.$$

Here the sum is over all conjugacy classes of D_4, and $\Theta^{\mathrm{K}_{n-4}[x]}_{[y]}$ denotes the K_{n-4} conjugacy class ϑ-function associated with the D_4 conjugacy class collection $[x]$ in the E_n conjugacy class collection $[y]$.

There exist Weyl rotations of the E_n lattices which act like triality rotations on the D_4 conjugacy classes, and which permute the K_{n-4} ϑ-functions in a corresponding way. Using such a Weyl rotation one can easily derive identities for both factors in the decomposition. The ones for D_4 are nothing new, since we know already that triality rotations generate the Riemann identities. For K_{n-4} we do get something new, namely

$$\Theta^{\mathrm{K}_{n-4}[x]}_{[y]}(z|\Omega) = \Theta^{\mathrm{K}_{n-4}t[x]}_{[y]}(T_{\mathrm{E}_n} z|\Omega)\,. \tag{C.20}$$

Here z_i is a collection of K_{n-4} character parameters, of which there are $n-4$ for each handle of the Riemann surface. For $n=8$ there is just one collection of conjugacy classes $[y]$ at each genus, since E_8 has just one conjugacy class. Furthermore $\mathrm{K}_4 = \mathrm{D}_4$, so that the K_4 identity is again simply the usual Riemann identity.

The identities for K_3 and K_2 are relevant for lower-dimensional superstrings. One can write them down more explicitly by choosing a specific Weyl reflection. Let $\boldsymbol{\lambda}$ be a vector of length 1 belonging to conjugacy class (x) of SO(8). There is a pair of E_n roots of the form $\boldsymbol{\beta}_1 = (\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $\boldsymbol{\beta}_2 = (\boldsymbol{\lambda}, -\boldsymbol{\mu})$. Note that $\boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2 = 0$. Define T_{E_n} to be the composition of Weyl reflections generated by $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, i.e. the matrix T_{E_n} is given by

$$T_{\mathrm{E}_n} z = z - (\boldsymbol{\beta}_1 \cdot z)\boldsymbol{\beta}_1 - (\boldsymbol{\beta}_2 \cdot z)\boldsymbol{\beta}_2 \,.$$

This transformation fixes the conjugacy class (x) and interchanges the other two. Moreover, if z is orthogonal to the D_4 factor then

$$T_{\mathrm{E}_n} z = z - 2(z \cdot \boldsymbol{\mu})\boldsymbol{\mu} = (\mathbf{1} - 2\boldsymbol{\mu} \otimes \boldsymbol{\mu}) z \,,$$

where $\mathbf{1}$ is the identity matrix. From this we can read off the rotation matrix T_{E_n}. For example to obtain the triality rotation that interchanges (v) and (s) we take $\boldsymbol{\lambda} \in (c)$, and for E_6 we can take $\boldsymbol{\mu} = (-\frac{1}{2}, \frac{1}{2}\sqrt{3})$, for E_7 $\boldsymbol{\mu} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2})$ and for E_8 $\boldsymbol{\mu} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence, restricting to the part K_{n-4} of E_n orthogonal to D_4 one obtains the following matrices

$$T_{\mathrm{E}_6} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}, \qquad T_{\mathrm{E}_7} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \end{pmatrix} \tag{C.21}$$

As mentioned already, for E_8 we get $T_{\mathrm{E}_8} = T$, with T as in (C.19).

C.5. Eisenstein functions

In the rest of this appendix we restrict ourselves to genus 1, although generalizations may often be possible. For proofs of some of the results stated in this sub-section we refer to [100, 119, 120], and for a more extensive summary of these results to ref. [91].

The Eisenstein series is defined as

$$G_{2k} = \sum_{m,n \in \mathbb{Z}}' (m\tau + n)^{-2k} \,,$$

where k is a positive integer, and the sum is over all integers m and n except $m = n = 0$. The sum is only well-defined when it is absolutely convergent, that is for $k \geq 2$, and defines a holomorphic function on the complex upper half plane, i.e. it has no poles. In fact it does not even have poles at $\tau = \mathrm{i}\infty$, because

$$G_{2k}(\mathrm{i}\infty) = 2\zeta(2k) = -\frac{(2\pi \mathrm{i})^{2k}}{\cancel{(2k)!}} B_{2k} \,,$$

where ζ is the Riemann ζ-function and B_{2k} are the Bernoulli numbers.

It is elementary to derive the modular properties of G_{2k}. By simply reordering the sum one gets, for $k \geq 2$

$$G_{2k}(\tau + 1) = G_{2k} \,, \qquad G_{2k}(-1/\tau) = \tau^{2k} G_{2k}(\tau) \,.$$

By combining these transformations one can show that

$$G_{2k}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{2k}G_{2k}(\tau)\,.$$

The Eisenstein function is thus an entire modular function of weight $2k$.

Since $G_{2k}(\tau+1)=G_{2k}(\tau)$ it has a Fourier series. Indeed, one can show that for $k\geq 2$

$$G_{2k}(\tau)=\frac{2(2\pi\mathrm{i})^{2k}}{(2k-1)!}\left[-\frac{B_{2k}}{4k}+\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n\right],$$

where $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$ and

$$\sigma_k(n)=\sum_{d|n,\,d>0}d^k\,,$$

i.e. $\sigma_k(n)$ is the sum of the kth powers of the divisors of n. For example

$$G_4(\tau)=(1/45)\pi^4\left[1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right],$$

$$G_6(\tau)=(2/945)\pi^6\left[1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right],$$

$$G_8(\tau)=(16/4725)\pi^8\left[1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right].$$

The functions G_{2k} are not all independent. In fact they form a ring of modular functions generated by G_4 and G_6. This means that all other Eisenstein functions can be written as polynomials in G_4 and G_6, for example $G_8=(144/21)G_4^2$.

The partition function of the even self-dual lattices can all be expressed in terms of this polynomial ring. For example

$$\Theta^{\mathrm{E}_8}_{[0]}(0|\tau)=(45/\pi^4)G_4(\tau)\,.$$

For sixteen-dimensional lattices the partition function has weight 8, and there is just one independent Eisenstein function of that weight, so that the $\mathrm{E}_8\times\mathrm{E}_8$ and $\mathrm{D}_{16}^{(0)+(s)}$ partition functions are both equal to a multiple of G_4^2. For 24-dimensional lattices one can have linear combinations of two independent functions, namely G_4^3 and G_6^2. There exists an interesting relation between these two functions and the Dedekind η-function:

$$\eta(\tau)^{24}=\frac{675}{256\pi^{12}}\,[20G_4^3-49G_6^2]\,.$$

Finally, we should mention the function

$$j(\tau)=\frac{3^6 5^3}{\pi^{12}}\,\frac{G_4^3}{\eta^{24}}=\frac{1}{q}+744+196\,884q+21\,493\,760q^2+\cdots.$$

This is the partition function (consisting of the lattice contribution plus the oscillators) of one sector of a 26-dimensional bosonic string theory compactified to two dimensions on E_8^3. By modifying the constant term (i.e. choosing it equal to the number of gauge-bosons) one gets the analogous partition function for any other Niemeier lattice; the Leech lattice partition function is obtained by replacing 744 by 24. The function j is an absolute modular invariant, and has the amusing property that all of its coefficients (except the constant term) are related to dimensions of monster group representations.

The function $G_2(\tau)$ plays an interesting role in string anomalies. Since it is not defined as it stands, we have to provide a prescription for regulating the sums. Two obvious possibilities are

$$G_2(\tau) = 2\zeta(2) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(m\tau+n)^{-2}$$

$$\hat{G}_2(\tau) = \lim_{s\to 0}(m\tau+n)^{-2}|m\tau+n|^{-s}\,.$$

The first of these definitions leads to a manifestly holomorphic, but not manifestly modular invariant function (since one cannot re-arrange the sums), while for the second one it is just the other way around. The suggested violations of modular invariance and holomorphicity do indeed occur, since

$$\hat{G}_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\hat{G}_2(\tau)\,,$$

but

$$\hat{G}_2(\tau) = G_2(\tau) - \frac{\pi}{\operatorname{Im}\tau}\,. \tag{C.22}$$

Clearly $\hat{G}_2$ is not holomorphic, and G_2 not modular covariant (i.e. invariant up to weight factors). From this relation it follows that G_2 transforms as follows

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - 2\pi\mathrm{i}c(c\tau+d)\,.$$

The Fourier-expansion of G_2 is

$$G_2(\tau) = \tfrac{1}{3}\pi^3\left[1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right]$$

C.6. Some properties of genus-1 ϑ-functions

For the one-loop ϑ-functions it is convenient to define (with arguments suppressed)

$$\vartheta_1 = \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}, \qquad \vartheta_2 = \vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}$$

$$\vartheta_3 = \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}, \qquad \vartheta_4 = \vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}.$$

The modular transformations of these functions are

$$\vartheta_1(z|\tau+1) = e^{i\pi/4}\vartheta_1(z|\tau)\,, \qquad \vartheta_2(z|\tau+1) = e^{i\pi/4}\vartheta_2(z|\tau)\,,$$
$$\vartheta_3(z|\tau+1) = \vartheta_4(z|\tau)\,, \qquad \vartheta_4(z|\tau+1) = \vartheta_3(z|\tau)\,; \tag{C.23}$$

$$\vartheta_1\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = (-i)\, e^{i\pi z^2/\tau}\, \sqrt{-i\tau}\vartheta_1(z|\tau)\,,$$
$$\vartheta_2\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = e^{i\pi z^2/\tau}\, \sqrt{-i\tau}\vartheta_4(z|\tau)\,,$$
$$\vartheta_3\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = e^{i\pi z^2/\tau}\, \sqrt{-i\tau}\vartheta_3(z|\tau)\,, \tag{C.24}$$
$$\vartheta_4\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = e^{i\pi z^2/\tau}\, \sqrt{-i\tau}\vartheta_2(z|\tau)\,,$$

where $\sqrt{-i\tau}$ is defined to lie in the right half of the complex plane. These relations can all be derived easily using the techniques discussed earlier in this appendix.

It is often useful to define the ratio's

$$P\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \frac{\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau)}{\eta(\tau)}\,. \tag{C.25}$$

The η-function transforms as follows

$$\eta(\tau+1) = e^{i\pi/12}\,\eta(\tau)\,, \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)\,. \tag{C.26}$$

The functions $P[{a \atop b}]$ are nothing but the one-loop determinants for two spin-$\frac{1}{2}$ Majorana–Weyl fermions (with a certain choice for the overall phase of these chiral determinants). Using the Hamiltonian formulation of one-loop path-integrals, explained in section 3.2, one can derive the following formula for $P[{a \atop b}]$

$$P\begin{bmatrix} a \\ b \end{bmatrix}(0|\tau) = e^{2\pi i ab} q^{a^2/2-1/24} \prod_{n=1}^{\infty} (1 + q^{n+a-1/2}\, e^{2\pi i b})(1 + q^{n-a-1/2}\, e^{-2\pi i b})\,,$$

with $0 \le a, b \le 1$.

Another useful expression for the ϑ-functions is the expansion of the dependence on their first argument in terms of Eisenstein functions:

$$\frac{\vartheta_1(z|\tau)}{z\vartheta_1'(0|\tau)} = \exp\left\{-\sum_{k=1}^{\infty} \frac{z^{2k}}{2k}\, G_{2k}(q)\right\}, \tag{C.27}$$

$$\frac{\vartheta_2(z|\tau)}{\vartheta_2(0|\tau)} = \exp\left\{-\sum_{k=1}^{\infty} \frac{z^{2k}}{2k}\, [G_{2k}(q) - 2^{2k} G_{2k}(q^2)]\right\}, \tag{C.28}$$

$$\frac{\vartheta_3(z|\tau)}{\vartheta_3(0|\tau)} = \exp\left\{-\sum_{k=1}^{\infty} \frac{z^{2k}}{2k}\,[G_{2k}(q) - G_{2k}(-\sqrt{q})]\right\}, \tag{C.29}$$

$$\frac{\vartheta_4(z|\tau)}{\vartheta_4(0|\tau)} = \exp\left\{-\sum_{k=1}^{\infty} \frac{z^{2k}}{2k}\,[G_{2k}(q) - G_{2k}(\sqrt{q})]\right\}, \tag{C.30}$$

where we have used $z\vartheta_1'$ in the first expression because $\vartheta_1(0|\tau) = 0$. By definition $\vartheta_1'(0|\tau) \equiv \partial/\partial z\ \vartheta_1(z|\tau)|_{z=0}$, and $\sqrt{q} = \exp i\pi\tau$.

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